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Analysis of Pressure Variation of Fluid in Bounded Circular Reservoirs: Steady State Case

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Abstract

In this work, we investigate the pressure distribution of fluid in a bounded circular reservoir. The diffusivity equation was used in the analysis.The finite element method (FEM) was used as a mathematical tool in the analysis. The domain was discretized into ten Lagrange quadratic elements and was assembled to represent the cross section of the reservoir. The analysis was done with the assumption that before the well begins production, there was uniform distribution of pressure throughout the reservoir and that the well has been producing long enough to attain the steady state flow. Thus, this work covers the steady state case when the pressure at different locations in the reservoir is constant at all time, i.e., the change in pressure with time is zero.The result shows that there is an increase in pressure from the wellbore to the external boundary of the reservoir. This increase was very pronounced around the vicinity of the wellbore and flattens out within the region of the external boundary. The results obtained from this analysiswere compared with the results obtained from the exact differential equation method. The comparison shows that there wasa strong agreement between both methods.

Keywords: Bounded circular reservoir, steady state, diffusivity equation, dimensionless variables and Finite element formulation

Nomenclature:

B	Formation volume factor, RB/STB
c	Compressibility, psia-1
h	Thickness, ft
K	Stiffness matrix
k	Permeability, md
n	Number of elements
P	Pressure, psi
P_D	Dimensionless pressure
P_i	Initial reservoir pressure, psi
Q	Terminal flow rate
q	Volumetric flow rate, STB/D
r	Radius, ft
r_D	Dimensionless radius
r_e	External radius, ft
r_{eD}	Dimensionless external radius
r_w	Wellbore radius, ft
t	Time, hr
t_D	Dimensionless time
w	Weight function
$\forall$	For all

Greek letters

ϕ	Porosity, fraction
μ	Viscosity, cp
π	Pi
ψ	Interpolation function

1.0 Introduction

There are basically three types of flow regimes that exist in describing the flow behaviour of fluids and pressure distribution as a function of time in reservoirs. These flow regimes are: steady-state flow, pseudo-steady-state flow, and unsteady-state flow.

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In pseudo-steady-state flow regime, the pressure at different locations in the reservoir is declining linearly as a function of time, i.e., at a constant declining rate. Mathematically, this definition states that the rate of change of pressure with respect to time at every position is constant[1].

The unsteady-state flow frequently called transient flow is a fluid flow condition that occurs when the rate of change of pressure with respect to time at any position in the reservoir is neitherzeronor constant. This definition suggests that the pressure derivative with respect to time is essentially a function of both position and time. In the unsteady-state flow cases, it is assumed that a well is located in a very large reservoir and producing at a constant flow rate. This rate creates a pressure disturbance in the reservoir that travels throughout this infinite-size reservoir. During this transient flow period, reservoir boundaries have no effect on the pressure behaviour and this is often very short in duration.

As soon as the pressure disturbance reaches all drainage boundaries, it ends the transient (unsteady-state) flow regime. A different flow regime begins that is called pseudo-steady state flow. As soon as the entire reservoir pressure has been affected, an unexpected situation arises. The change in pressure with respect to time at all radii in the reservoir becomes uniform. Therefore, the pressure distributions at subsequent times are parallel [2]. It is necessary at this point to impose different boundary conditions on the diffusivity equation and derive an appropriate solution to this flow regime.

In literature on Petroleum Engineering Research,solutions to analytical flow equations for an elliptical flow domain with vertical fracture at the wellbore are abundant [3 – 10]. However, much work has not been done for circular reservoir[11].

Van der Ploeg et al.[12]developed a closed-form solution for steady saturated flow into a fully penetrating well in elliptical flow geometry. Van der Ploeg et al.'s work was related to water flow in a confined elliptical aquifer. Steady state solutions were developed for various well locations using gravity flow. Results and flow nets were presented for several cases. The essence of the approach was to derive orthonormal functions for the specific problems using method of Powers et al.[13]. Although Van der Ploeg et al. presented solution for different well locations, only the solution for a well at the centre is considered here.

The flow regime is said to be identified as a steady-state flow if the pressure at every location in the reservoir remains constant, i.e., does not change with respect to time. In reservoirs, the steady-state flow condition can only occur when the reservoir is completely recharged and supported by strong aquifer or pressure maintenance operations.

In well testing analysis, there are four solutions that are useful: solution for a bounded circular reservoir; the solution for an ideal reservoir with a well consider to be a line with zero wellbore radius; the pseudo-steady state solution; and the solution that includes wellbore radius for a well in an infinite reservoir. This paperaddresses the case of a fluid in steady state flow regime.Research in the field of reservoir using FEM is sparse. Therefore, the finite element method (FEM) was used in this work to analyse thecase of a steady state flow of fluid in abounded circular reservoir.

