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Development of 2D Models for Velocities and Pressure Distribution in Viscous Flow Between two Parallel Co-Rotating Discs

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Abstract

The study of flow of fluid between two parallel co-rotating discs continues to be of great concern to engineers and mathematicians. Previous works on this physical phenomenon have extensively applied experimental, analytical and numerical methods such as finite difference, spectral, and boundary element methods of analysis. Consequently, the present study is on the use of finite element method to develop models for predicting velocities and pressure distribution for viscous flow between two parallel discs co-rotating at the same angular velocity.

Finite element method an element-wise and powerful numerical tool is used in this study to solve Continuity and Navier-Stokes equations. The first approach was to reduce the non-linear Navier-Stokes equations to solvable non-dimensional governing differential equations for radial velocity, tangential velocity and pressure by order of magnitude analysis; Galerkin-weighted-residual finite element is then used to discretize the governing equations into element equations which are solved to obtain models to predict velocities and pressures distribution. The resulting finite element solutions for radial velocity, tangential velocity and pressure are compared to the closed-form analytical solutions obtained from the present study's governing equations. Finally, different values of Reynolds number, radius ratio, swirl velocity, and angular velocity are used to determine their effects in the flow domain.

The results obtained showed that finite element method models converged fast to closed-form analytical models as the number of elements are increased, and the more the radial and tangential velocity profiles assumed better parabolic profile. More so, radial and tangential velocities are seen to depend on Reynolds number, radial location, and angular velocity with radial velocity increasing significantly with swirl ratio; while pressure is observed to depend on all the parameters except Reynolds number.

Therefore in this study, two dimensional models for velocity components and pressure distribution have been developed with the intension of aiding in finite element-based Tesla pump/turbine design of experimental set-up and/or software since with the models design geometry and flow parameters can easily be varied.

Keywords: Tesla pump, 2D viscous flow, Navier-Stokes, discs-gap, close-formed, analytical solution, finite element solution

1.0 Introduction

Nikola Tesla in 1913 patented a turbine, referred to as bladeless turbine. This bladeless turbine is based on the principle of centripetal fluid flow over a bladeless disc surfacewith the fluid and disc interface being governed by the adhesion and

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boundary layer effects.
Today, the flow of fluid in Tesla systemis categorized into two main configurations: turbine and pump configurations, which are sometimes simplified into radial inflow or outflow, with or without swirling, between two closelyspaced parallel co-rotating discs. Present study is on the pump configuration, which involves the fluid entering discs gap through a small inlet hole near the discs centre from where it makes contact with the rotating discs to flow radially outwards.The tangential and radial velocity components in this flow are mainly due to the effects of adhesion, boundary layer effects and centrifugal forces.
Fluid flows are known to be governed by Continuity and Navier-Stokes (C-NS) equations. And the solutions to the Navier-Stokes equations by analytical manipulations are often a formidable task because of the inherent non-linear nature of the equations. Therefore, these governing equations are usually simplified into a more workable form, notably by order of magnitude analysis.Consequently, in previous studies [1 – 5] on flow between two rotating discs closed-form analytical solutions were formulated from continuity and Navier-Stokes equations using order of magnitude analysis, and the resulting governing equations are numerically solved (by integral method, Taylor's series) and then compared to Fluent 12 and/or experimental results. Their results approximated closely to previous analytical and/or experimental results. Whereas in other previous studies [6 – 8], and **[**8] the closed-form analytical governing equations are solved by numerical methods, such as finite element method to solve 2D Navier-Stokes equations between porous discs or channel, Other researchers [9 – 12] used finite element method in 1D between parallel plates. Their results are found to converge well with exact or experimental results.
In this present study, therefore, we formulated close-formed analytical equations from C-NS equations and then used the finite element method to solve the close-formed analytical equations in order to develop velocities and pressure models in two-dimensional viscous flow between two parallel co-rotating discs. These models have been developed with the intension of aiding in finite element-based Tesla pump/turbine design of experimental set-up and/or software since with the models design geometry and flow parameters can easily be varied

2.0 Methodology

2.1 Model Geometry

The model geometry for the study is a two dimensional domain in cylindrical coordinates with point O at the midpoint between the two discs (Figure 1).

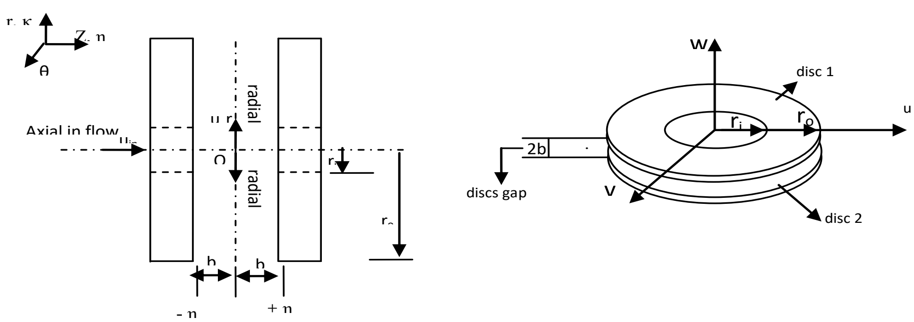


Fig. 1: Model Geometry

Figure 2shows that as the fluid enters at r_i, the flow sustains a uniform flow that becomes laminar significant distant from the inlet.

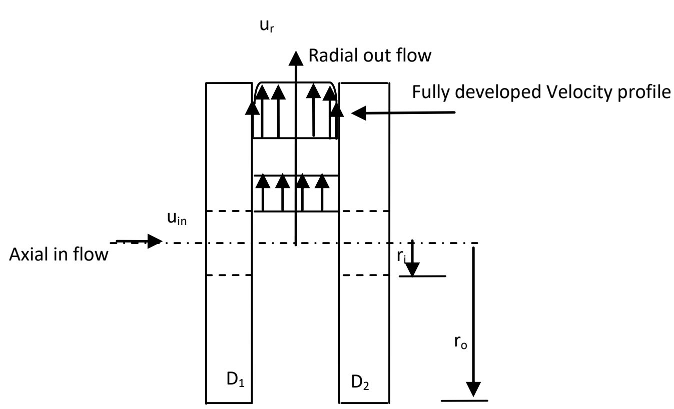


Fig. 2: Radial Outflow Laminar Profile.

2.2 Domain Discretization

The domain Ω (0 ≤ κ ≤ 1;-1 ≤ η ≤ +1) is subdivided into 4, 6 and 8 rectangular elements mesh along the r- and z-axes respectively (see Appendix A).

2.3 Mathematical Formulations

Present study flow problem is governed by Continuity and Navier-Stokes equations (1), (2), (3) and (4)in cylindrical coordinates:

Continuity equation:

$$\frac{1}{r^*}\frac{\partial\left(r^*u_r\right)}{\partial r^*}+\frac{1}{r^*}\frac{\partial v_t}{\partial\theta^*}+\frac{\partial w_a}{\partial z^*}=0 \quad (1)$$

r-momentum equation:

$$u_r\frac{\partial u_r}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial u_r}{\partial\theta^*}-\frac{v_t^2}{r^*}+w_a\frac{\partial u_r}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial r^*}+\nu^*\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^*u_r\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2u_r}{\partial\theta^2}-\frac{2}{r^{*2}}\frac{\partial v_t}{\partial\theta}+\frac{1}{r^*}\frac{\partial^2\left(u_r\right)}{\partial z^{*2}}\right]+\rho g_r \quad (2)$$

θ-momentum equation:

$$u_r\frac{\partial v_t}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial v_t}{\partial\theta^*}+\frac{u_rv_t}{r^*}+w_a\frac{\partial v_t}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{1}{r^*}\frac{\partial p^*}{\partial\theta^*}+\nu^*\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^*v_t\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2v_t}{\partial\theta^{*2}}+\frac{2}{r^{*2}}\frac{\partial u_r}{\partial\theta^*}+\frac{1}{r^*}\frac{\partial^2v_t}{\partial z^{*2}}\right]+\rho g_\vartheta \quad (3)$$

z-momentum equation:

$$u_r\frac{\partial w_a}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial w_a}{\partial\theta^*}+w_a\frac{\partial w_a}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial z^*}+\nu\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^*w_a\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2w_a}{\partial\theta^{*2}}+\frac{1}{r^*}\frac{\partial^2w_a}{\partial z^{*2}}\right]+\rho^*g_z^* \quad (4)$$

2.4 Relevant Assumptions and Boundary Conditions

In order to simplify the non-linear C-NS equations aboveto workable level, the following assumptions are made:the flow is defined under cylindrical polar coordinates (r^*, θ, z^*), the flow domain is symmetrical over θ-coordinate (i.e. $\left(\frac{\partial}{\partial\theta}=0\right)$) with very large discs radii and small gap, the flow is steady-laminar-fully-developed, the fluid is viscous-Newtonian-incompressible-isotropic, the body forces (gravitational and inertia) are negligible,the flow is two-dimensional in the radial and tangential directions but with the flow significantly in radial direction and due to angular velocity, ω ($\left(\frac{\partial p}{\partial\kappa}=0\right)$), no-slip condition exists at discs faces, that because flow profile between any two discs are the same two discs are model,that analysis are carried out with discs angular velocities equal ($\omega^*_{d1}=\omega^*_{d2}$), the source strength Q at inlet is assumed to be zero, with assumptions (i) and (ix) flow is taken to be pure Couette- Poiseuille flowdisc-driven with constant pressure gradient is imposed on the flow.

Applying Coriolis effects, order magnitude analysis and transformation to non-dimensional parameters to equations (1), (2), (3) and (4), the resulting equations are (see Appendix B):

Continuity equation:

$$\frac{du}{d\kappa}=0 \quad (5)$$

r-momentum equation:

$$-2U_o\omega\alpha^2\kappa=-\kappa\frac{dp}{d\kappa}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2u}{\partial\eta^2}\right) \quad (6)$$

θ-momentum equation:

$$2V_o\omega\kappa + V_o\kappa\frac{\partial v}{\partial\kappa} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial\eta^2}\right) \quad (7)$$

z-momentum equation:

$$\frac{dp}{dz} = 0 \quad (8)$$

2.5 Subject to Non-Dimensional Boundary Conditions:

$$\kappa_o : \begin{cases} u^*_{R^*_o(-b)} = 0, u^*_{R^*_o(CL)} = U^*_o, u^*_{R^*_o(+b)} = 0 \\ v^*_{\theta R^*_o(-b)} = R^*_o\omega^*, v^*_{\theta R^*_o(CL)} = V_o, v^*_{\theta R^*_o(+b)} = R_o\omega \end{cases}$$

$$\kappa_i : \begin{cases} u^*_{r^*_i(-b)} = 0, u^*_{r^*_i(CL)} = \dfrac{du_{ri(CL)}}{dz^*}, u^*_{r^*_i(+b)} = 0 \\ v^*_{\theta ri(-b)} = 0, v^*_{\theta ri(CL)} = \dfrac{dv_{\theta ri(CL)}}{dz^*} = 0, v^*_{\theta ri(+b)} = 0 \end{cases} \quad (9)$$

$$p_{\kappa i(\eta=0)} = 0$$

3.0 Method of Solution

We used finite element method to solve the above equations (5), (6) and (7) over domains of (4), (6) and (8) rectangular elements mesh along the r- and z-axes respectively(see Appendix C).

3.1 Determination of Radial Velocity Model

For radial velocity model determination the governing equation is:

$$-2\alpha^2 U_o\omega\kappa = -\kappa\frac{dp}{d\kappa} + \frac{1}{\mathrm{Re}_r}\left(\frac{\partial^2 u}{\partial\eta^2}\right) \quad (10)$$

Subject to these boundary conditions:

$$\kappa = \kappa_o : u_{\kappa o(-\eta)} = 0, U_{o(CL)} = U_{\max}, u_{\kappa o(+\eta)} = 0$$

$$\kappa = \kappa_i : u_{\kappa i(-\eta)} = 0, \frac{dU_{\max}}{d\eta} = 0, u_{\kappa i(+\eta)} = 0 \quad (11)$$

The residual, R, of the governing equation (10)is

$$0 = -\frac{1}{\mathrm{Re}_r}\left[\frac{d^2 u}{d\eta^2}\right] + \kappa\frac{dp}{d\kappa} - 2\omega\alpha^2 U_o\kappa \quad (12)$$

The Galerkin scheme is therefore given as

$$\int_{\Omega^e} wRd\Omega = 0 \quad (13)$$

Where the Galerkin-weighted residual integral is

$$\int_{\Omega^e} w\left[-\frac{1}{\mathrm{Re}_r}\left[\frac{d^2 u}{d\eta^2}\right] + \kappa\frac{dp}{d\kappa} - 2\omega\alpha^2 U_o\kappa\right]d\Omega = 0 \quad (14)$$

The weak form of equation (14)is

$$0 = \int_{\Omega^e} w\left[-\frac{1}{\mathrm{Re}_r}\left[\frac{d^2 u}{d\eta^2}\right] + \kappa\frac{dp}{d\kappa} - 2\omega\alpha^2 U_o\kappa\right]d\Omega$$

$$= \int_{\Omega^e}\left[\frac{1}{\mathrm{Re}_r}\frac{dw}{d\eta}\frac{du}{d\eta} + w\kappa\frac{dp}{d\kappa} - 2w\omega\alpha^2 U_o\kappa\right]d\Omega - \frac{w}{\mathrm{Re}_r}\frac{du}{d\eta}\bigg|_\Gamma \quad (15)$$

Where,
Ω = domain of fluid body between discs = dκdη
w = weight function which depends on κ and η.
Let exact solution u(κ, η) be approximated by the finite element solution u^e over the element domain Ω^e, then,

$$u(\kappa,\eta) = u^e(\kappa,\eta) = \sum_{j=1}^{n} u_j^e \psi_j^e(\kappa,\eta)$$

$$\frac{du}{d\eta} = \sum_{j=1}^{n} u_j^e(\eta)\frac{d\psi_j^e(\eta)}{d\kappa} \qquad (16)$$

Also let w the weight function be

$$w(\kappa,\eta) = \psi_i(\kappa,\eta) \qquad (17)$$

So that

$$\frac{dw}{d\kappa} = \frac{d\psi_i(\kappa)}{d\kappa}$$

$$\frac{dw}{d\eta} = \frac{d\psi_i(\eta)}{d\eta} \qquad (18)$$

Where,
i, j = 1,2,3,…n,
$\psi_j^e(\kappa,\eta)$ = the jth interpolation functions of element e
By substituting equations (16) through (18) into (15) yields,

$$0 = \int_{A^e}\left[\frac{1}{\mathrm{Re}_r}\frac{dw}{d\eta}\frac{du}{d\eta} + w\kappa\frac{dp}{d\kappa} - 2w\alpha^2\omega U_o\kappa\right]dA - \frac{w}{\mathrm{Re}_r}\frac{du}{d\eta}\bigg|_\Gamma$$

$$= \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\frac{1}{\mathrm{Re}_r}\frac{dw}{d\eta}\frac{du}{d\eta} + w\kappa\frac{dp}{d\kappa} - 2w\alpha^2\omega U_o\kappa\right]d\kappa d\eta\theta - \frac{w}{\mathrm{Re}_r}\frac{du}{d\eta}\bigg|_\Gamma$$

$$= \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\frac{1}{\mathrm{Re}_r}\frac{d\psi_i}{d\eta}\sum_{j=1}^{n}u_j\frac{d\psi_j}{d\eta} - \kappa\psi_i\frac{dp}{d\kappa} - 2\kappa\psi_i\alpha^2\omega U_o\right]d\kappa d\eta - \frac{\psi_i}{\mathrm{Re}_r}\frac{du}{d\eta}\bigg|_\Gamma \qquad (19)$$

$$= \sum_{j=1}^{n}U_j\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{b}\left[\frac{1}{\mathrm{Re}_r}\frac{d\psi_i}{d\eta}\frac{d\psi_j}{d\eta}\right]d\kappa d\eta - \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\kappa\psi_i\frac{dp}{d\kappa} + 2\kappa\psi_i\alpha^2\omega U_o\right]d\kappa d\eta - \frac{q_n}{\mathrm{Re}_r}\oint_\Gamma \kappa\psi_i ds$$

$$= \sum_{j=1}^{n}\left[M_{ij}^e\right]u_j^e - \left\{f_i^e\right\} - \left\{Q_i^e\right\}$$

Where,

$$q_n = \left[\frac{du}{d\eta} - \frac{du}{d\eta}\right]_{ri}^{ro} = \frac{du}{d\eta}\bigg|_\Gamma ;\Gamma = elementsurface \qquad (20)$$

In matrix form

$$\left[M_{ij}^e\right]U_j^e = \left\{f_i^e\right\} + \left\{Q_i^e\right\} \qquad (21)$$

Where,

$$M_{ij}^e = \sum_{j=1}^{n}u_j\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\frac{1}{\mathrm{Re}_r}\frac{d\psi_i}{d\eta}\frac{d\psi_j}{d\eta}\right]d\kappa d\eta$$

$$f_i^e = \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\kappa\psi_i\frac{\partial p}{\partial\kappa} + 2\kappa\psi_i\alpha^2 U_o\right]d\kappa d\eta$$

$$Q_i^e = \frac{q_n}{\mathrm{Re}_r}\oint_\Gamma \kappa\psi_i ds \qquad (22)$$

3.2 Determination of Tangential Velocity Model

For tangential velocity model determination the governing equation is:

$$2V_o\omega\kappa + V_o\kappa\frac{\partial v}{\partial\kappa} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial\eta^2}\right) \tag{23}$$

Subject to

$$\kappa = \kappa_o : v_{(-\eta)} = \kappa_o\omega, V_{o(CL)} = V_{\max}, v_{\theta(+\eta)} = \kappa_o\omega$$
$$\kappa = \kappa_i : v_{\theta(-\eta)} = 0, \frac{dV_{\max}}{d\eta} = 0, v_{(+\eta)} = 0 \tag{24}$$

The steps in determination of radial velocity solutions for 4, 6 and 8 elements are followed in determination of tangential velocity solutions from the θ-momentum governing equation(23).

3.3 Determination of Pressure Distribution Model

For pressure distribution model, we have determined governing equation for pressure distribution from r-momentum equation as follows:

$$-2\alpha^2 U_o\omega\kappa = -\kappa\frac{dp}{d\kappa} + \frac{1}{\mathrm{Re}_r}\left(\frac{\partial^2 u}{\partial\eta^2}\right) \tag{25}$$

Pressure in the present study is stated, as one of our assumptions that it does not vary with η only with κ. Hence at the centreline between discs space, $\frac{d}{d\eta}\left(\frac{du}{d\eta}\right) = 0$,of which equation (25) reduces to a one non-dimensional equation:

$$-2\alpha^2 U_o\omega = -\frac{dp}{d\kappa} \tag{26}$$

Subject to

$$p_{\kappa i(\eta=0)} = p_1 = 0 \tag{27}$$

Therefore for pressure distribution, we used linear elements four our analyses as two, three and four linear elements as follows (see Appendix D).

The residual, R, of the governing equation (26) is

$$0 = \frac{dp}{d\kappa} - 2\alpha^2 U_o\omega \tag{28}$$

The Galerkin scheme is therefore given as

$$\int_{\Omega^e} wRd\Omega = 0 \tag{29}$$

Where the Galerkin integral is

$$\int_{\Omega^e} w\left[\frac{dp}{d\kappa} - 2\alpha^2 U_o\omega\right]d\Omega = 0 \tag{30}$$

The weak form of equation 3.86 is

$$0 = \int_{\Omega^e} w\left[\frac{dp}{d\kappa} - 2\alpha^2 U_o\omega\right]d\Omega$$
$$= \int_{\Omega^e}\left[w\frac{dp}{d\kappa} - 2w\alpha^2 U_o\omega\right]d\Omega \tag{31}$$

Where,

Ω^e = linear element length

w = weight function which depends on κ.

Let the exact solution p(κ) be approximated by the finite element solution p^e over the element domain Ω^e, then,

$$p(\kappa) = p^e(\kappa) = \sum_{j=1}^{m} p_j^e\psi_j^e(\kappa)$$
$$\frac{dp}{d\kappa} = \sum_{j=1}^{n} p_j^e(\kappa)\frac{d\psi_j^e(\kappa)}{d\kappa} \tag{32}$$

Also let w the weight function be

$$w(\kappa)=\psi_i(\kappa) \tag{33}$$

So that

$$\frac{dw_i}{d\kappa}=\frac{d\psi_i(\kappa)}{d\kappa} \tag{34}$$

Where,
i, j = 1,2,3,…m,
$\psi_i^e(\kappa)$ = the i[th] interpolation functions of element e
$\psi_j^e(\kappa)$ = the j[th] interpolation functions of element e

By substituting equations (32) through (34) into (30) yields,

$$0=\int_{\Omega^e}\left[w\frac{dp}{d\kappa}-2w\alpha^2U_o\omega\right]d\Omega$$

$$=\int_{\kappa_i}^{\kappa_{i+1}}\left(w\frac{dp}{d\kappa}-2w\alpha^2U_o\omega\right)d\kappa$$

$$=\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\sum_{j=1}^{n}p_j\frac{d\psi_j}{d\kappa}-2\psi_i\alpha^2U_o\omega\right)d\kappa$$

$$=\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\sum_{j=1}^{n}p_j\frac{d\psi_j}{d\kappa}\right)d\kappa-\int_{\kappa_i}^{\kappa_{i+1}}\left(2\psi_i\alpha^2U_o\omega\right)d\kappa$$

$$=\sum_{j=1}^{n}p_j\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\frac{d\psi_j}{d\kappa}\right)d\kappa-2\alpha^2U_o\omega\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\right)d\kappa$$

$$=\left[M_{ij}^e\right]p_j^e-\left\{f_i^e\right\} \tag{35}$$

Where,

$$k_{ij}^e=\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\frac{d\psi_j}{d\kappa}\right)d\kappa$$

$$f_i^e=-2\alpha^2\omega U_o\int_{\kappa_i}^{\kappa_{i+1}}\left(\psi_i\right)d\kappa \tag{36}$$

4.0 Results and Discussion

Table 1: Radial Velocity Centreline Nodal Values

Radial velocity centreline nodal values					
Node number	Node 2	Node 5	Node 8	Node 11	Node 14
Four elements	0.0	0.1137S	0.2049S	-	-
Six elements	0.0	0.0577S	0.1627S	0.2783S	-
Eight elements	0.0	0.1330S	0.1440S	0.1540S	0.3330S

Table 2: Tangential Velocity Centreline Nodal Values

Tangential velocity centreline nodal values					
Node numbers	Nodes 1, 2, 3	Node 4, 5, 6	Node 7, 8, 9	Node 10, 11, 12	Node 13, 14, 15
Four elements	0.0	0.1137T	0.2049T	-	-
Six elements	0.0	0.0577T	0.1627T	0.2783T	-
Eight elements	0.0	0.1330T	0.1440T	0.1540T	0.3330T

Table 3: Pressure Distribution Centreline Nodal Values

Pressure Distribution centreline nodal values				
Node number	Nodes 1	Node 2	Node 3	Node 4
Two elements	-0.500J	-1.000J	-	-
Three elements	-0.333J	-0.667J	-1.000J	-
Four elements	-0.500J	-1.000J	-1.500J	-2.000J

Where, $S = 6a^2b^2\alpha^2U_o\omega Re_r$; $T = 6a^2b^2V_o\omega Re_r$; $J = a\alpha^2U_o\omega$

We have assumed that for any number of elements N, domain equation can be modelled as:

For radial velocity:

$$U^{\Omega}(\kappa,\eta) = U_{\max,N}\left(6a^2b^2U_o\omega\alpha^2\,\mathrm{Re}_r\right) \tag{37}$$

For tangential velocity:

$$V^{\Omega}(\kappa,\eta) = V_{\max,N}\left(6a^2b^2V_o\omega\,\mathrm{Re}_r\right) \tag{38}$$

Where $U_{max,N}$, $V_{max,N}$, $P_{max,N}$ = radial velocity, tangential velocity and pressure distribution maximum nodal value along the centreline.

For pressure distribution:

$$P^{\Omega}(\kappa) = -P_{\max,N}\left(aU_o\omega\alpha^2\right) \tag{39}$$

As a result of the inherent problem encounter in fluid transiting from free stream to fully-developed flow at discs gap inlet, asymptotic analytical solutions to continuity and Navier-Stokes equations have been used [11] and [1] to describe parabolic velocity profile between two discs gap with good accuracy. Consequently, in the present study we have similarly adopted the same approach of developing closed-form analytical solutions to continuity and Navier-Stokes equations subject to this present study's boundary conditions and assumptions (see Appendix D). The resulting asymptotic analytical solutions are:

For radial velocity close-formed approximations are:

$$\begin{aligned} Ue(\kappa,\eta) &= 2b^2U_o\omega\alpha^2\,\mathrm{Re}_r\,\kappa\left(1-\eta^2\right) \to at:\left(-1\le\eta\le+1\right) \\ Ue(\kappa,\eta) &= 2b^2U_o\omega\alpha^2\,\mathrm{Re}_r\,\kappa \to at:\eta=0 \end{aligned} \tag{40}$$

For tangential velocity close-formed approximations are:

$$\begin{aligned} Ve(\kappa,\eta) &= 2b^2V_o\omega\,\mathrm{Re}_r\,\kappa\left(1-\eta^2\right) \to at:-1\le\eta\le+1 \\ Ve(\kappa,\eta) &= 2b^2V_o\omega\,\mathrm{Re}_r\,\kappa \to at:\eta=0; \end{aligned} \tag{41}$$

For pressure distribution:

$$Pe(\kappa,0) = -2U_o\omega\alpha^2\kappa \to at:\eta=0 \tag{42}$$

Where the subscript e stands for analytical solution

Following [1], the present study models with respect to the above asymptotic approximations can be written as:

Radial velocity models:

$$U^{\Omega}(\kappa,\eta) = 6Na^2b^2U_o\omega\alpha^2\,\mathrm{Re}_r\,\kappa\left(1-\eta^2\right) \tag{43}$$

Tangential velocity models:

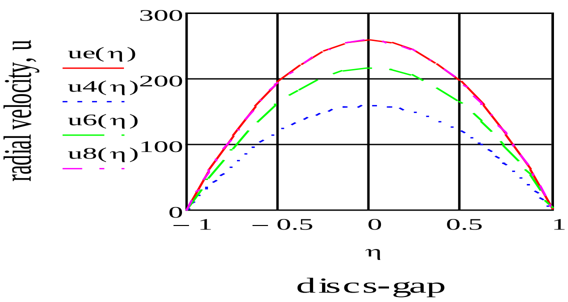
$$V^{\Omega}(\kappa,\eta) = 6Na^2b^2V_o\omega\,\mathrm{Re}_r\,\kappa\left(1-\eta^2\right) \tag{44}$$

Pressure distribution models:

$$P^{\Omega}(\kappa) = -NaU_o\omega\alpha^2\kappa \tag{45}$$

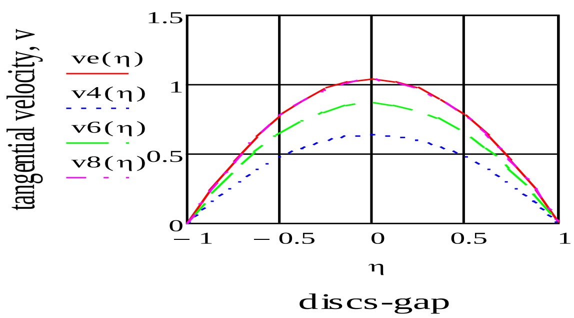
The above results are discussedbased on the following used arbitrary values: a = b =1, $U_o = 5.0$, $V_o = 0.02$, $\omega = 52$, $\alpha = 1$ and $\kappa = 1$ except when they are varied for further investigation.

Figure 4 and Figure 5compare analytical result with present models results for 4, 6 and 8 elements for radial velocity against disc-gap and tangential velocity against disc-gaprespectively. These graphs show that for both analytical and present models results, u and v at the discs surface are zero, thus satisfy the no-slip boundary conditions at discs walls; also since $\omega_1 = \omega_2$ curves converge with the ideal parabolic profile curve. Lastly, from Figures 4 and 5 it is reveal that present models results approximate well the analytical result as the number of elements is increased from 4 to 8.



Legend	
Red	Analytical solution
Blue	Finite element solution (4 elements)
Green	Finite element solution (6 elements)
Pink	Finite element solution (8 elements)

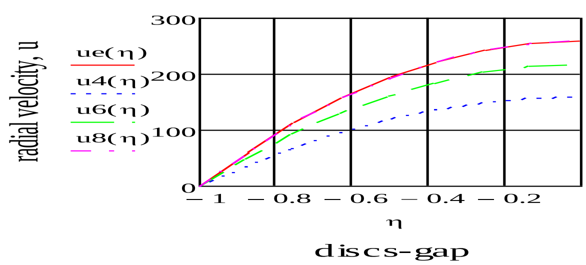
Fig. 4: Comparison of Analytical and Present Modelssolutions Along Discs Gap



Legend	
Red	Analytical solution
Blue	Finite element solution (4 elements)
Green	Finite element solution (6 elements)
Pink	Finite element solution (8 elements)

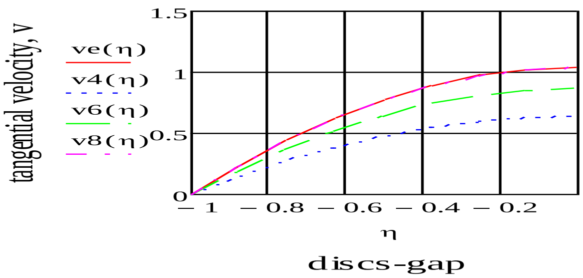
Fig. 5: Comparison of Analytical And present Modelssolutions Between Discs Gap

Figure 6and figure 7 are graphs of radial velocity against half discs gap and tangential velocity against half discs gap respectively. The graph compare analytical result with present models results for 4, 6 and 8 elements. From Figures6and 7, it is revealed that u and v increases from zero at the disc surface towards the mid-section of disc gap due to the effects of boundary layer, centrifugal and adhesion forces, and Coriolis effect.



Legend	
Red	Analytical solution (ue)
Blue	Four element (u4)
Green	Six element (u6)
Pink	Eight element (u8)

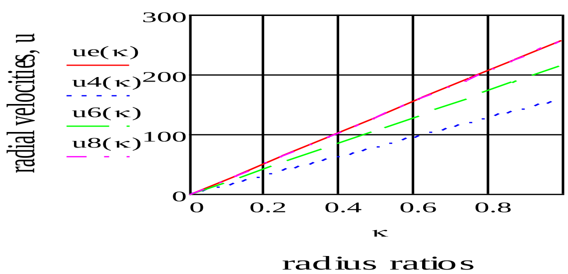
Fig. 6: Comparison of Analytical and Present Model Solutions Between Half Discs Gap



Legend	
Red	Analytical solution (ue)
Blue	Four element (u4)
Green	Six element (u6)
Pink	Eight element (u8)

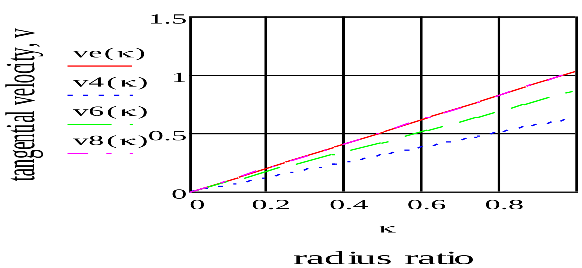
Fig. 7: Comparison of Analytical And Present Model Solutions Between Half Discs Gap

Figure 8and Figure 9 are graphs of radial and tangential velocities against radii ratio respectively. In the graphs analytical solutions are compared with present models solutions for 4, 6 and 8 elements. And the results depict that radial and tangential velocity both increase as radiusratio is increased from zero at the fluid inlet to maximum at disc outlet (i.e. disc periphery).



Legend	
Red	Analytical solution (ue)
Blue	Four element (u4)
Green	Six element (u6)
Pink	Eight element (u8)

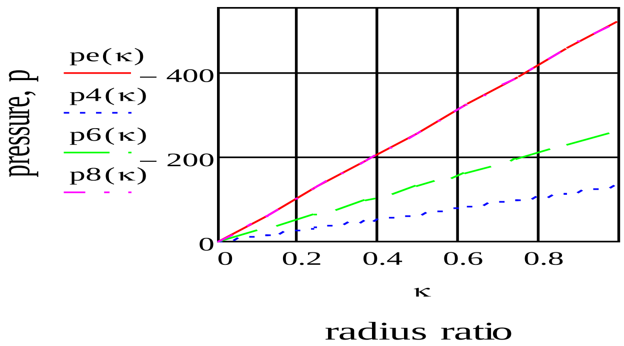
Fig. 8: Comparison of Analytical and Present Model Solutions Along Centreline



Legend	
Red	Analytical solution (ue)
Blue	Four element (u4)
Green	Six element (u6)
Pink	Eight element (u8)

Fig. 9: Comparison of Analytical and Present Model Solutions Along Centreline

Figure 10 is a graph of pressure against radius ratio. The graph compares analytical result with present models results for 2, 3 and 4 elements. From Figure 10 it is revealed that pressure increases as the radius ratio increases from domain inlet to outlet which can be attributed to deceleration of fluid in radial direction and the effect of centrifugal force.



Legend	
Red	Analytical solution (ue)
Blue	Four element (u4)
Green	Six element (u6)
Pink	Eight element (u8)

Fig. 10: Comparison of Analytical and Present Model Solutions Along Centreline

From Figures 11, 12, and 13, it is revealed that pressure increases with flow swirl ratio (α), discs angular velocity (ω), and flow mean velocity(U_o) and independent of Reynolds number.From Figures 11 and 12, since flow is disc-driven both α and ω zero at inlet,but as the discs rotation increases, swirling develops from both discs thus causing pressure to build up proportionately.

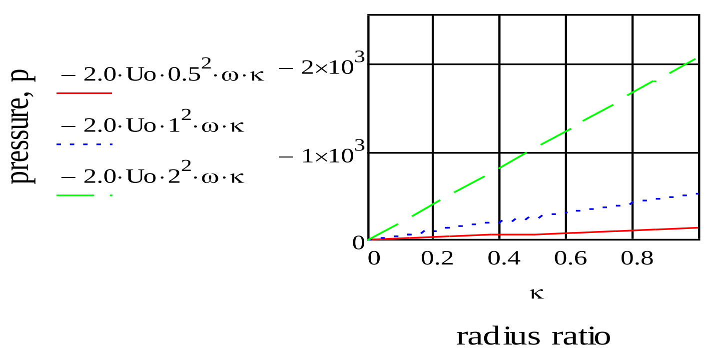


Fig. 11: Effect of Variation of Swirl Ratio On Pressure

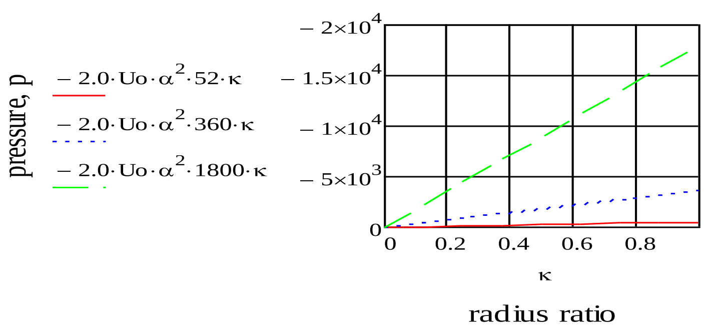


Fig. 12: Effect of Variation of Angular Velocity on Pressure

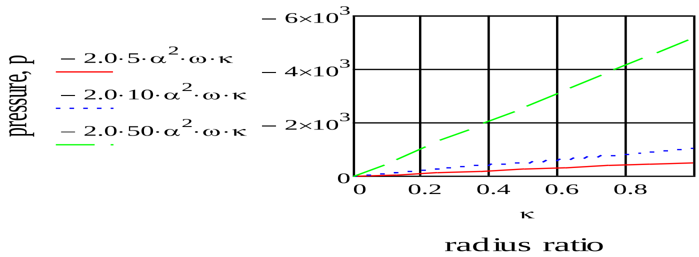
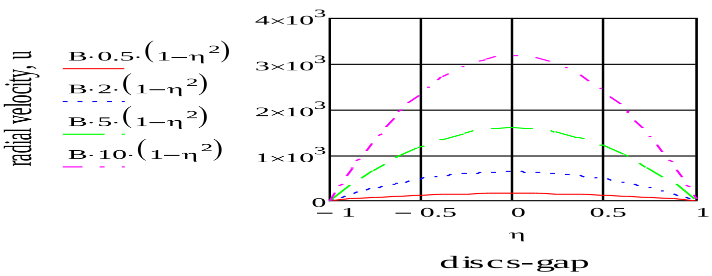


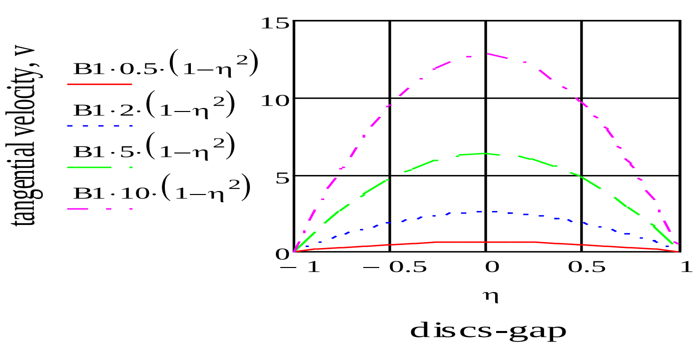
Fig. 13: Effect of Variation of Mean Velocity on Pressure

From Figures14 and 15 radial and tangential velocities can be seen to increase as Reynolds number (R_e) increases from 0.5 to 10. And at R_e = 10, a parabolic profile curve is represented, which can be attributed to high boundary layer and adhesion effects resulting from the geometry of the domain (discs-gap).



Legend	
Red	Re_r = 0.5
Blue	Re_r = 2.0
Green	Re_r = 5.0
Pink	Re_r = 10.0

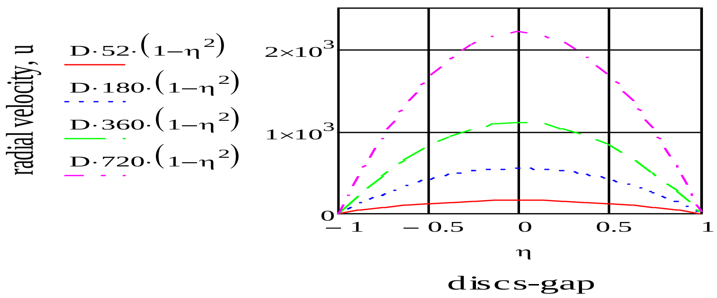
Fig. 14: Variation of Reynolds Number on Radial Velocity-Discs Gap Profile



Legend	
Red	Re_r = 0.5
Blue	Re_r = 2.0
Green	Re_r = 5.0
Pink	Re_r = 10.0

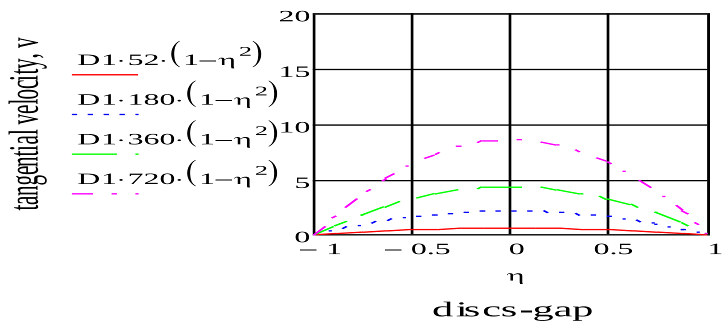
Fig. 15: Variation Reynolds no. on Tangential Velocity-Discs Gap Profile

Figures16 and 17 reveal that as ω increases from 52rad/s to 720rad/s both radial and tangential velocities increase. Since angular velocity is observed to depend on radius location is expected that v increases with ω; also since flow is discs-driven in the direction of increasing radius ratio, u will tend to increase with ω.