2.0 Theory

The law of conservation of mass, Darcy's law and the equation of state has been combined to obtain the following partial differential equation:

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} = \frac{\phi \mu c}{0.000264k}\frac{\partial P}{\partial t} \qquad (1)$$

with the assumptions that compressibility, c, is small and independent of pressure, P; permeability, k, is constant and isotropic; viscosity, μ, is independent of pressure; porosity, ϕ, is constant; and that certain terms in the basic differential equation (including pressure gradients squared) are negligible. Eq. (1) is called the diffusivity equation and the term $\frac{\phi \mu c}{0.000264k}$ is the inverse of the diffusivity constant, η.

In this work, the diffusivity equation is analysed for bounded circular reservoirs, the case in which the well is assumed to be located in the centre of a cylindrical reservoir and also, the flow is assumed to be in steady state where the pressure does not vary with respect to time.

3.0 Governing Equation

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} = \frac{\phi \mu c}{0.000264k}\frac{\partial P}{\partial t}$$

Initial and boundary conditions:

i. $P = P_i$ at t = 0 $\forall$ r (2)

ii. $\left(r\frac{\partial P}{\partial r}\right)_{r_w} = \frac{qB\mu}{2\pi kh}$ for t > 0 (3)

iii. $\left(\frac{\partial P}{\partial r}\right)_{r_c} = 0 \ \forall \ t$ (4)

3.1 Dimensionless Variables

Eqs.(1) – (4) incorporate physical parameters such as permeability, and it would be irrelevant to solve this problem for a particular combination of values for these parameters. Dimensionless variables are designed to eliminate the physical parameters that affect quantitatively, but not qualitatively, the reservoir response. Eq. (1)is in Darcy units, and the dimensionless terms will render the system of units employed irrelevant. For this model, three dimensionless variables are required. In US Oilfield units, distance, time and pressure are replaced as follows:

Dimensionless time $t_D = \frac{0.0002637\,kt}{\phi \mu c r_w^2}$ (5)

Dimensionless distance $r_D = \frac{r}{r_w}$ (6)

Dimensionless pressure $P_D = \frac{kh}{141.2 qB\mu}\left(P_i - P\right)$ (7)

By defining dimensionless variables in this way, it is possible to create an analytical model of the well and reservoir, or theoretical 'type-curve', that provides a 'global' description of the pressure response that is independent of the flow rate or of the actual values of the well and reservoir parameters.

Eq. (1) can be transformed by substituting the following dimensionless variables in Eqs.(5) – (7) into eq. (1) and this becomes:

$$\frac{\partial^2 P_D}{\partial r_D{}^2} + \frac{1}{r_D}\frac{\partial P_D}{\partial r_D} = \frac{\partial P_D}{\partial t_D} \quad (8)$$

In this case, the governing equation is analysed assuming that the flow is steady i.e.,

$$\frac{\partial P_D}{\partial t_D} = 0 \quad (9)$$

By substituting eq. (9) into eq. (8), we have:

$$\frac{\partial^2 P_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial P_D}{\partial r_D} = 0 \quad (10)$$

The boundary conditions become:

At $r_D = r_{Dw}, P_D = P_{Dw}$ and (11)

$r_D = r_{De}, P_D = P_{De}$ (12)

Eq. (10) can also be written in a condensed form as:

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial P_D}{\partial r_D}\right) = 0 \quad (13)$$

Eq. (13) represents the steady state equation of the dimensionless form of the diffusivity equation.

4.0 Finite Element Formulation

4.1 Weak Formulation

In the development of the weak form, we assumed a quadratic element mesh and placed it over the domain and applied the following steps:

Multiply eq. (13) by the weight function (w) and integrate the final equation over the domain.

$$\int_v w\left[\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial P_D}{\partial r_D}\right)\right]dv = 0 \quad (14)$$

Mathematically,

$$\int_v dv = \int_0^1 \int_0^{2\pi} \int_{r_{DA}}^{r_{DB}} r_D dr_D d\theta dz \tag{15}$$

Substitute eq. (15) into eq. (14)

$$\int_0^1 \int_0^{2\pi} \int_{r_{DA}}^{r_{DB}} w\left[\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial P_D}{\partial r_D}\right)\right] r_D dr_D d\theta dz = 0 \tag{16}$$

Integrating eq. (16) with respect to z , then θ , over the limits, we have;

$$2\pi \int_{r_A}^{r_B} w\left[\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D \frac{\partial P_D}{\partial r_D}\right)\right] r_D dr_D = 0 \tag{17}$$

Eq. (17)can be simplified by integrating by part,

$$w\left[r_D \frac{\partial P_D}{\partial r_D}\right]_{r_{DA}}^{r_{DB}} - \int_{r_{DA}}^{r_{DB}} r_D \frac{\partial w}{\partial r_D}\frac{\partial P_D}{\partial r_D} dr_D = 0 \tag{18}$$

$$wQ_{DA} + wQ_{DB} - \int_{r_{DA}}^{r_{DB}} r_D \frac{\partial w}{\partial r_D}\frac{\partial P_D}{\partial r_D} dr_D = 0 \tag{19}$$