Legend	
Red	ω= 52
Blue	ω= 180
Green	ω= 360
Pink	ω= 720

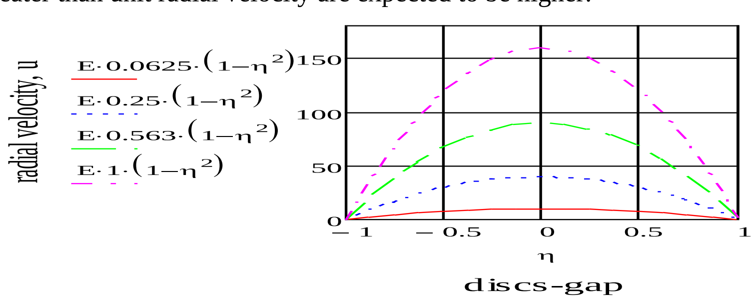
Fig. 16: Variation of Angular Velocity on Radial Velocity-Discs Gap Profile



Legend	
Red	ω= 52
Blue	ω= 180
Green	ω= 360
Pink	ω= 720

Fig. 17: Variation of Angular Velocity on Tangential Velocity-Discs Gap Profile

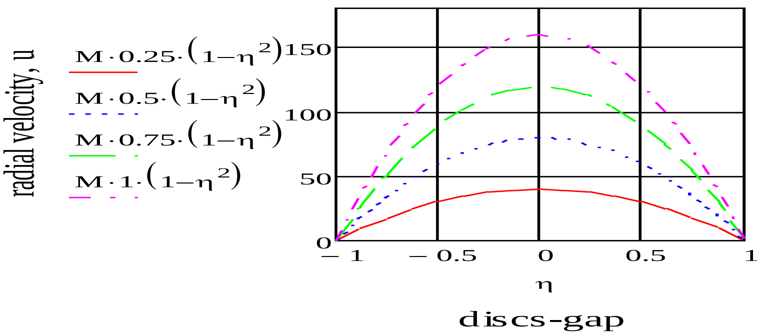
Figure 18 indicate that radial velocity increaseswith swirl ratio.Radial velocity is expected to be low at swirl ratio less than unity this because under this condition slip flow is experienced which results in low radial velocity. At swirl ratio equal to or greater than unit radial velocity are expected to be higher.



Legend	
Red	α = 0.25
Blue	α = 0.50
Green	α = 0.75
Pink	α = 1

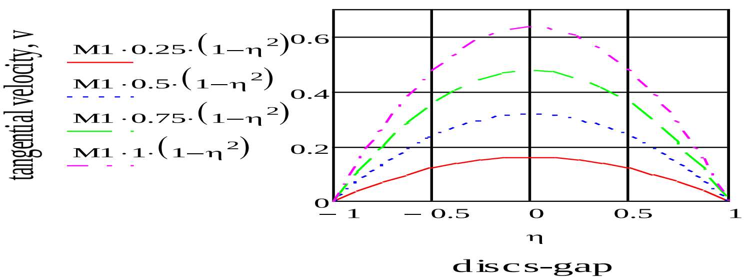
Fig. 18: Variation of Swirl Ratio on Radial Velocity-Discs Gap Profile

From Figures19 and 20, different values of radius ratio (0.25, 0.50, 0.75 and 1.0.) are applied to present models and the result shows radial and tangential velocity both increase as radii ratio increases.



Legend	
Red	κ = 0.25
Blue	κ = 0.50
Green	κ = 0.75
Pink	κ = 1.00

Fig. 19: Variation of Radii Ratio On Radial Velocity-Discs Gap Profile



Legend	
Red	κ = 0.25
Blue	κ = 0.50
Green	κ = 0.75
Pink	κ = 1.00

Fig. 20: Variation of Radii Ratio On Tangential Velocity-Discs Gap Profile

Tables4 and 5 depict comparison of analytical and present study models nodal and percentage error values of 4, 6 and 8 elements for radial and tangential velocities respectively. Tables4 and 5 show how fast the present study model solutions converge with analytical solution for radial and tangential velocities; while Tables6 and 7 show that the error between present model and analytical decreases fast as number of elements is increased.

Table 4: Comparison of Nodal Values of Analytical and Present Models Solutions

Radius locations (κ, 0)	Four elements U4	Six elements U6	Eight elements U8	Analytical solution Ue
0	**0**	**0**	**0**	**0**
0.125	19.976	27.129	32.468	32.500
0.250	39.952	54.259	**64.935**	**65.000**
0.333	53.216	**72.273**	86.493	86.580
0.375	59.928	81.388	97.403	97.500
0.500	**79.904**	108.518	**129.870**	**130.000**
0.625	99.880	135.647	162.338	162.500
0.667	106.592	**144.762**	173.247	173.420
0.750	119.856	162.776	**194.805**	**195.000**
0.875	139.832	189.906	227.273	227.500
1.000	**159.808**	**217.035**	**259.740**	**260.000**

Table 5: Comparison of nodal values of analytical and present models solutions

Radius locations (κ, 0)	Four elements V4	Six elements V6	Eight elements V8	Analytical solution Ve
0	**0**	**0**	**0**	**0**
0.125	0.080	0.109	0.129	0.130
0.250	0.160	0.217	**0.259**	**0.260**
0.333	0.213	**0.289**	0.345	0.346
0.375	0.240	0.326	0.389	0.390
0.500	**0.320**	0.434	**0.519**	**0.520**
0.625	0.399	0.543	0.649	0.650
0.667	0.426	**0.579**	0.693	0.694
0.750	0.479	0.651	**0.779**	**0.780**
0.875	0.559	0.760	0.909	0.910
1.000	**0.639**	**0.868**	**1.039**	**1.040**

Table 6: Percentage error of Table 4 values

Radius locations (κ, 0)	Four elements U4	Six elements U6	Eight elements U8
0	0%	0%	0%
0.125	38.5354	16.5261	0.0985
0.250	38.5354	16.5246	0.1000
0.333	38.5355	16.5246	0.1005
0.375	38.5354	16.5251	0.0995
0.500	38.5354	16.5246	0.1000
0.625	38.5354	16.5249	0.0996
0.667	38.5353	16.5251	0.0997
0.750	38.5354	16.5251	0.1000
0.875	38.5354	16.5248	0.0997
1.000	38.5354	16.5250	0.1000

Table 7: Percentage error of Table 5 nodal values

Radius locations (κ, 0)	Four elements V4	Six elements V6	Eight elements V8
0	0%	0%	0%
0.125	38.4615	16.1538	0.7692
0.250	38.4615	16.5385	0.3846
0.333	38.4393	16.4740	0.2890
0.375	38.4615	16.4103	0.2564
0.500	38.4615	16.5385	0.1923
0.625	38.6154	16.4615	0.1538
0.667	38.6167	16.5706	0.1440
0.750	38.5897	16.5385	0.1282
0.875	38.5714	16.4835	0.1099
1.000	38.5577	16.5385	0.0961

Table 8 shows comparison at the nodes of analytical and present study model solutions for pressure distributions. Present study model can be seen to have converged very fast with the analytical solution; while Table 9 shows that the percentage error between present study models and analytical decreases very fast to zero for 8 elements.

Table 8: Comparison of nodal values of analytical and present models solutions for pressure distribution

Radius locations (κ, 0)	Two elements P2	Three elements P3	Four elements P4	Analytical solution Pe
0	**0**	**0**	**0**	**0**
0.125	-16.25	-32.50	-65.00	-65.00
0.250	-32.50	-65.00	**-130.00**	**-130.00**
0.333	-43.29	**-86.58**	-173.16	-173.16
0.375	-48.75	-97.50	-195.00	-195.00
0.500	**-65.00**	-130.00	**-260.00**	**-260.00**
0.625	-81.25	-162.50	-325.00	-325.00
0.667	-86.71	**-173.42**	-346.84	-346.84
0.750	-97.50	-195.00	**-390.00**	**-390.00**
0.875	-113.75	-227.50	-455.00	-455.00
1.000	**-130.00**	**-260.00**	**-520.00**	**520.00**

Table 9: Percentage error of Table 8 values

Radius locations (κ, 0)	Two elements P2	Three elements P3	Four elements P4
0	0%	0%	0%
0.125	75.0000	50.0000	0.0000
0.250	75.0000	50.0000	0.0000
0.333	75.0000	50.0000	0.0000
0.375	75.0000	50.0000	0.0000
0.500	75.0000	50.0000	0.0000
0.625	75.0000	50.0000	0.0000
0.667	75.0000	50.0000	0.0000
0.750	75.0000	50.0000	0.0000
0.875	75.0000	50.0000	0.0000
1.000	75.0000	50.0000	0.0000

It can be seen in Tables 10 and 11 that present model solutions converges fast with the analytical solution as the number of elements increases for radial and tangential velocities. Tables 12 and 13 show that the errors between analytical and present study models decrease with increase in number of elements.

Table 10: Comparison of analytical and present models solutions for half discs gap

Discs gap $-1 \leq \eta \leq 0$	Four elements U4	Six elements U6	Eight elements U8	Exact solution Ue
1.000	0	0	0	0
0.875	37.455	50.868	60.486	60.938
0.750	69.916	94.953	113.579	113.750
0.625	97.384	132.256	158.200	158.438
0.500	119.857	162.776	194.708	195.000
0.375	137.336	186.514	223.102	223.438
0.250	149.821	203.470	243.384	243.750
0.125	157.312	213.644	255.554	255.938
0	159.809	217.035	259.61	260.000

Table 11: Comparison of analytical and present models solutions for half discs gap

Discs gap $-1 \leq \eta \leq 0$	Four elements V4	Six elements V6	Eight elements V8	Exact solution Ve
1.000	0	0	0	0
0.875	0.150	0.203	0.243	0.244
0.750	0.280	0.380	0.454	0.455
0.625	0.390	0.529	0.633	0.634
0.500	0.479	0.651	0.779	0.780
0.375	0.549	0.746	0.893	0.894
0.250	0.599	0.814	0.974	0.975
0.125	0.629	0.855	1.022	1.024
0	0.639	0.868	1.038	1.040

Table 12: Percentage error of Table 10 values

Discs gap $-1 \leq \eta \leq 0$	Four elements U4	Six elements U6	Eight elements U8
1.000	0.000%	0.000%	0.000%
0.875	38.5359	16.5250	0.7417
0.750	38.5354	16.5248	0.1503
0.625	38.5349	16.5251	0.1502
0.500	38.5349	16.5251	0.1497
0.375	38.5350	16.5254	0.1504
0.250	38.5350	16.5251	0.1502
0.125	38.5351	16.5250	0.1500
0	38.5350	16.5250	0.1500

Table 13: Percentage error of Table 11 values

Discs gap $-1 \leq \eta \leq 0$	Four elements V4	Six elements V6	Eight elements V8
1.000	0.000%	0.000%	0.000%
0.875	38.5245	16.8032	0.4098
0.750	38.4615	16.4835	0.2197
0.625	38.4858	16.5615	0.1577
0.500	38.5897	16.5384	0.1282
0.375	38.5906	16.5548	0.1118
0.250	38.5641	16.5128	0.1025
0.125	38.5742	16.5039	0.1953
0	38.5576	16.5384	0.1923

Tables 14 and 15 compare finite element solutions for 4, 6 and 8 elements with the analytical solution. From the tables it is clearly revealed that present models solution converges fast with the analytical solution as the number of elements increases.

Table 14: Comparison of variation of radial velocity along centreline for analytical and present models solutions

Radius locations (κ, η)	Four elements U4	Six elements U6	Eight elements U8	Analytical solution Ue
(1, 0)	0	0	0	0
(0.875, 0.0)	19.976	27.129	32.451	32.500
(0.750, 0.0)	39.952	54.259	64.903	65.000
(0.625, 0.0)	59.928	81.388	97.354	97.500
(0.500, 0.0)	79.905	108.517	129.805	130.000
(0.375, 0.0)	99.881	135.647	162.256	162.500
(0.250, 0.0)	119.857	162.776	194.708	195.000
(0.125, 0.0)	139.833	189.906	227.159	227.500
(0.0, 0.0)	159.809	217.035	259.61	260.000

Table 15: Comparison of variation of tangential velocity along centreline for analytical and present models solutions

Discs gap (κ, η)	Four elements V4	Six elements V6	Eight elements V8	Analytical solution Ve
(1, 0)	0	0	0	0
(0.875, 0.0)	0.080	0.109	0.130	0.130
(0.750, 0.0)	0.160	0.217	0.260	0.260
(0.625, 0.0)	0.240	0.326	0.389	0.390
(0.500, 0.0)	0.320	0.434	0.519	0.520
(0.375, 0.0)	0.400	0.543	0.649	0.650
(0.250, 0.0)	0.479	0.651	0.779	0.780
(0.125, 0.0)	0.559	0.760	0.909	0.910
(0.0, 0.0)	0.639	0.868	1.038	1.040

Table 16: Percentage error of Table 14 values

Radius locations (κ, η)	Four elements U4	Six elements U6	Eight elements U8
(1, 0)	0	0	0
(0.875, 0.0)	38.5353	16.5261	0.1507
(0.750, 0.0)	38.5353	16.5246	0.1492
(0.625, 0.0)	38.5353	16.5251	0.1497
(0.500, 0.0)	38.5346	16.5253	0.1500
(0.375, 0.0)	38.5347	16.5249	0.1501
(0.250, 0.0)	38.5348	16.5251	0.1497
(0.125, 0.0)	38.5349	16.5248	0.1498
(0.0, 0.0)	38.5350	16.5250	0.1500

Table 17: Percentage error of Table 15 values

Discs gap (κ, η)	Four elements V4	Six elements V6	Eight elements V8
(1, 0)	0	0	0
(0.875, 0.0)	38.4615	16.1538	0.0000
(0.750, 0.0)	38.4615	16.5384	0.0000
(0.625, 0.0)	38.4615	16.4102	0.2564
(0.500, 0.0)	38.4615	16.5384	0.1923
(0.375, 0.0)	38.4615	16.4615	0.1538
(0.250, 0.0)	38.5897	16.5384	0.1282
(0.125, 0.0)	38.5714	16.4835	0.1098
(0.0, 0.0)	38.5576	16.5384	0.1923

In Table 18 is revealed that radial velocity increases at a fast rate even with small increment of Reynolds number, while for tangential velocity the rate of increase of tangential velocity with Reynolds number lower as shown in Table 19.

Table 18: Effect of variation of Reynolds number on radial velocity

η	Re_r = 0.5	Re_r = 2	Re_r = 5	Re_r = 10
0	0.00	0.00	0-00	0.00
0.125	37.455	149.821	374.552	749.105
0.250	69.916	279.666	699.164	$1.398x10^3$
0.375	97.384	389.534	973.836	$1.948x10^3$
0.500	119.857	479.427	$1.199x10^3$	$2.397x10^3$
0.625	137.336	549.343	$1.373x10^3$	$2.747x10^3$
0.750	149.821	599.284	$1.498x10^3$	$2.996x10^3$
0.875	157.312	629.248	$1.573x10^3$	$3.146x10^3$
1.000	159.809	639.236	$1.598x10^3$	$3.196x10^3$

Table 19: Effect of variation of Reynolds number on tangential velocity

η	Re_r = 0.5	Re_r = 2	Re_r = 5	Re_r = 10
0	0.00	0.00	0-00	0.00
0.125	0.15	0.60	1.50	3.00
0.250	0.28	1.12	2.80	5.60
0.375	0.39	1.56	3.90	7.80
0.500	0.48	1.92	4.80	9.60
0.625	0.55	2.20	5.50	11.00
0.750	0.60	2.40	6.00	12.00
0.875	0.63	2.52	6.30	12.60
1.000	0.64	2.56	6.40	12.80

Tables 20 and 21 show that for both radial and tangential velocity that the rate of increase of both velocities with angular velocity is almost double. This implies that both velocities depend much on angular velocity.

Table 20: Effect of variation of angular velocity on radial velocity

η	ω = 52	ω = 180	ω = 360	ω = 720
0	0.00	0.00	0-00	0.00
0.125	37.452	129.642	259.284	518.569
0.250	69.911	241.999	483.998	967.995
0.375	97.376	337.070	674.139	$1.348\text{x}10^3$
0.500	119.847	414.855	829.710	$1.659\text{x}10^3$
0.625	137.325	475.355	950.709	$1.901\text{x}10^3$
0.750	149.809	518.569	$1.037\text{x}10^3$	$2.074\text{x}10^3$
0.875	157.299	544.497	$1.089\text{x}10^3$	$2.178\text{x}10^3$
1.000	159.796	553.140	$1.106\text{x}10^3$	$2.213\text{x}10^3$

Table 21: Variation of angular velocity on tangential velocity

η	ω = 52	ω = 180	ω = 360	ω = 720
0	0.00	0.00	0.00	0.00
0.125	0.146	0.506	1.013	2.025
0.250	0.273	0.945	1.890	3.780
0.375	0.380	1.316	2.633	5.265
0.500	0.468	1.620	3.240	6.480
0.625	0.536	1.856	3.713	7.425
0.750	0.585	2.025	4.050	8.100
0.875	0.614	2.126	4.253	8.505
1.000	0.624	2.160	4.320	8.640

Table 22 shows that radial velocity increase with swirl ratio (V/U). For swirl ratio equal to zero, it implies flow is pure through flow (no swirl condition). Hence, for swirling flow, flow in the radial direction is enhanced thereby increasing radial velocity.

Table 22: Effect of variation of swirl ratio on radial velocity

η	α = 0.25	α = 0.5	α = 0.75	α = 1
0	0.00	0.00	0.00	0.00
0.125	2.341	9.364	21.087	37.455
0.250	4.370	17.479	39.363	69.916
0.375	6.086	24.346	54.827	97.384
0.500	7.491	29.964	67.479	119.857
0.625	8.583	34.334	77.320	137.336
0.750	9.364	37.455	84.349	149.821
0.875	9.832	39.328	88.567	157.312
1.000	9.988	39.952	89.972	159.809

Tables 23 and 24 both show that the rate of increase of radial and tangential velocities with radial location is almost double. This implies that for high radial and tangential velocities very large radius should be considered.

Table 23: Effect of variation of radii ratio on radial velocity

η	κ = 0.25	κ = 0.5	κ = 0.75	κ = 1
0	0.00	0.00	0-00	0.00
0.125	9.364	18.728	28.091	37.455
0.250	17.479	34.958	52.437	69.916
0.375	24.346	48.692	73.038	97.384
0.500	29.964	59.928	89.893	119.857
0.625	34.334	68.668	103.002	137.336
0.750	37.455	74.910	112.336	149.821
0.875	39.328	78.656	117.984	157.312
1.000	39.952	79.904	119.857	159.809

Table 24: Effect of variation of radii ratio on tangential velocity

η	κ = 0.25	κ = 0.5	κ = 0.75	κ = 1
0	0.00	0.00	0-00	0.00
0.125	0.037	0.075	0.112	0.150
0.250	0.070	0.140	0.210	0.280
0.375	0.097	0.195	0.292	0.389
0.500	0.120	0.240	0.359	0.479
0.625	0.137	0.275	0.412	0.549
0.750	0.150	0.300	0.449	0.599
0.875	0.157	0.315	0.472	0.629
1.000	0.160	0.320	0.479	0.639

5.0 Conclusion

Considered in this present study are the comparisons of analytical solutions and finite element models for viscous flow through the gap between two co-rotating discs. From the study the following conclusions were drawn:

1. Since the NS equations for fluid flow has the inherent difficult of being solved directly relevant assumptions are made on the continuity and Navier-Stokes equations to simplify them using order of magnitude analysis. Thereby three asymptotic analytical governing equations for radial and tangential velocity as well as for pressure distribution were developed.
2. The developed asymptotic analytical formulations were solved analytically and the result used as bases of comparison with developed models.
3. The models developed reveal confirmed that finite element method can used as good approximation to analytical solution as the results obtained from the present study are highly accurate and converge well with the analytical solutions as the number of elements in the domain is increased. More so, the results obtained show that the radial and tangential velocities depend on Reynolds number, radial location, swirl ratio, and angular velocity; while pressure depends on all parameters except Reynolds number.

5.1 Recommendation

1. It was observed that finite element method has not been fully applied to fluid flow between parallel co-rotating discs especially for 2D and 3D problems. Therefore, it is recommended that other element shapes and sizes be analyzed.
2. Finite element procedure employed in this present work should be applied to turbine configuration using quadratic elements.
3. Numerical simulation should be carried out to validate models developed.

5.2 Contribution to Knowledge

In this study, two dimensional models have been developed for velocity components and pressure distribution for flow between two parallel co-rotating discs in a Tesla pump by using the finite element method. Thesemodels have been developed with the intension of aiding in finite element-based Tesla pump/turbine design of experimental set-up and/or software since with the models design geometry and flow parameters can easily be varied

6.0 References

[1.] Sengupta, S. and Guha, A. (2012): A theory of Tesla disc turbines. Journal of Power and Energy Vol. 266, No. 5 (pp 650-663).

[2.] Tsifourdaris, P. (2003). On the flows developed within the gap of two parallel discs. Phdthesis, Concordia University, Quebec. Available online: spectrum.library.concordia.ca/1998/1/NQ77909. Retrieved: 22/10/2012

[3.] Podergajs, M. (2011): The Tesla Turbine. Seminar paper. University of Ljubljana, Faculty of Mathematics and Physics. Available online: www.f1.ijs.si/~rudi/sola/Tesla_Turbine. Retrieved: 22/10/2012

[4.] Batista, M. (2007): A note on steady flow of incompressible fluid between two co-rotating disks. Available online: arXiv: physics/ 0703005. Retreived: 22/10/2012

[5.] Ghaly, W. S. and Vatistas, G. H. (2001): A parametric study of axisymmetric swirling flows in a low aspect ratio vortex chamber. Proceedings of ICFDP7: International Conference on Fluid Dynamics and Propulsion, Egypt. Available online: http://icfd11.org/ICFD7/2001041.PDF. Retrieved: 22/10/2012

[6.] Takhar, H. S.,Bhargava,R., Agrawal, R.S. and Balaji, A.V.S. (2000): Finite element solution of micropolar fluid flow and heat transfer between two porous discs. International Journal of Engineering Science 38 (pp 1907-1922).

[7.] Yu, X., Lu, H. Y., Sun, J. N., Luo, X., and Xu, G. Q.(2012): Study of the Pressure Drop for Radial Inflow between Co-Rotating Disks, ICAS 2012, 28th International Congress of The Aeronautical Sciences. Available online: www.icas.org/ICAS_ARCHIVE/ICAS2012/PAPERS/622.PDF. Retrieved: 9/6/2014

[8.] Torii, S. and Yang, W. (2008): Thermal-Fluid Transport Phenomena Between twin Rotating Parallel Disks. International Journal of Rotating Machinery, Vol. 2008, Article ID 406809, 6 pages. Available online: www.hindawi.com/journals/ijrm/2008/406809. Retrieved: 2/9/2010).

[9.] Akpobi, J. A. and Akpobi, E. D. (2007): A finite element analysis of the distribution of velocity in viscous incompressible fluids using the Lagrange interpolation function. Journal of Applied Science, Environmental and Management Vol. 11(1) 31 – 38

[10.] Jang, Y. (2009): Boundary layer analysis with Navier-Stokes equation in 2D channel flow. Proceedings of the 7th IASME/WSEAS International Conference on Fluid Mechanics and Aerodynamics, Jordan. (Pp 101-106). Available online: www.ecs.umass.edu/mie/labs/mda/fea/fealib/jang/jangReport). Retrieved: 2/9/2010).

[11.] Oliveira, M. D. C. and Pascoa, J. C. (2009): Analytical and experimental modelling of a viscous disc pump for MEMS applications. III Conferencia Nacional em Mecanica de Fluidos, Termodinamica e Energia (MEFTE-BRAGANCA 09). Retrieved: 22/7/2011.

[12.] Breiter, M. C. and Pohlhausen, K. (1962): Laminar flow between two parallel rotating discs. USA: Aeronautical Research Laboratory Office of Aerospace Research United States Air Force. Available online: www.dtic.mil/dtic/tr/fulltext/u2/275562. Retrieved: 2/5/2014.

[13.] Takhar, H. S., Bhargava, R., and Agrawal, R.S. (2001): Finite element solution of micropolar fluid flow from an enclosed rotating disc with suction and injection. International Journal of Engineering Science 39 (pp913-927).

[14.] Reddy, J. N. (1993): An Introduction To The Finite Element Method 2nd Ed. New York: McGraw-Hill, Inc.

[15.] Reddy, K. K. (2010). Analytical and numerical Solutions of Flow Problems. 9th Indo-German Winter Academy. Indian Institute of Technology, Guwahati. Available online: www.leb.eei.uni-erlangen.de/winterakademie/2010/.../pdf. Retrieved: 2/9/2013.

[16.] Allen, J. S. (1990):A model for fluid flow between parallel co-rotating annular disks. Masters Thesis. University of Dayton, Ohio. Available online: www.me.mtu.edu/~jstallen/publications/reports/jsa_msthesis. Retrieved: 2/6/2009.

[17.] Corsini, A. (1999): A finite element method for the computational fluid dynamics of turbomachinery. Lecture Notes. Technical University of Budapest. Available online: https:/www.ara.bme.hu/oktatas/letolt/sln-tubx99. Retrieved: 2/9/2013.

[18.] Buchanan, G. R. (1995): Schaum's Outline Series Finite Element Analysis.New York: McGraw-Hill, Inc.

[19.] Dechaumphal, P., Triputtarat, J., and Sikkhabandit, S. (1998): A finite element method for viscous incompressible flow analysis. Thammasat International Journal of Technology Vol. 3, No. 2 (pp 60-68)

[20.] Dechaumphal, P. and Kanjanakijkasem, W. (1999): A finite element method for viscous incompressible thermal flows. ScienceAsia Vol. 25 (pp 165-172)

[21.] Lawn, M. J. and Rice, W. (1974): Calculated design data for the multiple-disk turbine using incompressible fluid. Transactions of the ASME (pp 252-258)

[22.] Attia, H. A. (2010): Asymptotic solution for rotating disc flow in a porous medium. Journal of mechanics and Mechanical Engineering Vol. 14, No. 1 (pp 119-136)

[23.] Huang, J., Wei, J. and Qiu, M. (2011):Laminar Flow in the Gap between Two Rotating Parallel Frictional Plates in Hydro-viscous Drive. Chinese Journal of Mechanical Engineering, Vol. 24, No.*. (pp 1-9).Available online: www.cjmenet.com; www.cjmenet.com.cn. Retrieved: 4/4/2012

[24.] Jawarneh, A. M. (2009): The flow characteristic in a sink-swirl flow within two discs. Proceedings of the 7th IASME/WSEAS International Conference on Fluid Mechanics and Aerodynamics, Jordan. (Pp 101-106). Available online: www.wseas.us/e-library/conferences/2009/moscow/FMA/FMA14. Retrieved: 2/9/2013.

[25.] Kumar, A., Agrawal, S.P., and Kaur, P.P. (2013): Finite element Galerkin's approach for viscous incompressible fluid flow through a porous medium in a coaxial cylinders. Journal of Mathematical Sciences and Applications Vol. 1, No. 3 (pp 39-42).

[26.] Wikipedia (2013). Coriolis effect. Available online: (http://3n.wikipedia.org/wiki/Coriolis_effect). Retrieved: 26/2/2013

[27.] Wikipedia (2013). Boundary layer. Available online: (http://3n.wikipedia.org/wiki/Boundary_layer). Retrieved: 26/2/2013

[28.] Wikipeedia (2014). Tesla turbine. Available on;line: (http://en.wikipedia.org/wiki/Tesla_turbine). Retrieved: 10/11/2014

[29.] Nikola Tesla, "Our Future Motive Power". Available online: (http:/www.tfcbooks.com/tesla/1931-12-00.htm). Retrieved: Retrieved: 2/2/2010.

Appendix A

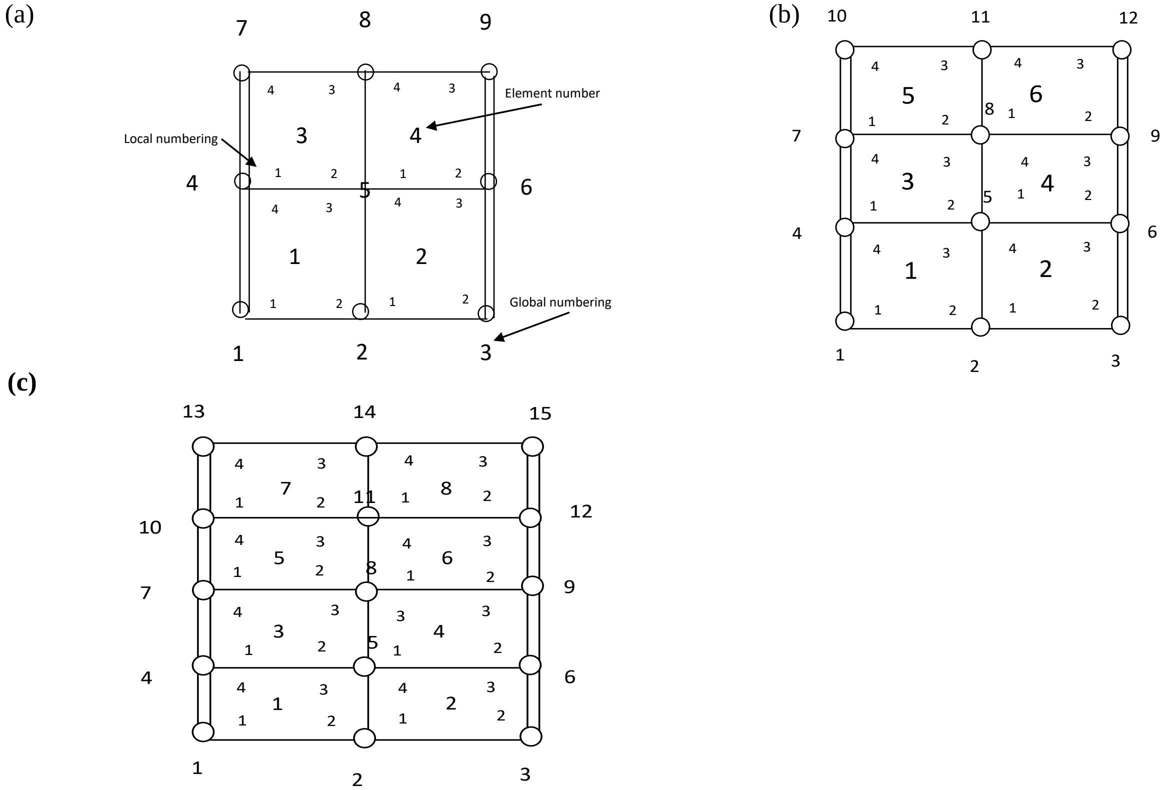


Fig. 1: A four, six and eight elements mesh

Appendix B

DERIVATION OF NON-DIMENSIONAL GOVERNING EQUATIONS FROM CONTINUITY AND NAVIER-STOKES EQUATIONS IN POLAR COORDINATES:

Continuity equation (CE)

$$\frac{1}{r^*}\frac{\partial\left(r^* u_r\right)}{\partial r^*}+\frac{1}{r^*}\frac{\partial v_t}{\partial\theta^*}+\frac{\partial w_a}{\partial z^*}=0 \tag{1}$$

r-momentum equation(r-ME)

$$u_r\frac{\partial u_r}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial u_r}{\partial\theta^*}-\frac{v_t^2}{r^*}+w_a\frac{\partial u_r}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial r^*}+v^*\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^* u_r\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^{*2}}\frac{\partial v_t}{\partial\theta}+\frac{1}{r^*}\frac{\partial^2\left(u_r\right)}{\partial z^{*2}}\right]+\rho g_r \tag{2}$$

θ-momentum equation(θ-ME)

$$u_r\frac{\partial v_t}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial v_t}{\partial\theta^*}+\frac{u_r v_t}{r^*}+w_a\frac{\partial v_t}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{1}{r^*}\frac{\partial p^*}{\partial\theta^*}+v^*\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^* v_t\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 v_t}{\partial\theta^{*2}}+\frac{2}{r^{*2}}\frac{\partial u_r}{\partial\theta^*}+\frac{1}{r^*}\frac{\partial^2 v_t}{\partial z^{*2}}\right]+\rho g_\vartheta \tag{3}$$

z-momentum equation(z-ME)

$$u_r\frac{\partial w_a}{\partial r^*}+\frac{v_t}{r^*}\frac{\partial w_a}{\partial\theta^*}+w_a\frac{\partial w_a}{\partial z^*}$$
$$=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial z^*}+v\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^* w_a\right)}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 w_a}{\partial\theta^{*2}}+\frac{1}{r^*}\frac{\partial^2 w_a}{\partial z^{*2}}\right]+\rho^* g_z^* \tag{4}$$

By applying the assumptions and boundary conditions to above equations themreduce to
Continuity equation (CE)

$$\frac{1}{r^*}\frac{\partial\left(r^* u_r\right)}{\partial r^*}=0 \tag{5}$$

r-momentum equation(r-ME)

$$-\frac{v_t^2}{r^*}=-\frac{1}{\rho^*}\frac{\partial p^*}{\partial r^*}+\nu^*\left[\frac{1}{r^*}\frac{\partial^2\left(u_r\right)}{\partial z^{*2}}\right] \tag{6}$$

θ-momentum equation(θ-ME)

$$u_r\frac{\partial v_t}{\partial r^*}+\frac{u_r v_t}{r^*}=\nu^*\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\frac{\partial\left(r^* v_t\right)}{\partial r^*}\right)+\frac{1}{r^*}\frac{\partial^2 v_t}{\partial z^{*2}}\right] \tag{7}$$

z-momentum equation(z-ME)

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{8}$$

Applying Coriolis effects and then ignoring all quadratic terms in v, u and ω the equations above reduce as follow:

$$v_\theta=v+\bar{v}=+r\omega \tag{9}$$

Where,
v_θ (r, θ) = disc absolute relative velocity
v = fluid (azimuthal) velocity with reference to frame (disc)
$\bar{v}$ = disc angular velocity
r = the disc radius
ω = angular velocity of the disk.
r-momentum:

$$\nu^*\left[\frac{1}{r^*}\frac{\partial^2\left(u_r\right)}{\partial z^{*2}}\right]-\frac{1}{\rho^*}\frac{\partial p^*}{\partial r^*}=-\frac{\left(v_\theta^2+2vr^*\omega^*+r^{*2}\omega^{*2}\right)}{r^*}=-2v\omega^* \tag{10}$$

θ-momentum:

$$\nu^*\left(\frac{\partial^2\left(v_\theta+r^*\omega^*\right)}{\partial z^{*2}}\right)=u_r\frac{\partial\left(v_\theta+r^*\omega^*\right)}{\partial r^*}+\frac{u_r\left(v_\theta+r^*\omega^*\right)}{r^*}$$

$$\nu^*\left(\frac{\partial^2 v_\theta}{\partial z^{*2}}\right)=u_r\frac{\partial v_\theta}{\partial r^*}+2u_r\omega^* \tag{11}$$

(3) To enable larger domain space, we transform to Non-dimensional Parameters using:

$$b=\frac{b^*}{z^*},\, a=\frac{a^*}{R_o^*}\eta=\frac{z^*}{b^*},\, \kappa=\frac{r^*}{R_o^*},\, u=\frac{u_r}{U_o^*},\, v=\frac{v_\theta}{V_o^*},\, \omega=\frac{\omega^*}{\Omega_o^*},\, \Omega_o^*=\frac{V_o^*}{R_o^*},\, p=\frac{p^*}{\rho^* U_o^{*2}},$$

$$U_o=\frac{U_o^*}{\omega^* R_o^*}=\frac{Q_b}{2\pi\omega R_o^{*2}b^*},\, V_o=\frac{V_o^*}{\omega^* R_o^*}=\frac{Q_b}{2\pi\omega^* R_o^{*2}b^*},\, \xi=\frac{b^*}{R_o^*},$$

$$\mathrm{Re}=\frac{\rho^* U_o^* b^{*2}}{\mu^*},\, \mathrm{Re}_r=\xi\,\mathrm{Re},\, \alpha=\frac{V_o^*}{U_o^*}$$

Where u_r, v_θ, ω^*, p^*, U^*_o, V^*_o, r^*, z^*, b^*, R^*_o, Ω^*_o and p^*_o are dimensional parameters while a, b, η, r, u, v, p, U_o, and V_o, Re, α and ξ (<<1 aspect ratio) are the dimensionless parameters. Note that U_o, V_o and p_o are the dimensionless mean radial velocity, mean tangential velocity and mean pressure respectively. These non-dimensional parameters are substituted into NS equations which reduced them to the following simplified asymptotic equations:
Continuity equation (CE)

$$\frac{\partial\left(\kappa R_o^* u U_o^*\right)}{\partial\left(\kappa R_o^*\right)} = \frac{\kappa R_o^* U_o^*}{\kappa R_o^*}\frac{\partial u}{\partial \kappa} = \frac{\partial u}{\partial \kappa} = 0 \tag{12}$$

r-momentum equation

$$\frac{\mu^*}{\rho^*}\left[\frac{\partial^2\left(uU_o^*\right)}{\kappa R_o^*\partial\eta^2\left(b^{*2}\right)^2}\right] - \frac{\partial p\left(\rho^* U_o^{*2}\right)}{\rho^*\partial\kappa(R_o^*)} = -2vV_o^*\omega\Omega_o = -\frac{2vV_o^*\omega V_o^*}{R_o^*}$$

$$\frac{\mu^* U_o^*}{\rho^* b^{*2}\kappa R_o^*}\left[+\frac{\partial^2 u}{\partial\eta^2}\right] - \frac{\rho^* U_o^{*2}}{\rho^* R_o^*}\frac{\partial p}{\partial\kappa} = -\frac{2vV_o^{*2}\omega}{R_o^*} \tag{13}$$

Multiplying thru by $\frac{R_o^*}{U_o^{*2}}$ yields

$$-\frac{1}{\mathrm{Re}_r}\left(\frac{\partial^2 u}{\partial\eta^2}\right) = -\kappa\frac{\partial p}{\partial\kappa} + 2U_o\omega\alpha^2\kappa \tag{14}$$

$U_o =$ constant = is non-dimensional maximum velocity at centreline.

θ-momentum equation

$$\frac{\mu^*}{\rho^*}\frac{1}{R_o^*}\left(\frac{1}{\kappa}\frac{\partial^2 v\left(V_o^*\right)}{\partial\eta^2\left(b^{*2}\right)}\right) = uU_o^*\frac{\partial v\left(V_o^*\right)}{\partial\kappa\left(R_o^*\right)} + \frac{2uU_o^*\omega V_o^*}{R_o^*}$$

$$\frac{\mu^* V_o^*}{\rho^*\kappa R_o^* b^{*2}}\left(\frac{\partial^2 v}{\partial\eta^2}\right) = \frac{uU_o^* V_o^*}{R_o^*}\frac{\partial v}{\partial\kappa} + \frac{2uU_o^*\omega V_o^*}{R_o^*} \tag{15}$$

And multiplying thru by $\frac{R_o^*}{U_o^* V_o^*}$ yields

$$\frac{1}{\mathrm{Re}_r}\left(\frac{\partial^2 v}{\partial\eta^2}\right) = \kappa\frac{\partial v}{\partial\kappa} + 2U_o\omega\alpha^2\kappa \tag{16}$$

Where, α = swirl ratio = V^*_o/U^*_o = 1; U^*_o and V^*_o are dimensional radial and tangential centreline average (or maximum) velocities respectively since this is where the maximum centrifugal force occurs.

Therefore, simplified governing continuity and Navier-Stokes equations are:

Continuity equation:

$$\frac{\partial u}{\partial\kappa} = 0 \tag{17}$$

r-momentum equation:

$$-\frac{1}{\mathrm{Re}_r}\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\eta}\right) = -\kappa\frac{\partial p}{\partial\kappa} + \kappa H \tag{18}$$

Where,

$$H = 2U_o\alpha^2\omega$$

θ-momentum equation:

$$\frac{1}{\mathrm{Re}_r}\frac{\partial}{\partial\eta}\left(\frac{\partial v}{\partial\eta}\right) = \kappa\frac{\partial v}{\partial\kappa} + \kappa G \tag{19}$$

Where,

$$G = 2V_o\omega$$

Subject to the following non-dimensional boundary conditions:

$$R_o : \begin{cases} u_{\kappa o(-z)} = 0, U_{(CL)} = U_{\max}, u_{\kappa o(+z)} = 0 \\ v_{(ro,-b)} = r_o\omega, V_{(CL)} = V_{\max}, v_{(ro,+z)} = r_o\omega \end{cases}$$

$$R_i : \begin{cases} u_{\kappa i(-z)} = 0, \dfrac{dU_{\max}}{dz} = 0, u_{\kappa i(+z)} = 0 \\ v_{\theta(ri,-b)} = 0, \dfrac{dV_{\max}}{dz} = 0, v_{(ri,+b)} = 0 \end{cases}$$

$$p_{\kappa i(z=0)} = 0$$

(20)

Appendix C

RADIAL AND TANGENTIAL VELOCITIES FINITE ELEMENT ANALYSIS

For four (4) linear rectangular elements, the following approximation functions are used (Reddy, 1998):

$$\psi_1^e = \left(1 - \frac{r}{a}\right)\left(1 - \frac{z}{b}\right) = 1 - \frac{z}{b} - \frac{r}{a} + \frac{rz}{ab}$$

$$\frac{\partial \psi_1^e}{\partial r} = -\frac{1}{a} + \frac{z}{ab} \quad \frac{\partial \psi_1^e}{\partial z} = -\frac{1}{b} + \frac{r}{ab}$$

$$\psi_2^e = \frac{r}{a}\left(1 - \frac{z}{b}\right) = \frac{r}{a} - \frac{rz}{ab}$$

$$\frac{\partial \psi_2^e}{\partial r} = \frac{1}{a} - \frac{z}{ab} \quad \frac{\partial \psi_2^e}{\partial z} = -\frac{r}{ab}$$

$$\psi_3^e = \frac{rz}{ab}$$

$$\frac{\partial \psi_3^e}{\partial r} = \frac{z}{ab} \quad \frac{\partial \psi_3^e}{\partial z} = \frac{r}{ab}$$

$$\psi_4^e = \frac{z}{b}\left(1 - \frac{r}{a}\right) = \frac{z}{b} - \frac{rz}{ab}$$

$$\frac{\partial \psi_4^e}{\partial r} = -\frac{z}{ab} \quad \frac{\partial \psi_4^e}{\partial z} = \frac{1}{b} - \frac{r}{ab}$$

(1) Four Elements

Derivation of stiff matrices: $k_{ij}^e = \dfrac{1}{\mathrm{Re}_r} \int_{\kappa_i}^{\kappa_{i+1}} \int_{\eta_i}^{\eta_{i+1}} \left[\dfrac{d\psi_i}{d\eta} \dfrac{d\psi_j}{d\eta} \right] d\eta d\kappa$

$$k_{11}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{7a}{48b}\quad k_{31}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}$$

$$k_{12}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}\quad k_{32}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{48b}$$

$$k_{13}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}\quad k_{33}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{48b}$$

$$k_{14}^{1}=\int_{0}^{a/}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-7a}{48b}\quad k_{34}^{1}=\int_{0}^{a/2}\int_{0}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}$$

$$k_{21}^{1}=\int_{0}^{a/}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}\quad k_{41}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7a}{48b}$$

$$k_{22}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{48b}\quad k_{42}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}$$

$$k_{23}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{48b}\quad k_{43}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}$$

$$k_{24}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}\quad k_{44}^{1}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{7a}{48b}$$

$$k_{11}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{7a}{48b}\quad k_{31}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}$$

$$k_{12}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}\quad k_{32}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{48b}$$

$$k_{13}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}\quad k_{33}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{48b}$$

$$k_{14}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-7a}{48b}\quad k_{34}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}$$

$$k_{21}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}\quad k_{41}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7a}{48b}$$

$$k_{22}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{48b}\quad k_{42}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}$$

$$k_{23}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{48b}\quad k_{43}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{24b}$$

$$k_{24}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-a}{24b}\quad k_{44}^{2}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{7a}{48b}$$

$$k_{11}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{48b} \quad k_{31}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b}$$

$$k_{12}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{24b} \quad k_{32}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{48b}$$

$$k_{13}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b} \quad k_{33}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{48b}$$

$$k_{14}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{48b} \quad k_{34}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{a}{24b}$$

$$k_{21}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{24b} \quad k_{41}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{48b}$$

$$k_{22}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{48b} \quad k_{42}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b}$$

$$k_{23}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{48b} \quad k_{43}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{24b}$$

$$k_{24}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b} \quad k_{44}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{a}{48b}$$

$$k_{11}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{48b} \quad k_{31}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b}$$

$$k_{12}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{24b} \quad k_{32}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{48b}$$

$$k_{13}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b} \quad k_{33}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{48b}$$

$$k_{14}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{48b} \quad k_{34}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{a}{24b}$$

$$k_{21}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{24b} \quad k_{41}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{48b}$$

$$k_{22}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{48b} \quad k_{42}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b}$$

$$k_{23}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{48b} \quad k_{43}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{24b}$$

$$k_{24}^{3} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{24b} \quad k_{44}^{3} = \int_{a/2}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{a}{48b}$$

Derivation of force matrices: $N_{ij}^{e} = \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\kappa\psi_i \frac{d\psi_j}{d\kappa}\right)d\eta d\kappa$

$$N_{11}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_1 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-7ab}{288} \quad N_{31}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288}$$

$$N_{12}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_1 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7ab}{288} \quad N_{32}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288}$$

$$N_{13}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_1 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{144} \quad N_{33}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{ab}{576}$$

$$N_{14}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_1 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{ab}{144} \quad N_{34}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{576}$$

$$N_{21}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_2 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-7ab}{576} \quad N_{41}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{144}$$

$$N_{22}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_2 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7ab}{576} \quad N_{42}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{144}$$

$$N_{23}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_2 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288} \quad N_{43}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288}$$

$$N_{24}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_2 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288} \quad N_{44}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288}$$

$$N_{11}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_1 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288} \quad N_{31}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_3 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288}$$

$$N_{12}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_1 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288} \quad N_{32}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_3 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288}$$

$$N_{13}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_1 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{ab}{144} \quad N_{33}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_3 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{7ab}{576}$$

$$N_{14}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_1 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{144} \quad N_{34}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_3 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-7ab}{576}$$

$$N_{21}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_2 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{576} \quad N_{41}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_4 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{144}$$

$$N_{22}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_2 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{576} \quad N_{42}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_4 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{ab}{144}$$

$$N_{23}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_2 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{ab}{288} \quad N_{43}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_4 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{7ab}{288}$$

$$N_{24}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_2 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-ab}{288} \quad N_{44}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\kappa\psi_4 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{7ab}{288}$$

$$N_{11}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}\quad N_{11}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}$$

$$N_{12}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{1}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}\quad N_{12}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}$$

$$N_{13}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{1}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{ab}{144}\quad N_{13}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{576}$$

$$N_{14}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{1}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{144}\quad N_{14}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{576}$$

$$N_{21}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-49ab}{576}\quad N_{41}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{144}$$

$$N_{22}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{49ab}{576}\quad N_{42}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{ab}{144}$$

$$N_{23}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}\quad N_{43}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{ab}{288}$$

$$N_{24}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}\quad N_{44}^{3}=\int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{288}$$

$$N_{11}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{288}\quad N_{31}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}$$

$$N_{12}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{1}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{ab}{288}\quad N_{32}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}$$

$$N_{13}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{1}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{ab}{144}\quad N_{33}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{49ab}{576}$$

$$N_{14}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{1}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{144}\quad N_{34}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-49ab}{576}$$

$$N_{21}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{576}\quad N_{41}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\eta}\right)d\eta d\kappa=\frac{-ab}{144}$$

$$N_{22}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{576}\quad N_{42}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\eta}\right)d\eta d\kappa=\frac{ab}{144}$$

$$N_{23}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}\quad N_{43}^{4}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\eta}\right)d\eta d\kappa=\frac{7ab}{288}$$

$$N_{24}^{3}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}\quad N_{44}^{3}=\int_{a/2}^{a}\int_{b/2}^{b}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\eta}\right)d\eta d\kappa=\frac{-7ab}{288}$$

Derivation of stiffness matrix: $M_{ij}^{e} = \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\psi_i \frac{d\psi_j}{d\kappa}\right)d\eta d\kappa$

$$M_{11}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_1 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-7b}{64} \quad M_{31}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{96}$$

$$M_{12}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_1 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7b}{64} \quad M_{32}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{b}{96}$$

$$M_{13}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_1 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{32} \quad M_{33}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{192}$$

$$M_{14}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_1 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{32} \quad M_{34}^{1} = \int_0^{a/3}\int_0^{b/2}\left(\kappa\psi_3 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{192}$$

$$M_{21}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_2 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-7b}{192} \quad M_{41}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{32}$$

$$M_{22}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_2 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7b}{192} \quad M_{42}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{b}{32}$$

$$M_{23}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_2 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{96} \quad M_{43}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{64}$$

$$M_{24}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\psi_2 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{96} \quad M_{44}^{1} = \int_0^{a/2}\int_0^{b/2}\left(\kappa\psi_4 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{64}$$

$$M_{11}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{64} \quad M_{31}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{96}$$

$$M_{12}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{b}{64} \quad M_{32}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7b}{192}$$

$$M_{13}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{32} \quad M_{33}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{-7b}{192}$$

$$M_{14}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{32} \quad M_{34}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{b}{96}$$

$$M_{21}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{192} \quad M_{41}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_1}{d\kappa}\right)d\eta d\kappa = \frac{-b}{32}$$

$$M_{22}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{b}{192} \quad M_{42}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_2}{d\kappa}\right)d\eta d\kappa = \frac{7b}{64}$$

$$M_{23}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{b}{96} \quad M_{43}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_3}{d\kappa}\right)d\eta d\kappa = \frac{-7b}{64}$$

$$M_{24}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{-b}{96} \quad M_{44}^{2} = \int_0^{a/2}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_4}{d\kappa}\right)d\eta d\kappa = \frac{b}{32}$$

$$M_{11}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_1 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-7b}{192} \quad M_{11}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_3 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{32}$$

$$M_{12}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_1 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{7b}{192} \quad M_{12}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_3 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{32}$$

$$M_{13}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_1 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{96} \quad M_{13}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_3 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{64}$$

$$M_{14}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_1 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{96} \quad M_{14}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_3 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{64}$$

$$M_{21}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_2 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-7b}{64} \quad M_{41}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_4 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{96}$$

$$M_{22}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_2 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{7b}{64} \quad M_{42}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_4 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{96}$$

$$M_{23}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_2 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{32} \quad M_{43}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_4 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{192}$$

$$M_{24}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\kappa\psi_2 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{32} \quad M_{44}^{3} = \int_{a/2}^{a}\int_{0}^{b/2}\left(\psi_4 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{192}$$

$$M_{11}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{192} \quad M_{31}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{32}$$

$$M_{12}^{4} = \int_{a/}^{a}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{192} \quad M_{32}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{32}$$

$$M_{13}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{96} \quad M_{33}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{7b}{64}$$

$$M_{14}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_1 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{96} \quad M_{34}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_3 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-7b}{64}$$

$$M_{21}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{64} \quad M_{41}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_1}{d\eta}\right)d\eta d\kappa = \frac{-b}{96}$$

$$M_{22}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{64} \quad M_{42}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_2}{d\eta}\right)d\eta d\kappa = \frac{b}{96}$$

$$M_{23}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{b}{32} \quad M_{43}^{4} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_3}{d\eta}\right)d\eta d\kappa = \frac{7b}{192}$$

$$M_{24}^{3} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_2 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-b}{32} \quad M_{44}^{3} = \int_{a/2}^{a}\int_{b/2}^{b}\left(\psi_4 \frac{d\psi_4}{d\eta}\right)d\eta d\kappa = \frac{-7b}{192}$$

Derivation of local force matrices: $f_i^e = H\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\psi_i\kappa\right)d\eta d\kappa$

$$f_1^1 = H\int_0^{a/2}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{a^2b}{32} \quad f_1^3 = H\int_{a/2}^{a}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{a^2b}{32}$$

$$f_2^1 = H\int_0^{a/2}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{a^2b}{64} \quad f_2^3 = H\int_{a/2}^{a}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{7a^2b}{64}$$

$$f_3^1 = H\int_0^{a/3}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{a^2b}{192} \quad f_3^3 = H\int_{a/2}^{a}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{7a^2b}{192}$$

$$f_4^e = H\int_0^{a/2}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{a^2b}{96} \quad f_4^3 = H\int_{a/2}^{a}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{a^2b}{96}$$

$$f_1^2 = H\int_0^{a/2}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{a^2b}{96} \quad f_1^4 = H\int_{a/2}^{a}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{a^2b}{96}$$

$$f_2^2 = H\int_0^{a/2}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{a^2b}{192} \quad f_2^4 = H\int_{a/2}^{a}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{7a^2b}{192}$$

$$f_3^2 = H\int_0^{a/2}\int_{b/2}^{b}(\psi_3\kappa)\,d\eta d\kappa = \frac{a^2b}{64} \quad f_3^4 = H\int_{a/2}^{a}\int_{b}^{2b}(\psi_3\kappa)\,d\eta d\kappa = \frac{7a^2b}{64}$$

$$f_4^2 = H\int_0^{a/2}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{a^2b}{32} \quad f_4^4 = H\int_{a/2}^{a}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{a^2b}{32}$$

$$\left[K^e\right] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}$$

$$\left[K^1\right] = \frac{1}{576b\,\mathrm{Re}_r}\begin{bmatrix} 77 & 31 & -22 & -86 \\ 17 & 8.5 & -13 & -25 \\ -25 & -11 & 12.5 & 23.5 \\ -86 & -22 & 25 & 83 \end{bmatrix} \quad \left[K^2\right] = \frac{1}{576b\,\mathrm{Re}_r}\begin{bmatrix} 83 & 25 & -22 & -86 \\ 23.5 & 12.5 & -11 & -25 \\ -25 & -11 & 15.5 & 20.5 \\ -86 & -22 & 31 & 77 \end{bmatrix}$$

$$\left[K^3\right] = \frac{1}{576b\,\mathrm{Re}_r}\begin{bmatrix} 5 & 25 & -22 & -14 \\ -0.5 & 108.5 & -77 & -31 \\ -31 & -77 & 87.5 & 20.5 \\ -14 & -22 & 25 & 11 \end{bmatrix} \left[K^4\right] = \frac{1}{576b\,\mathrm{Re}_r}\begin{bmatrix} 11 & 25 & -22 & -14 \\ 20.5 & 87.5 & -77 & -25 \\ -31 & -77 & 108.5 & 0.5 \\ -14 & -22 & 31 & -5 \end{bmatrix}$$

Force vectors

$$\left[f^1\right] = H\begin{Bmatrix} \frac{a^2b}{32} \\ \frac{a^2b}{64} \\ \frac{a^2b}{64} \\ \frac{a^2b}{96} \end{Bmatrix} \left[f^2\right] = H\begin{Bmatrix} \frac{a^2b}{96} \\ \frac{a^2b}{192} \\ \frac{a^2b}{64} \\ \frac{a^2b}{32} \end{Bmatrix} \left[f^3\right] = H\begin{Bmatrix} \frac{a^2b}{32} \\ \frac{7a^2b}{64} \\ \frac{7a^2b}{192} \\ \frac{a^2b}{96} \end{Bmatrix} \left[f_4^1\right] = H\begin{Bmatrix} \frac{a^2b}{96} \\ \frac{7a^2b}{192} \\ \frac{7a^2b}{64} \\ \frac{a^2b}{32} \end{Bmatrix}$$

Assembling of element equations to obtain domain equations for the four elements:

$$K_{11}=k_{11}^{1} \quad K_{24}=k_{24}^{1} \quad K_{67}=k_{23}^{3} \quad K_{58}=k_{14}^{3}+k_{32}^{4} \quad K_{75}=k_{31}^{3}$$
$$K_{12}=k_{12}^{1} \quad K_{32}=k_{21}^{2} \quad K_{68}=k_{24}^{3} \quad K_{59}=k_{24}^{4} \quad K_{77}=k_{33}^{3}$$
$$K_{15}=k_{13}^{1} \quad K_{33}=k_{22}^{2} \quad K_{51}=k_{31}^{1} \quad K_{41}=k_{41}^{1} \quad K_{78}=k_{34}^{3}$$
$$K_{14}=k_{14}^{1} \quad K_{36}=k_{23}^{2} \quad K_{52}=k_{32}^{1}+k_{41}^{2} \quad K_{42}=k_{42}^{1} \quad K_{86}=k_{42}^{3}$$
$$K_{21}=k_{21}^{1} \quad K_{35}=k_{24}^{2} \quad K_{53}=k_{42}^{2} \quad K_{45}=k_{43}^{1}+k_{12}^{4} \quad K_{85}=k_{41}^{3}+k_{32}^{4}$$
$$K_{22}=k_{22}^{1}+k_{11}^{2} \quad K_{62}=k_{31}^{2} \quad K_{56}=k_{42}^{2}+k_{12}^{3} \quad K_{44}=k_{44}^{1}+k_{11}^{4} \quad K_{84}=k_{31}^{4}$$
$$K_{23}=k_{12}^{2} \quad K_{63}=k_{32}^{2} \quad K_{55}=k_{33}^{1}+k_{44}^{2}+k_{11}^{3}+k_{22}^{4} \quad K_{48}=k_{13}^{4} \quad K_{87}=k_{43}^{3}$$
$$K_{26}=k_{13}^{2} \quad K_{66}=k_{33}^{2}+k_{22}^{3} \quad K_{56}=k_{34}^{1}+k_{21}^{4} \quad K_{49}=k_{14}^{4} \quad K_{88}=k_{44}^{3}+k_{33}^{4}$$
$$K_{25}=k_{23}^{1}+k_{14}^{2} \quad K_{65}=k_{34}^{2}+k_{21}^{3} \quad K_{57}=k_{13}^{3} \quad K_{76}=k_{32}^{3} \quad K_{89}=k_{34}^{4}$$
$$K_{95}=k_{42}^{4} \; K_{94}=k_{41}^{4} \quad K_{98}=k_{43}^{4} \quad K_{99}=k_{44}^{4}$$

$$\left[K^{\Omega}\right]=\begin{bmatrix} k_{11}^{1} & k_{12}^{1} & 0 & k_{14}^{1} & k_{13}^{1} & 0 & 0 & 0 & 0 \\ k_{21}^{1} & k_{22}^{1}+k_{11}^{2} & k_{12}^{2} & k_{24}^{1} & k_{23}^{1}+k_{14}^{2} & k_{13}^{2} & 0 & 0 & 0 \\ 0 & k_{21}^{2} & k_{22}^{2} & 0 & k_{24}^{2} & k_{23}^{2} & 0 & 0 & 0 \\ k_{41}^{1} & k_{42}^{1} & 0 & k_{44}^{1}+k_{11}^{4} & k_{43}^{1}+k_{12}^{4} & 0 & k_{14}^{4} & k_{13}^{4} & 0 \\ k_{31}^{1} & k_{32}^{1}+k_{41}^{2} & k_{42}^{2} & k_{34}^{1}+k_{21}^{4} & k_{33}^{1}+k_{44}^{2}+k_{11}^{3}+k_{22}^{4} & k_{43}^{2}+k_{12}^{3} & k_{24}^{4} & k_{23}^{4}+k_{14}^{3} & k_{13}^{3} \\ 0 & k_{31}^{2} & k_{32}^{2} & 0 & k_{34}^{2}+k_{21}^{3} & k_{33}^{2}+k_{22}^{3} & 0 & k_{24}^{3} & k_{23}^{3} \\ 0 & 0 & 0 & k_{41}^{4} & k_{42}^{4} & 0 & k_{44}^{4} & k_{43}^{4} & 0 \\ 0 & 0 & 0 & k_{31}^{4} & k_{32}^{4}+k_{41}^{3} & k_{42}^{3} & k_{34}^{4} & k_{33}^{4}+k_{44}^{3} & k_{43}^{3} \\ 0 & 0 & 0 & 0 & k_{31}^{3} & k_{32}^{3} & 0 & k_{34}^{3} & k_{33}^{3} \end{bmatrix}$$

$$\left[K^{\Omega}\right]=\begin{bmatrix} 77 & 31 & 0 & -86 & -22 & 0 & 0 & 0 & 0 \\ 20.5 & 32 & 25 & -25 & -99 & -22 & 0 & 0 & 0 \\ 0 & -25 & -11 & 0 & -25 & -11 & 0 & 0 & 0 \\ -86 & -22 & 0 & 94 & 50 & 0 & -14 & 26 & 0 \\ -25 & -97 & -22 & -49 & 182 & 51 & -25 & -101 & -22 \\ 0 & -25 & -11 & 0 & 20 & 124 & 0 & -31 & -77 \\ 0 & 0 & 0 & -14 & 26 & 0 & 5 & 31 & 0 \\ 0 & 0 & 0 & -31 & -101 & -22 & -0.5 & 119.5 & 25 \\ 0 & 0 & 0 & 0 & -31 & -77 & 0 & 20.5 & 87.5 \end{bmatrix}$$

For a = b = 1 and $Re_r = 0.5$

Assembling of local elements equations to obtain domain force vector equation:

$$\left[F^{\Omega}\right]=H\begin{Bmatrix} f_1^{1} \\ f_2^{1}+f_1^{2} \\ f_2^{2} \\ f_4^{1}+f_1^{3} \\ f_3^{1}+f_4^{2}+f_2^{3}+f_1^{4} \\ f_3^{2}+f_2^{4} \\ f_4^{3} \\ f_3^{3}+f_4^{4} \\ f_3^{4} \end{Bmatrix}=H\begin{Bmatrix} \frac{a^2b}{32} \\ \frac{5a^2b}{192} \\ \frac{a^2b}{192} \\ \frac{8a^2b}{192} \\ \frac{40a^2b}{192} \\ \frac{10a^2b}{192} \\ \frac{a^2b}{96} \\ \frac{13a^2b}{192} \\ \frac{7a^2b}{64} \end{Bmatrix}=\frac{a^2bH}{192}\begin{Bmatrix} 6 \\ 5 \\ 1 \\ 8 \\ 40 \\ 10 \\ 2 \\ 13 \\ 21 \end{Bmatrix}=\frac{a^2bH}{192}\begin{Bmatrix} 6 \\ 5 \\ 1 \\ 8 \\ 0 \\ 10 \\ 2 \\ 13 \\ 21 \end{Bmatrix}$$

Since no flux is specified at node 5,

$$f_3^1 + f_4^2 + f_2^3 + f_1^4 = 0$$

(2) Six Elements

Derivation of stiff matrices: $k_{ij}^e = \frac{1}{\mathrm{Re}_r}\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\frac{d\psi_i}{d\eta}\frac{d\psi_j}{d\eta}\right]d\eta d\kappa$

$$k_{11}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b} \quad k_{31}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{12}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{32}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b}$$

$$k_{13}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{33}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{162b}$$

$$k_{14}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b} \quad k_{34}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{21}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{41}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b}$$

$$k_{22}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{162b} \quad k_{42}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{23}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad k_{43}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{24}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{44}^1 = \int_0^{a/3}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b}$$

$$k_{11}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b} \quad k_{31}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{12}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{32}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b}$$

$$k_{13}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{33}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{162b}$$

$$k_{14}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b} \quad k_{34}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{21}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{41}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b}$$

$$k_{22}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{162b} \quad k_{42}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{23}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad k_{43}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{24}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{44}^2 = \int_0^{a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b}$$

$$k_{11}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad k_{31}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$k_{12}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad k_{32}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$k_{13}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad k_{33}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$k_{14}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad k_{34}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$k_{21}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad k_{41}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$k_{22}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad k_{42}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$k_{23}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad k_{43}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$k_{24}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad k_{44}^{3} = \int_{a/3}^{2a/3}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$k_{11}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad k_{31}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$k_{12}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad k_{32}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$k_{13}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad k_{33}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$k_{14}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad k_{34}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$k_{21}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad k_{41}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$k_{22}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad k_{42}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$k_{23}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad k_{43}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$k_{24}^{3} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad k_{44}^{4} = \int_{a/3}^{2a/3}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$k_{11}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{162b} \quad k_{31}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{12}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{32}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b}$$

$$k_{13}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{33}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b}$$

$$k_{14}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad k_{34}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{21}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{41}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{162b}$$

$$k_{22}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b} \quad k_{42}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{23}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b} \quad k_{43}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{324b}$$

$$k_{24}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{44}^5 = \int_{2a/3}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b}$$

$$k_{11}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{a}{162b} \quad k_{31}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{12}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{32}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b}$$

$$k_{13}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{33}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b}$$

$$k_{14}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad k_{34}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{21}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b} \quad k_{41}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b}$$

$$k_{22}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{19a}{162b} \quad k_{42}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b}$$

$$k_{23}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-19a}{162b} \quad k_{43}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{324b}$$

$$k_{24}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{324b} \quad k_{44}^6 = \int_{2a/3}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{a}{162b}$$

Derivation of force matrices: $N_{ij}^e = \int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\kappa\psi_i\frac{d\psi_j}{d\kappa}\right)d\eta d\kappa$

$$N_{11}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_1\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{3888}\quad N_{31}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_3\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{972}$$

$$N_{12}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_1\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{3888}\quad N_{32}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_3\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{ab}{972}$$

$$N_{13}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_1\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\quad N_{33}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_3\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{ab}{1944}$$

$$N_{14}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_1\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\quad N_{34}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_3\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{1944}$$

$$N_{21}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_2\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\quad N_{41}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_4\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}$$

$$N_{22}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_2\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\quad N_{42}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_4\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{23}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_2\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{ab}{972}\quad N_{43}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_4\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{3888}$$

$$N_{24}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_2\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{972}\quad N_{44}^{1}=\int_{0}^{a/3}\int_{0}^{b/2}\left(\kappa\psi_4\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{3888}$$

$$N_{11}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_1\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{3888}\quad N_{31}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_3\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{972}$$

$$N_{12}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_1\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{3888}\quad N_{32}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_3\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{ab}{972}$$

$$N_{13}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_1\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\quad N_{33}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_3\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{14}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_1\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\quad N_{34}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_3\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}$$

$$N_{21}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_2\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{1944}\quad N_{41}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_4\frac{d\psi_1}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}$$

$$N_{22}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_2\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{ab}{1944}\quad N_{42}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_4\frac{d\psi_2}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{23}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_2\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{ab}{972}\quad N_{43}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_4\frac{d\psi_3}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{3888}$$

$$N_{24}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_2\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-ab}{972}\quad N_{44}^{2}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_4\frac{d\psi_4}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{3888}$$

$$N_{11}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-91ab}{3888}\quad N_{31}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{972}$$

$$N_{12}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{91ab}{3888}\quad N_{32}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{972}$$

$$N_{13}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{1944}\quad N_{33}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{14}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{1944}\quad N_{34}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}$$

$$N_{21}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{1944}\quad N_{41}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{1944}$$

$$N_{22}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{1944}\quad N_{42}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{1944}$$

$$N_{23}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{972}\quad N_{43}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{3888}$$

$$N_{24}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{972}\quad N_{44}^{3}=\int_{0}^{a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{3888}$$

$$N_{11}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{3888}\quad N_{31}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{972}$$

$$N_{12}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{3888}\quad N_{32}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{972}$$

$$N_{13}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{1944}\quad N_{33}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{1944}$$

$$N_{14}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{1944}\quad N_{34}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{1944}$$

$$N_{21}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\quad N_{41}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-13ab}{1944}$$

$$N_{22}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\quad N_{42}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{13ab}{1944}$$

$$N_{23}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{972}\quad N_{43}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{91ab}{3888}$$

$$N_{24}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{972}\quad N_{44}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-91ab}{3888}$$

$$N_{11}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{3888}\qquad N_{31}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-19ab}{972}$$

$$N_{12}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{3888}\qquad N_{32}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{972}$$

$$N_{13}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\qquad N_{33}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{1944}$$

$$N_{14}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\qquad N_{34}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-19ab}{1944}$$

$$N_{21}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-133ab}{1944}\qquad N_{41}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-17ab}{1944}$$

$$N_{22}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{133ab}{1944}\qquad N_{42}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{23}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{972}\qquad N_{43}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{3888}$$

$$N_{24}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-719ab}{972}\qquad N_{44}^{5}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{3888}$$

$$N_{11}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{1}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{3888}\qquad N_{31}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-19ab}{972}$$

$$N_{12}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{3888}\qquad N_{32}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{972}$$

$$N_{13}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}\qquad N_{33}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{133ab}{1944}$$

$$N_{14}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}\qquad N_{34}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{3}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-133ab}{1944}$$

$$N_{21}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-19ab}{1944}\qquad N_{41}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{1}}{d\kappa}\right)d\eta d\kappa=\frac{-7ab}{1944}$$

$$N_{22}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{1944}\qquad N_{42}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{2}}{d\kappa}\right)d\eta d\kappa=\frac{7ab}{1944}$$

$$N_{23}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{19ab}{972}\qquad N_{43}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{3}}{d\kappa}\right)d\eta d\kappa=\frac{49ab}{3888}$$

$$N_{24}^{6}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{2}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-19ab}{972}\qquad N_{44}^{4}=\int_{a}^{2a}\int_{b}^{2b}\left(\kappa\psi_{4}\frac{d\psi_{4}}{d\kappa}\right)d\eta d\kappa=\frac{-49ab}{3888}$$

$$\left[K^{e}\right]=\begin{bmatrix}k_{11}&k_{12}&k_{13}&k_{14}\\k_{21}&k_{22}&k_{23}&k_{24}\\k_{31}&k_{32}&k_{33}&k_{34}\\k_{41}&k_{42}&k_{43}&k_{44}\end{bmatrix}+\begin{bmatrix}N_{11}&N_{12}&N_{13}&N_{14}\\N_{21}&N_{22}&N_{23}&N_{24}\\N_{31}&N_{32}&N_{33}&N_{34}\\N_{41}&N_{42}&N_{43}&N_{44}\end{bmatrix}$$

$$\left[K^1\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}454 & 108.5 & -77 & -463\\ 77 & 131 & -22 & -86\\ -86 & -22 & 85 & 85\\ -463 & -77 & 87.5 & 449\end{bmatrix}$$

$$\left[K^2\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}449 & 87.5 & -91 & -463\\ 83 & 25 & -22 & -86\\ -86 & 25 & -17 & 77\\ -463 & 91 & -80.5 & 560.5\end{bmatrix}$$

$$\left[K^3\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}122.5 & 201.5 & -143 & -181\\ 107 & 217 & -154 & -170\\ -170 & -154 & 175 & 149\\ -181 & -143 & 162.5 & 161.5\end{bmatrix}$$

$$\left[K^4\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}161.5 & 162.5 & -143 & -181\\ 149 & 175 & -154 & -170\\ -170 & -154 & 217 & 107\\ -181 & -143 & 201.5 & 122.5\end{bmatrix}$$

$$\left[K^5\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}-0.5 & 108.5 & -77 & -31\\ -49 & 589 & -418 & -122\\ -122 & -418 & 475 & 65\\ -31 & -77 & 87.5 & 20.5\end{bmatrix}$$

$$\left[K^6\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}20.5 & 87.5 & -77 & -31\\ 65 & 475 & -418 & -122\\ -122 & -418 & 589 & -49\\ -31 & -77 & 108.5 & -0.5\end{bmatrix} \qquad 3.67$$

Assembling of element equations to obtain domain equations for the four elements:

$$\left[K^{\Omega}\right]=\begin{bmatrix}
k^1_{11} & k^1_{12} & 0 & k^1_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
k^1_{21} & k^1_{22}+k^2_{11} & k^2_{23} & 0 & k^1_{23}+k^2_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & k^2_{21} & k^2_{33} & 0 & 0 & k^2_{23} & 0 & 0 & 0 & 0 & 0 & 0\\
k^1_{41} & 0 & 0 & k^1_{44}+k^3_{11} & k^1_{43}+k^3_{12} & k^3_{14} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & k^1_{32}+k^2_{41} & 0 & k^1_{34}+k^3_{21} & k^1_{33}+k^2_{44}+k^3_{22}+k^4_{11} & k^2_{43}+k^4_{12} & 0 & k^3_{23}+k^4_{14} & 0 & 0 & 0 & 0\\
0 & 0 & k^2_{32} & 0 & k^2_{34}+k^4_{21} & k^2_{33}+k^4_{22} & 0 & 0 & k^4_{23} & 0 & 0 & 0\\
0 & 0 & 0 & k^3_{41} & 0 & 0 & k^3_{44}+k^5_{11} & k^3_{43}+k^5_{12} & 0 & k^5_{14} & 0 & 0\\
0 & 0 & 0 & 0 & k^3_{32}+k^4_{41} & 0 & k^3_{34}+k^5_{21} & k^3_{33}+k^4_{44}+k^5_{22}+k^6_{11} & k^4_{43}+k^6_{12} & 0 & k^5_{23}+k^6_{41} & 0\\
0 & 0 & 0 & 0 & 0 & k^4_{32} & 0 & k^4_{34}+k^6_{21} & k^4_{33}+k^6_{22} & 0 & 0 & k^6_{23}\\
0 & 0 & 0 & 0 & 0 & 0 & k^5_{41} & 0 & 0 & k^5_{44} & k^5_{43} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k^5_{32}+k^6_{41} & 0 & k^5_{34} & k^5_{33}+k^6_{44} & k^6_{43}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k^6_{32} & 0 & k^3_{44}+k^6_{34} & k^6_{33}
\end{bmatrix}$$

$K_{11}=k_{11}^{1}$ $\quad K_{36}=k_{23}^{2}$ $\quad K_{63}=k_{32}^{2}$ $\quad K_{88}=k_{33}^{3}+k_{44}^{4}+k_{22}^{5}+k_{11}^{6}$ $\quad K_{10,11}=k_{43}^{5}$

$K_{12}=k_{12}^{1}$ $\quad K_{41}=k_{41}^{1}$ $\quad K_{65}=k_{34}^{2}+k_{21}^{4}$ $\quad K_{89}=k_{43}^{4}+k_{12}^{6}$ $\quad K_{11,8}=k_{32}^{5}+k_{41}^{6}$

$K_{13}=0$ $\quad K_{44}=k_{44}^{1}+k_{11}^{3}$ $\quad K_{66}=k_{33}^{2}+k_{22}^{4}$ $\quad K_{8,11}=k_{23}^{5}+k_{14}^{6}$ $\quad K_{11,10}=k_{34}^{5}$

$K_{14}=k_{14}^{1}$ $\quad K_{45}=k_{43}^{1}+k_{12}^{3}$ $\quad K_{69}=k_{23}^{4}$ $\quad K_{96}=k_{32}^{4}$ $\quad K_{11,11}=k_{33}^{5}+k_{44}^{6}$

$K_{21}=k_{21}^{1}$ $\quad K_{47}=k_{14}^{3}$ $\quad K_{74}=k_{41}^{3}$ $\quad K_{98}=k_{34}^{4}+k_{21}^{6}$ $\quad K_{11,12}=k_{43}^{6}$

$K_{22}=k_{22}^{1}+k_{11}^{2}$ $\quad K_{52}=k_{32}^{1}+k_{41}^{2}$ $\quad K_{77}=k_{44}^{3}+k_{11}^{5}$ $\quad K_{99}=k_{33}^{4}+k_{22}^{6}$ $\quad K_{12,9}=k_{32}^{6}$

$K_{23}=k_{23}^{2}$ $\quad K_{54}=k_{34}^{1}+k_{21}^{3}$ $\quad K_{78}=k_{43}^{3}+k_{12}^{5}$ $\quad K_{9,12}=k_{23}^{6}$ $\quad K_{12,11}=k_{34}^{6}$

$K_{25}=k_{23}^{1}+k_{14}^{2}$ $\quad K_{56}=k_{43}^{2}+k_{12}^{4}$ $\quad K_{7,10}=k_{14}^{5}$ $\quad K_{10,7}=k_{41}^{5}$ $\quad K_{12,12}=k_{33}^{6}$

$K_{32}=k_{21}^{2}$ $\quad K_{58}=k_{23}^{3}+k_{14}^{4}$ $\quad K_{85}=k_{32}^{3}+k_{41}^{4}$ $\quad K_{76}=k_{32}^{3}$

$K_{33}=k_{33}^{2}$ $\quad K_{55}=k_{33}^{1}+k_{44}^{2}+k_{22}^{3}+k_{11}^{4}$ $\quad K_{87}=k_{34}^{3}+k_{21}^{5}$ $\quad K_{10,10}=k_{44}^{5}$

Where K_{ij} are the global matrices and k^{e}_{ij} are the local matrices.

$$\left[K^{\Omega}\right]=\frac{1}{3888b\,\mathrm{Re}_r}\begin{bmatrix}
454 & 108.5 & 0 & -463 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
77 & 483.5 & -22 & 0 & -483.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 83 & -17 & -22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-463 & 0 & 0 & 205 & 289 & 0 & -181 & 0 & 0 & 0 & 0 & 0\\
0 & -485 & 0 & 190 & 944 & 82 & 0 & -335 & 0 & 0 & 0 & 0\\
0 & 0 & 26 & 0 & 226 & 156 & 0 & 0 & -154 & 0 & 0 & 0\\
0 & 0 & 0 & -181 & 0 & 0 & -199 & 271 & 0 & -31 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 100 & 887 & 289 & 0 & -449 & 0\\
0 & 0 & 0 & 0 & 0 & -154 & 0 & 172 & 692 & 0 & 0 & -418\\
0 & 0 & 0 & 0 & 0 & 0 & -31 & 0 & 0 & 20.5 & 87.5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -449 & 0 & 65 & 474.5 & 108.5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -418 & 0 & -49 & 589
\end{bmatrix}$$

Determination of force vector $F_i^1 = H\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\kappa\psi_i\right)d\eta\, d\kappa$

$$f_1^1 = H\int_0^{a/3}\int_0^{b/2}\left(\psi_1\kappa\right)d\eta\, d\kappa = \frac{7a^2b}{432} \qquad f_1^2 = H\int_0^{a/3}\int_{b/2}^{b}\left(\psi_1\kappa\right)d\eta\, d\kappa = \frac{7a^2b}{1296}$$

$$f_2^1 = H\int_0^{a/3}\int_0^{b/2}\left(\psi_2\kappa\right)d\eta\, d\kappa = \frac{a^2b}{216} \qquad f_2^2 = H\int_0^{a/3}\int_{b/2}^{b}\left(\psi_2\kappa\right)d\eta\, d\kappa = \frac{a^2b}{648}$$

$$f_3^1 = H\int_0^{a/3}\int_0^{b/2}\left(\psi_3\kappa\right)d\eta\, d\kappa = \frac{a^2b}{648} \qquad f_3^2 = H\int_0^{a/3}\int_{b/2}^{b}\left(\psi_3\kappa\right)d\eta\, d\kappa = \frac{a^2b}{216}$$

$$f_4^1 = H\int_0^{a/3}\int_0^{b/2}\left(\psi_4\kappa\right)d\eta\, d\kappa = \frac{7a^2b}{1296} \qquad f_4^2 = H\int_0^{a/3}\int_{b/2}^{b}\left(\psi_4\kappa\right)d\eta\, d\kappa = \frac{7a^2b}{432}$$

$$f_1^3 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_1\kappa)\,d\eta\,d\kappa = \frac{13a^2b}{432} \quad f_1^4 = H\int_{a/3}^{2a/3}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta\,d\kappa = \frac{13a^2b}{1296}$$

$$f_2^3 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_2\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{216} \quad f_2^4 = H\int_{a/3}^{2a/3}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{648}$$

$$f_3^3 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_3\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{648} \quad f_3^4 = H\int_{a/3}^{2a/3}\int_{b/2}^{b}(\psi_3\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{216}$$

$$f_4^3 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_4\kappa)\,d\eta\,d\kappa = \frac{13a^2b}{1296} \quad f_4^4 = H\int_{a/3}^{2a/3}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta\,d\kappa = \frac{13a^2b}{432}$$

$$f_1^5 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_1\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{432} \quad f_1^6 = H\int_{2a/3}^{a}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{1296}$$

$$f_2^5 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_2\kappa)\,d\eta\,d\kappa = \frac{19a^2b}{216} \quad f_2^6 = H\int_{2a/3}^{a}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta\,d\kappa = \frac{19a^2b}{648}$$

$$f_3^5 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_3\kappa)\,d\eta\,d\kappa = \frac{19a^2b}{648} \quad f_3^6 = H\int_{2a/3}^{a}\int_{b/2}^{b}(\psi_3\kappa)\,d\eta\,d\kappa = \frac{19a^2b}{216}$$

$$f_4^5 = H\int_{a/3}^{2a/3}\int_0^{b/2}(\psi_4\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{1296} \quad f_4^6 = H\int_{2a/3}^{a}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta\,d\kappa = \frac{7a^2b}{432}$$

$$\left[f^1\right] = H\begin{Bmatrix}\frac{7a^2b}{432}\\ \frac{a^2b}{216}\\ \frac{a^2b}{648}\\ \frac{7a^2b}{1296}\end{Bmatrix} \left[f^2\right] = H\begin{Bmatrix}\frac{7a^2b}{1296}\\ \frac{a^2b}{648}\\ \frac{a^2b}{216}\\ \frac{a^2b}{432}\end{Bmatrix} \left[f^3\right] = H\begin{Bmatrix}\frac{13a^2b}{432}\\ \frac{7a^2b}{216}\\ \frac{7a^2b}{648}\\ \frac{13a^2b}{1296}\end{Bmatrix} \left[f_4^1\right] = H\begin{Bmatrix}\frac{13a^2b}{1296}\\ \frac{7a^2b}{648}\\ \frac{7a^2b}{216}\\ \frac{13a^2b}{432}\end{Bmatrix}$$

$$\left[f^5\right] = H\begin{Bmatrix}\frac{7a^2b}{432}\\ \frac{19a^2b}{216}\\ \frac{19a^2b}{648}\\ \frac{7a^2b}{1296}\end{Bmatrix} \left[f^6\right] = H\begin{Bmatrix}\frac{7a^2b}{1296}\\ \frac{19a^2b}{648}\\ \frac{19a^2b}{216}\\ \frac{7a^2b}{432}\end{Bmatrix}$$

Assembling of local elements equations to obtain domain force vector equation:

$$\left[F^\Omega\right] = H\begin{Bmatrix} f_1^1\\ f_2^1+f_1^2\\ f_2^2\\ f_4^1+f_1^3\\ f_3^1+f_4^2+f_2^3+f_1^4\\ f_3^2+f_2^4\\ f_4^3+f_1^5\\ f_3^3+f_4^4+f_1^6+f_2^5\\ f_3^4+f_2^6\\ f_4^5\\ f_3^5+f_4^6\\ f_3^6\end{Bmatrix} = \frac{a^2bH}{1296}\begin{Bmatrix}21\\13\\2\\46\\60\\8\\51\\174\\80\\7\\59\\114\end{Bmatrix}$$

Since no flux is specified at node 5,

$f_3^1 + f_4^2 + f_2^3 + f_1^4 = 0$

(3) Eight Elements

Derivation of stiff matrices: $k_{ij}^e = \frac{1}{\mathrm{Re}_r}\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left[\frac{d\psi_i}{d\eta}\frac{d\psi_j}{d\eta}\right]d\eta d\kappa$

$$k_{11}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{37a}{384b} \quad k_{31}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-5a}{384b}$$

$$k_{12}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{5a}{384b} \quad k_{32}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{384b}$$

$$k_{13}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-5a}{384b} \quad k_{33}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{384b}$$

$$k_{14}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-37a}{384b} \quad k_{34}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{5a}{384b}$$

$$k_{21}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{5a}{384b} \quad k_{41}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-37a}{384b}$$

$$k_{22}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{384b} \quad k_{42}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-5a}{384b}$$

$$k_{23}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{384b} \quad k_{43}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{5a}{384b}$$

$$k_{24}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-5a}{384b} \quad k_{44}^1 = \int_0^{a/4}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{37a}{384b}$$

$$k_{11}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{61a}{384b} \quad k_{31}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{384b}$$

$$k_{12}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{384b} \quad k_{32}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-a}{384b}$$

$$k_{13}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{384b} \quad k_{33}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{a}{384b}$$

$$k_{14}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-61a}{384b} \quad k_{34}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{384b}$$

$$k_{21}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{384b} \quad k_{41}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-61a}{384b}$$

$$k_{22}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{a}{384b} \quad k_{42}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{384b}$$

$$k_{23}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-a}{384b} \quad k_{43}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{384b}$$

$$k_{24}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{384b} \quad k_{44}^2 = \int_0^{a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{61a}{384b}$$

$$k_{11}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{91a}{384b}\quad k_{31}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}$$

$$k_{12}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}\quad k_{32}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-7a}{384b}$$

$$k_{13}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}\quad k_{33}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{7a}{384b}$$

$$k_{14}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-91a}{384b}\quad k_{34}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}$$

$$k_{21}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}\quad k_{41}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-91a}{384b}$$

$$k_{22}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7a}{384b}\quad k_{42}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}$$

$$k_{23}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-7a}{384b}\quad k_{43}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}$$

$$k_{24}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}\quad k_{44}^{3}=\int_0^{a/2}\int_0^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{91a}{384b}$$

$$k_{11}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{91a}{384b}\quad k_{31}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}$$

$$k_{12}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}\quad k_{32}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-7a}{384b}$$

$$k_{13}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}\quad k_{33}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{7a}{384b}$$

$$k_{14}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-91a}{384b}\quad k_{34}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}$$

$$k_{21}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}\quad k_{41}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-91a}{384b}$$

$$k_{22}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7a}{384b}\quad k_{42}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-25a}{384b}$$

$$k_{23}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-7a}{384b}\quad k_{43}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{25a}{384b}$$

$$k_{24}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{9-25a}{384b}\quad k_{44}^{4}=\int_0^{a/2}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{91a}{384b}$$

$$k_{11}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{127a}{384b} \quad k_{31}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b}$$

$$k_{12}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b} \quad k_{32}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-19a}{384b}$$

$$k_{13}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b} \quad k_{33}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{19a}{384b}$$

$$k_{14}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-127a}{384b} \quad k_{34}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b}$$

$$k_{21}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b} \quad k_{41}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-127a}{384b}$$

$$k_{22}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{19a}{384b} \quad k_{42}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b}$$

$$k_{23}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-19a}{384b} \quad k_{43}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b}$$

$$k_{24}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b} \quad k_{44}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{127a}{384b}$$

$$k_{11}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{127a}{384b} \quad k_{31}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b}$$

$$k_{12}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b} \quad k_{32}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-19a}{384b}$$

$$k_{13}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b} \quad k_{33}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{19a}{384b}$$

$$k_{14}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-127a}{384b} \quad k_{34}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b}$$

$$k_{21}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b} \quad k_{41}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-127a}{384b}$$

$$k_{22}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{19a}{384b} \quad k_{42}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b}$$

$$k_{23}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-19a}{384b} \quad k_{43}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{49a}{384b}$$

$$k_{24}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-49a}{384b} \quad k_{44}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{127a}{384b}$$

$$k_{11}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{169a}{384b}\quad k_{31}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}$$

$$k_{12}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}\quad k_{32}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-37a}{384b}$$

$$k_{13}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}\quad k_{33}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{37a}{384b}$$

$$k_{14}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-169a}{384b}\quad k_{34}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}$$

$$k_{21}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}\quad k_{41}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-169a}{384b}$$

$$k_{22}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{37a}{384b}\quad k_{42}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}$$

$$k_{23}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-37a}{384b}\quad k_{43}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}$$

$$k_{24}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}\quad k_{44}^{7}=\int_{3a/4}^{a}\int_{0}^{b/2}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{169a}{384b}$$

$$k_{11}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{169a}{384b}\quad k_{31}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}$$

$$k_{12}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}\quad k_{32}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-37a}{384b}$$

$$k_{13}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}\quad k_{33}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{37a}{384b}$$

$$k_{14}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_1}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-169a}{384b}\quad k_{34}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_3}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}$$

$$k_{21}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}\quad k_{41}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-169a}{384b}$$

$$k_{22}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{37a}{384b}\quad k_{42}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}$$

$$k_{23}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-37a}{384b}\quad k_{43}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{79a}{384b}$$

$$k_{24}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_2}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-79a}{384b}\quad k_{44}^{8}=\int_{3a/4}^{a}\int_{b/2}^{b}\left[\frac{d\psi_4}{d\eta}\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{169a}{384b}$$

Derivation of force matrices: $N_{ij}^{e}=\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\kappa\psi_i\frac{d\psi_j}{d\kappa}\right)d\eta d\kappa$

$$N_{11}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_1\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-35ab}{4608}\quad N_{31}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_3\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-ab}{2304}$$

$$N_{12}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_1\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{35ab}{4608}\quad N_{32}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_3\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{ab}{2304}$$

$$N_{13}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_1\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{5ab}{2304}\quad N_{33}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_3\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{ab}{4608}$$

$$N_{14}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_1\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-5ab}{2304}\quad N_{34}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_3\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-ab}{4608}$$

$$N_{21}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_2\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7ab}{4608}\quad N_{41}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_4\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-5ab}{2304}$$

$$N_{22}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_2\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7ab}{4608}\quad N_{42}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_4\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{5ab}{2304}$$

$$N_{23}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_2\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{ab}{2304}\quad N_{43}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_4\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{5ab}{4608}$$

$$N_{24}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_2\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-ab}{2304}\quad N_{44}^{1}=\int_0^{a/4}\int_0^{b/2}\left[\kappa\psi_4\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-5ab}{4608}$$

$$N_{11}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}\quad N_{31}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$

$$N_{12}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{32}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}$$

$$N_{13}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{33}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}$$

$$N_{14}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}\quad N_{34}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$

$$N_{21}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{41}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}$$

$$N_{22}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}\quad N_{42}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$

$$N_{23}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}\quad N_{43}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$

$$N_{24}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{44}^{2}=\int_0^{a/4}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}$$

$$N_{11}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_1\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}\quad N_{31}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_3\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$
$$N_{12}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_1\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{32}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_3\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}$$
$$N_{13}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_1\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{33}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_3\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}$$
$$N_{14}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_1\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}\quad N_{34}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_3\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$

$$N_{21}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_2\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{41}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_4\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}$$
$$N_{22}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_2\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}\quad N_{42}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_4\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$
$$N_{23}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_2\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}\quad N_{43}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_4\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$
$$N_{24}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_2\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{44}^{3}=\int_{0}^{a/2}\int_{0}^{b/2}\left[\kappa\psi_4\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}$$
$$N_{11}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}\quad N_{31}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$
$$N_{12}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{32}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}$$
$$N_{13}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{33}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}$$
$$N_{14}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_1\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}\quad N_{34}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_3\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$

$$N_{21}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}\quad N_{41}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_1}{d\eta}\right]d\eta d\kappa=\frac{-19a}{162b}$$
$$N_{22}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{a}{162b}\quad N_{42}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_2}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}$$
$$N_{23}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{-a}{162b}\quad N_{43}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_3}{d\eta}\right]d\eta d\kappa=\frac{7a}{324b}$$
$$N_{24}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_2\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{-7a}{324b}\quad N_{44}^{4}=\int_{0}^{a/2}\int_{b/2}^{b}\left[\kappa\psi_4\frac{d\psi_4}{d\eta}\right]d\eta d\kappa=\frac{19a}{162b}$$

$$N_{11}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad N_{31}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{12}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{32}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{13}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{33}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{14}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{34}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{21}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{41}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{22}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{42}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{23}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{43}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{24}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{44}^{5} = \int_{a/2}^{3a/4}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{11}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-a}{162b} \quad N_{31}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{12}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{32}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{13}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{33}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{14}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{34}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{21}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{41}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{22}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{42}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{23}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{43}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{24}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{44}^{6} = \int_{a/2}^{3a/4}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{11}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{31}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{12}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{32}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{13}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{33}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{14}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_1 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{34}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_3 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{21}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{41}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{22}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{42}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{23}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{43}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{24}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_2 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{44}^7 = \int_{3a/4}^{a}\int_{0}^{b/2}\left[\kappa\psi_4 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{11}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{31}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{12}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{32}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{13}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{33}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

$$N_{14}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_1 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{34}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_3 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{21}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b} \quad N_{41}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_1}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b}$$

$$N_{22}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b} \quad N_{42}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_2}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b}$$

$$N_{23}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{-7a}{162b} \quad N_{43}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_3}{d\eta}\right]d\eta d\kappa = \frac{13a}{324b}$$

$$N_{24}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_2 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{-13a}{324b} \quad N_{44}^8 = \int_{3a/4}^{a}\int_{b/2}^{b}\left[\kappa\psi_4 \frac{d\psi_4}{d\eta}\right]d\eta d\kappa = \frac{7a}{162b}$$

Derivation of local force matrices: $f_i^e = H\int_{\kappa_i}^{\kappa_{i+1}}\int_{\eta_i}^{\eta_{i+1}}\left(\psi_i\kappa\right)d\eta d\kappa$

$$f_1^1 = H\int_0^{a/4}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{5a^2b}{512} \quad f_1^3 = H\int_{a/3}^{a/2}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{11a^2b}{512}$$
$$f_2^1 = H\int_0^{a/4}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{a^2b}{512} \quad f_2^3 = H\int_{a/3}^{a/2}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{7a^2b}{512}$$
$$f_3^1 = H\int_0^{a/4}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{a^2b}{1536} \quad f_3^3 = H\int_{a/3}^{a/2}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{7a^2b}{1536}$$
$$f_4^e = H\int_0^{a/4}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{5a^2b}{1536} \quad f_4^3 = H\int_{a/3}^{a/2}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{11a^2b}{1536}$$

$$f_1^2 = H\int_0^{a/4}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{5a^2b}{1536} \quad f_1^4 = H\int_{a/3}^{a/2}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{11a^2b}{1536}$$
$$f_2^2 = H\int_0^{a/4}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{a^2b}{1536} \quad f_2^4 = H\int_{a/3}^{a/2}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{7a^2b}{1536}$$
$$f_3^2 = H\int_0^{a/4}\int_{b/2}^{b}(\psi_3\kappa)\,d\eta d\kappa = \frac{a^2b}{512} \quad f_3^4 = H\int_{a/3}^{a/2}\int_{b}^{2b}(\psi_3\kappa)\,d\eta d\kappa = \frac{7a^2b}{512}$$
$$f_4^2 = H\int_0^{a/4}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{5a^2b}{512} \quad f_4^4 = H\int_{a/3}^{a/2}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{11a^2b}{512}$$
$$f_1^5 = H\int_{a/2}^{3a/4}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{11a^2b}{512} \quad f_1^7 = H\int_{3a/4}^{a}\int_0^{b/2}(\psi_1\kappa)\,d\eta d\kappa = \frac{5a^2b}{512}$$
$$f_2^5 = H\int_{a/2}^{3a/4}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{19a^2b}{512} \quad f_2^7 = H\int_{3a/4}^{a}\int_0^{b/2}(\psi_2\kappa)\,d\eta d\kappa = \frac{37a^2b}{512}$$
$$f_3^5 = H\int_{a/2}^{3a/4}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{19a^2b}{1536} \quad f_3^7 = H\int_{3a/4}^{a}\int_0^{b/2}(\psi_3\kappa)\,d\eta d\kappa = \frac{37a^2b}{1536}$$
$$f_4^5 = H\int_{a/2}^{3a/4}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{11a^2b}{1536} \quad f_4^7 = H\int_{3a/4}^{a}\int_0^{b/2}(\psi_4\kappa)\,d\eta d\kappa = \frac{5a^2b}{1536}$$

$$f_1^6 = H\int_{a/2}^{3a/4}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{11a^2b}{1536} \quad f_1^8 = H\int_{3a/4}^{a}\int_{b/2}^{b}(\psi_1\kappa)\,d\eta d\kappa = \frac{5a^2b}{1536}$$
$$f_2^6 = H\int_{a/2}^{3a/4}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{19a^2b}{1536} \quad f_2^8 = H\int_{3a/4}^{a}\int_{b/2}^{b}(\psi_2\kappa)\,d\eta d\kappa = \frac{37a^2b}{1536}$$
$$f_3^6 = H\int_{a/2}^{3a/4}\int_{b}^{2b}(\psi_3\kappa)\,d\eta d\kappa = \frac{19a^2b}{512} \quad f_3^8 = H\int_{3a/4}^{a}\int_{b}^{b}(\psi_3\kappa)\,d\eta d\kappa = \frac{37a^2b}{512}$$
$$f_4^6 = H\int_{a/2}^{3a/4}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{11a^2b}{512} \quad f_4^8 = H\int_{3a/4}^{a}\int_{b/2}^{b}(\psi_4\kappa)\,d\eta d\kappa = \frac{5a^2b}{512}$$

Element matrices are:

$$\left[K^e\right] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} + \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{bmatrix}$$

$$\left[K^1\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}426.5 & 77.5 & -55 & -449\\ 56.5 & 15.5 & -11 & -61\\ -61 & -11 & 12.5 & 59.5\\ -449 & -55 & 62.5 & 446.5\end{bmatrix}$$

$$\left[K^2\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}729.5 & 86.5 & -79 & -737\\ 83.5 & 12.5 & -11 & -85\\ -85 & -11 & 15.5 & 80.5\\ -802 & -7 & 29.5 & 66.5\end{bmatrix}$$

$$\left[K^3\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}1053.5 & 338.5 & -289 & -1103\\ 275.5 & 108.5 & -77 & -307\\ -307 & -77 & 87.5 & 296.5\\ -1103 & -289 & 305.5 & 1086.5\end{bmatrix}$$

$$\left[K^4\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}1086.5 & 305.5 & -289 & -1103\\ 303.5 & 87.5 & -77 & -307\\ -307.5 & -77 & 108.5 & 275.5\\ -1103 & -289 & 338.5 & 1053.5\end{bmatrix}$$

$$\left[K^5\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}1485.5 & 626.5 & -577 & -1535\\ 521.5 & 294.5 & -209 & -607\\ -607 & -209 & 237.5 & 578.5\\ -1535 & -577 & 593.5 & 1518.5\end{bmatrix}$$

$$\left[K^6\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}1518.5 & 593.5 & -577 & -1535\\ 578.5 & 237.5 & -209 & -607\\ -607 & -209 & 294.5 & 521.5\\ -1535 & -577 & 626.5 & 1485.5\end{bmatrix}$$

$$\left[K^7\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}2010.5 & 965.5 & -943 & -2033\\ 818.5 & 573.5 & -407 & -985\\ -985 & -407 & 462.5 & 929.5\\ -2033 & -943 & 950.5 & 2025.5\end{bmatrix}$$

$$\left[K^8\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}2025.5 & 950.5 & -943 & -2033\\ 929.5 & 462.5 & -425.5 & -966.5\\ -966.5 & -425.5 & 573.5 & 818.5\\ -2030.5 & -945.5 & 965.5 & 2010.5\end{bmatrix}$$

And force vectors are

$$f_1^1 == \frac{5a^2b}{512} \quad f_1^3 == \frac{11a^2b}{512} \quad f_1^5 = \frac{11a^2b}{512} \quad f_1^7 = \frac{5a^2b}{512}$$

$$f_2^1 = \frac{a^2b}{512} \quad f_2^3 = \frac{7a^2b}{512} \quad f_2^5 = \frac{19a^2b}{512} \quad f_2^7 = \frac{37a^2b}{512}$$

$$f_3^1 = \frac{a^2b}{1536} \quad f_3^3 = \frac{7a^2b}{1536} \quad f_3^5 = \frac{19a^2b}{1536} \quad f_3^7 = \frac{37a^2b}{1536}$$

$$f_4^e = \frac{5a^2b}{1536} \quad f_4^3 = \frac{11a^2b}{1536} \quad f_4^5 = \frac{11a^2b}{1536} \quad f_4^7 = \frac{5a^2b}{1536}$$

$$f_1^2 = \frac{5a^2b}{1536} \quad f_1^4 = \frac{11a^2b}{1536} \quad f_1^6 = \frac{11a^2b}{1536} \quad f_1^8 = \frac{5a^2b}{1536}$$

$$f_2^2 = \frac{a^2b}{1536} \quad f_2^4 = \frac{7a^2b}{1536} \quad f_2^6 = \frac{19a^2b}{1536} \quad f_2^8 = \frac{37a^2b}{1536}$$

$$f_3^2 = \frac{a^2b}{512} \quad f_3^4 = \frac{7a^2b}{512} \quad f_3^6 = \frac{19a^2b}{512} \quad f_3^8 = \frac{37a^2b}{512}$$

$$f_4^2 = \frac{5a^2b}{512} \quad f_4^4 = \frac{11a^2b}{512} \quad f_4^6 = \frac{11a^2b}{512} \quad f_4^8 = \frac{5a^2b}{512}$$

Global matrix for the eight elements:

$K_{11}=k_{11}^1$ $\quad K_{36}=k_{23}^2$ $\quad K_{63}=k_{32}^2$ $\quad K_{88}=k_{33}^3+k_{44}^4+k_{22}^5+k_{11}^6$ $\quad K_{10,13}=k_{14}^7$ $\quad K_{13,10}=k_{41}^7$

$K_{12}=k_{12}^1$ $\quad K_{41}=k_{41}^1$ $\quad K_{65}=k_{34}^2+k_{21}^4$ $\quad K_{89}=k_{43}^4+k_{12}^6$ $\quad K_{11,8}=k_{32}^5+k_{41}^6$ $\quad K_{13,13}=k_{44}^7$

$K_{13}=0$ $\quad K_{44}=k_{44}^1+k_{11}^3$ $\quad K_{66}=k_{33}^2+k_{22}^4$ $\quad K_{8,11}=k_{23}^5+k_{14}^6$ $\quad K_{11,10}=k_{34}^5+k_{21}^7$ $\quad K_{13,14}=k_{43}^7$

$K_{14}=k_{14}^1$ $\quad K_{45}=k_{43}^1+k_{12}^3$ $\quad K_{69}=k_{23}^4$ $\quad K_{96}=k_{32}^4$ $\quad K_{11,11}=k_{33}^5+k_{44}^6+k_{22}^7+k_{11}^8$ $\quad K_{14,11}=k_{32}^7+k_{41}^8$

$K_{21}=k_{21}^1$ $\quad K_{47}=k_{14}^3$ $\quad K_{74}=k_{41}^3$ $\quad K_{98}=k_{34}^4+k_{21}^6$ $\quad K_{11,12}=k_{43}^6+k_{12}^8$ $\quad K_{14,13}=k_{34}^7$

$K_{22}=k_{22}^1+k_{11}^2$ $\quad K_{52}=k_{32}^1+k_{41}^2$ $\quad K_{77}=k_{44}^3+k_{11}^5$ $\quad K_{99}=k_{33}^4+k_{22}^6$ $\quad K_{11,14}=k_{23}^7+k_{14}^8$ $\quad K_{14,14}=k_{33}^7+k_{44}^8$

$K_{23}=k_{23}^2$ $\quad K_{54}=k_{34}^1+k_{21}^3$ $\quad K_{78}=k_{43}^3+k_{12}^5$ $\quad K_{9,12}=k_{23}^6$ $\quad K_{12,9}=k_{32}^6$ $\quad K_{14,15}=k_{43}^8$

$K_{25}=k_{23}^1+k_{14}^2$ $\quad K_{56}=k_{43}^2+k_{12}^4$ $\quad K_{7,10}=k_{14}^5$ $\quad K_{10,7}=k_{41}^5$ $\quad K_{12,11}=k_{34}^6+k_{21}^8$ $\quad K_{15,12}=k_{32}^8$

$K_{32}=k_{21}^2$ $\quad K_{58}=k_{23}^3+k_{14}^4$ $\quad K_{85}=k_{32}^3+k_{41}^4$ $\quad K_{10,10}=k_{44}^5+k_{11}^7$ $\quad K_{12,12}=k_{33}^6+k_{22}^8$ $\quad K_{15,14}=k_{34}^8$

$K_{33}=k_{33}^2$ $\quad K_{55}=k_{33}^1+k_{44}^2+k_{22}^3+k_{11}^4$ $\quad K_{87}=k_{34}^3+k_{21}^5$ $\quad K_{10,11}=k_{43}^5+k_{12}^7$ $\quad K_{12,15}=k_{23}^8$ $\quad K_{15,15}=k_{33}^8$

$$\left[K^{\Omega}\right]=\frac{1}{4608b\,\mathrm{Re}_r}\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1272a+4ab^2\,\mathrm{Re}_r&0&0&-1176a-8ab^2\,\mathrm{Re}_r&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&-1176a-8ab^2\,\mathrm{Re}_r&0&0&2929a+52ab^2\,\mathrm{Re}_r&0&0&-1752a+16ab^2\,\mathrm{Re}_r&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&-1752a+16ab^2\,\mathrm{Re}_r&0&0&2196a+196ab^2\,\mathrm{Re}_r&0&0&1536a+82ab^2\,\mathrm{Re}_r&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0& &0&0&0&0&0&-2472a+69ab^2\,\mathrm{Re}_r&0&0&2472a+2ab^2\,\mathrm{Re}_r&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

and for force vector is

$$F_i^e=\begin{Bmatrix}
f_1^1\\
f_2^1+f_1^2\\
f_2^2\\
f_4^1+f_1^3\\
f_3^1+f_4^2+f_2^3+f_1^4\\
f_3^2+f_2^4\\
f_4^3+f_1^5\\
f_3^3+f_4^4+f_2^5+f_1^6\\
f_3^4+f_2^6\\
f_4^5+f_1^7\\
f_3^5+f_4^6+f_2^7+f_1^8\\
f_3^6+f_2^8\\
f_4^7\\
f_3^7+f_4^8\\
f_3^8
\end{Bmatrix}=\frac{ab^2}{1536}\begin{Bmatrix}
15\\ 20\\ 1\\ 38\\ 42\\ 10\\ 44\\ 108\\ 40\\ 26\\ 168\\ 94\\ 5\\ 52\\ f_3^8
\end{Bmatrix}$$

Appendix D

PRESSURE DISTRIBUTION FINITE ELEMENT ANALYSIS

Therefore for pressure distribution, we used linear elements four our analyses as two, three and four linear elements as follows:

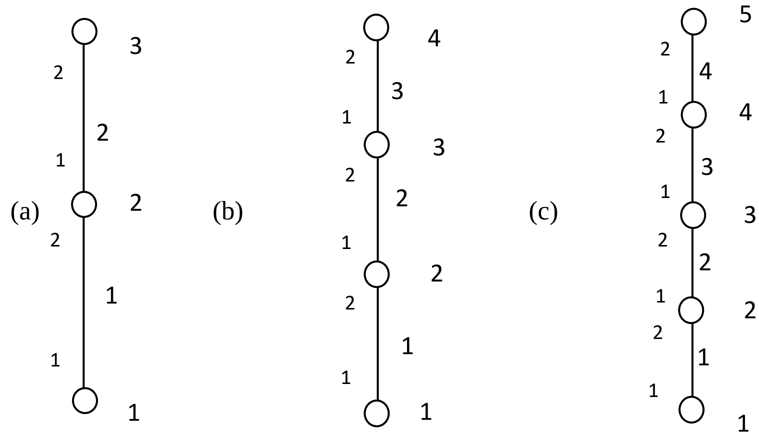


Fig. 2: 2, 3 and 4 Linear Elements

Applied linear interpolation functions

$$\psi_i = \frac{\kappa_{i+1}-\kappa}{\kappa_{i+1}-\kappa_i} = \frac{\kappa_{i+1}}{\kappa_{i+1}-\kappa_i} - \frac{\kappa}{\kappa_{i+1}-\kappa_i} = 1-\frac{\kappa}{h_e}$$

$$\psi_j = \frac{\kappa-\kappa_i}{\kappa_{i+1}-\kappa_1} = \frac{\kappa}{\kappa_{i+1}-\kappa_i} - \frac{\kappa_i}{\kappa_{i+1}-\kappa_i} = \frac{\kappa}{h_e} \tag{1}$$

Differentiating with respect to κ

$$\frac{d\psi_i}{d\kappa} = -\frac{1}{h_e}$$

$$\frac{d\psi_j}{d\kappa} = \frac{1}{h_e} \tag{2}$$

(1) Two Linear Elements

$$k_{ij}^e = \int_{\kappa_i}^{\kappa_{i+1}} \left(\psi_i \frac{d\psi_j}{d\kappa} \right) d\kappa \tag{3}$$

The local stiffness matrix is

$$k_{11}^1 = -\frac{1}{2} \quad k_{11}^2 = \frac{1}{2}$$

$$k_{12}^1 = \frac{1}{2} \quad k_{12}^2 = -\frac{1}{2}$$

$$k_{21}^1 = -\frac{1}{2} \quad k_{21}^2 = -\frac{3}{2}$$

$$k_{22}^1 = \frac{1}{2} \quad k_{22}^2 = \frac{3}{2}$$

$$k_{ij}^1 = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} k_{ij}^2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \tag{4}$$

And the local element force matrix is

$$f_i^e = -2\alpha^2 \omega U_o \int_{\kappa_i}^{\kappa_{i+1}} (\psi_i)\, d\kappa \tag{5}$$

$$f_1^1 = -2\alpha^2\omega V \frac{a}{4}$$
$$f_2^1 = -2\alpha^2\omega V \frac{a}{4}$$
$$f_1^2 = -2\alpha^2\omega V \frac{-a}{4} \qquad (6)$$
$$f_2^2 = -2\alpha^2\omega V \frac{3a}{4}$$

While the local element source matrix is

$$Q = 0 \qquad (7)$$

(2) Three Linear Elements

The local stiffness matrix is

$$k_{11}^1 = -\frac{1}{2}\; k_{11}^2 = \frac{1}{2} \quad k_{11}^3 = \frac{3}{2}$$
$$k_{12}^1 = \frac{1}{2} \quad k_{12}^2 = -\frac{1}{2}\; k_{12}^3 = -\frac{3}{2} \qquad (8)$$
$$k_{21}^1 = -\frac{1}{2}\; k_{21}^2 = -\frac{3}{2}\; k_{21}^3 = -\frac{5}{2}$$
$$k_{22}^1 = \frac{1}{2} \quad k_{22}^2 = \frac{3}{2} \quad k_{22}^3 = \frac{5}{2}$$

$$k_{ij}^1 = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} k_{ij}^2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} k_{ij}^3 = \frac{1}{2}\begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix} \qquad (9)$$

And the local element force matrix is

$$f_1^1 = -2\alpha^2\omega V \frac{a}{6}$$
$$f_2^1 = -2\alpha^2\omega V \frac{a}{6}$$
$$f_1^2 = -2\alpha^2\omega V \frac{-a}{6} \qquad (10)$$
$$f_2^2 = -2\alpha^2\omega V \frac{a}{2}$$
$$f_1^3 = -2\alpha^2\omega V \frac{-a}{2}$$
$$f_2^3 = -2\alpha^2\omega V \frac{5a}{6}$$

While the local element source matrix is

$$Q = 0 \qquad (11)$$

(3) Four Linear Elements

The local stiffness matrix is

$$k_{11}^1 = -\frac{1}{2}\; k_{11}^2 = \frac{1}{2} \quad k_{11}^3 = \frac{3}{2} \quad k_{11}^3 = \frac{5}{2}$$
$$k_{12}^1 = \frac{1}{2} \quad k_{12}^2 = -\frac{1}{2}\; k_{12}^3 = -\frac{3}{2}\; k_{12}^3 = -\frac{5}{2} \qquad (12)$$
$$k_{21}^1 = -\frac{1}{2}\; k_{21}^2 = -\frac{3}{2}\; k_{21}^3 = -\frac{5}{2}\; k_{21}^3 = -\frac{7}{2}$$
$$k_{22}^1 = \frac{1}{2} \quad k_{22}^2 = \frac{3}{2} \quad k_{22}^3 = \frac{5}{2} \quad k_{22}^3 = \frac{7}{2}$$

$$k_{ij}^{1}=\frac{1}{2}\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\; k_{ij}^{2}=\frac{1}{2}\begin{bmatrix}1 & -1\\ -3 & 3\end{bmatrix}\; k_{ij}^{3}=\frac{1}{2}\begin{bmatrix}3 & -3\\ -5 & 5\end{bmatrix}\; k_{ij}^{4}=\frac{1}{2}\begin{bmatrix}5 & -5\\ -7 & 7\end{bmatrix} \quad (13)$$

And the local element force matrix is

$$f_1^1=-2\alpha^2\omega V\,\frac{a}{8}$$
$$f_2^1=-2\alpha^2\omega V\,\frac{a}{8}$$
$$f_1^2=-2\alpha^2\omega V\,\frac{-a}{8}$$
$$f_2^2=-2\alpha^2\omega V\,\frac{3a}{8} \quad (14)$$
$$f_1^3=-2\alpha^2\omega V\,\frac{-3a}{8}$$
$$f_2^3=-2\alpha^2\omega V\,\frac{5a}{8}$$
$$f_1^4=-2\alpha^2\omega V\,\frac{-5a}{8}$$
$$f_2^4=-2\alpha^2\omega V\,\frac{7a}{8}$$

While the local element source matrix is

$$Q=0 \quad (15)$$

Appendix E

DERIVATION OF CLOSE-FORMED ANALYTICAL SOLUTION FOR RADIAL AND TANGENTIAL VELOCITIES AND PRESSURE DISTRIBUTION

(1) Close-formed Analytical Solution Formulation for Radial Velocity

$$-2U_o\omega\alpha^2\kappa=-\kappa\frac{\partial p}{\partial\kappa}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial\eta^2}\right)$$

Assume that $\left(\frac{dp}{d\kappa}=0\right)$, the governing equation reduces to:

$$-2U_o\omega\alpha^2\kappa=\frac{1}{\mathrm{Re}}\left(\frac{d^2u}{d\eta^2}\right) \quad (1)$$

Subject to boundary conditions:

$$u(\eta=+b)=u(\eta=-b)=0 \quad (2)$$

Where

$$G=2U_o\alpha^2\omega\mathrm{Re}_r \quad (3)$$

Integrating twice will yield

$$u(\eta)=G\eta^2+C_1\eta+C_2 \quad (4)$$

Applying boundary conditions to equation (4) yields

$$u(\eta=+b)=Gb^2+C_1b+C_2=0 \quad (5)$$

$$u(\eta=-b)=Gb^2-C_1b+C_2=0 \tag{6}$$

Adding equations (5) and (6) yields

$$C_2=-Gb^2 \tag{7}$$

By substituting for the constant C_2 into equation (5) gives

$$C_1=0 \tag{8}$$

Substituting the constants into equation (4) yields:

$$u(\eta)=-Gb^2\left(1-\eta^2\right)$$

$$=2U_o\alpha^2\omega\kappa\,\mathrm{Re}_r\,b^2\left(1-\eta^2\right) \tag{9}$$

Where b = element length or interval

(2) Asymptotic Analytical Solution Formulation for Tangential Velocity

Here we derived the analytical solution from the governing tangential velocity equation as follows.
Our governing equation for r-momentum is:

$$V_o\kappa\frac{\partial v}{\partial\kappa}+2V_o\omega\kappa=\frac{1}{\mathrm{Re}_r}\left(\frac{\partial^2 v}{\partial\eta^2}\right) \tag{10}$$

Subject to boundary conditions:

$$v(\kappa,\eta=+1)=v(\kappa,\eta=-1)=\kappa_i\omega=0$$

$$v(\kappa_0,\eta=+1)=v(\kappa_0,\eta=-1)=\kappa_0\omega \tag{11}$$

Assume that $\left(\frac{\partial v}{\partial\kappa}=0\right)$, the governing equation reduces to:

$$\frac{\partial^2 v}{\partial\eta^2}=-2V_o\omega\kappa\,\mathrm{Re}_r \tag{12}$$

Subject to boundary conditions:

$$v(\kappa,\eta=+1)=v(\kappa,\eta=-1)=\kappa_i\omega=0$$

$$v(\kappa_0,\eta=+1)=v(\kappa_0,\eta=-1)=\kappa_0\omega \tag{13}$$

Integrating (12) twice and imposing boundary conditions yields

$$v(\eta)=2V_o\omega\kappa\,\mathrm{Re}_r\,\eta^2+C_1\eta+C_2 \tag{14}$$

$$v(\eta=+b)=2V_o\omega\kappa\,\mathrm{Re}_r\,b^2+C_1b+C_2=0 \tag{15}$$

$$u(\eta=-b)=2V_o\omega\kappa\,\mathrm{Re}_r\,b^2-C_1b+C_2=0 \tag{16}$$

Adding equations (15) and (16) yields

$$C_2=-2V_o\omega\kappa\,\mathrm{Re}_r\,b^2 \tag{17}$$

By substituting the constant in (17) into equation (15) gives

$$C_1=0 \tag{18}$$

Substituting the constants into equation (14) yields:

$$v(\eta) = 2V_o \omega \kappa \operatorname{Re}_r b^2 \left(1-\eta^2\right) \tag{19}$$

Where b = element length or interval

(3) Asymptotic Analytical Solution for Pressure Distribution

Governing equation is;

$$-2\alpha^2 U_o \omega = -\frac{dp}{d\kappa} \tag{20}$$

Subject to:

$$p\left(\kappa_i = 0\right) = 0 \tag{21}$$

By integrating once yields:

$$p\left(\kappa\right) = -2\alpha^2 U_o \omega \kappa + C_1 \tag{22}$$

Applying boundary condition

$$p\left(\kappa\right) = -2\alpha^2 U_o \omega \left(0\right) + C_1 = 0 \tag{23}$$

So that

$$C_1 = 0 \tag{24}$$

Substituting for C_1 into (22) yields

$$p\left(\kappa\right) = -2\alpha^2 U_o \omega \kappa \tag{25}$$