Where $Q_D = r_D \dfrac{\partial P_D}{\partial r_D}$ (20)

4.2 Interpolation Function

The weak form in eq. (19) requires that the approximation chosen for P_D should be at least quadratic in r_D so that there are no terms in eq. (19) that are identically zero. Since the primary variable is simply the function itself, the Lagrange family of interpolation functions is admissible. We proposed that P_D is the approximation over a typical finite element domain by the expression:

$$P_D(r_D) = \sum_{j=1}^{n} P_{Dj}\psi_j^e(r_D) \ \ and \ \ w = \psi_i^e(r_D) \tag{21}$$

Substituting eq. (21) into eq. (20), we have;

$$Q_i^e - \int_{r_{DA}}^{r_{DB}} r_D \frac{d\psi_i^e}{dr_D}\frac{\partial}{dr_D}\sum_{j=1}^{n} P_{Dj}\psi_j^e(r_D) dr_D = 0 \tag{22}$$

Factor out $\sum_{j=1}^{n} P_{Dj}$

$$\sum_{j=1}^{n} P_{Dj} \int_{r_{DA}}^{r_{DB}} r_D \frac{d\psi_i^e}{dr_D}\frac{d\psi_j^e}{dr_D} dr_D = Q_i^e \tag{23}$$

In matrix form we can represent the semi-discrete finite element model thus,

$$\left[K_{ij}^e\right]\left\{P_{Dj}\right\} = \left\{Q_i^e\right\} \tag{24}$$

Where

$$K_{ij}^e = \int_{r_{DA}}^{r_{DB}} r_D \frac{d\psi_i^e}{dr_D}\frac{d\psi_j^e}{dr_D} dr_D \tag{25}$$

Here, eq. (24) is known as the finite element model.
Using Quadratic Lagrange Interpolation functions for a quadratic element:

$$\psi_1(r_D) = \frac{1}{h^2}(h + r_{DA} - r)(h - 2r_D + 2r_{DA}) \tag{26}$$

$$\psi_2(r_D)=\frac{4}{h^2}(r_D-r_A)(h+r_{DA}-r_D) \quad (27)$$

$$\psi_3(r_D)=\frac{-1}{h^2}(r_D-r_{DA})(h-2r_D+2r_{DA}) \quad (28)$$

The coefficient matrix can be easily derived by substituting the Lagrange interpolation functions into eq.(25)accordingly. The matrices are shown below:

$$\left[K^e\right]=\frac{1}{6h}\begin{bmatrix} 3h+14r_{DA} & -(4h+16r_{DA}) & h+2r_{DA} \\ -(4h+16r_{DA}) & 16h+32r_{DA} & -(12h+16r_{DA}) \\ h+2r_{DA} & -(12h+16r_{DA}) & 11h+14r_{DA} \end{bmatrix} \quad (29)$$

4.3 Shape Assembly

For the purpose of this work, 10 quadratic elementswasused to represent the entire reservoir,

$$r_{DA}=r_{Dw}+(n-1)h \quad (30)$$

Where n = number of elements

In this analysis, we have withheld the computational details of the finite element analysis (FEA) used. However, the authors would be glad to interact with researchers who may want to refer to the computational mathematics involved.

5.0 Results and Discussion

Eq. (13) can be analysed using exact differential equation method. This can be done by integrating twice and then imposing the boundary conditions in eqs. (11) and (12) [1]. The result of the analysis is shown in eq. (31).

$$P=P_{Dw}+\left[\frac{P_{De}-P_{Dw}}{\ln r_{De}/r_{Dw}}\right]\ln\frac{r_D}{r_{Dw}} \quad (31)$$

For the radial geometry in a reservoir, flow can be described under what is referred to as the steady state condition. This implies that, for a well producing at a constant rate q; the change in pressure with respect to time is zero at all points within the reservoir. Thus the outer boundary pressure P_{De} and the entire pressure profile remain constant with time. This condition may appear to be artificial but is realistic in the case of a pressure maintenance scheme, such as water injection, in which one of the aims is to keep the pressure constant. In such a case, the oil withdrawn from the radial cell is replaced by fluids crossing the outer boundary at $r_D=r_{De}$ .In addition, for simplicity, the reservoir will be assumed to be completely homogeneous in all its parameters and the well perforated across the entire formation thickness.

In reservoirs, the steady-state flow condition can only occur when the reservoir is completely recharged and supported by strong aquifer or pressure maintenance operations.

Table 1: Parameters for analysis

P_{Dw}	P_{De}	r_{Dw}	r_{De}
0.5	5	1	41

When the reservoir has been producing for quite a long time, a time will come where the fluid that is leaving the reservoir will be equal to the fluid entering the reservoir. At this time, the flow is said to be in its steady state. In this state the change in pressure is not a function of time but a function of position. This is to say that at this stage, the dimensionless pressure does not change with time but it varies in the radial direction.

Fig.1 shows the results obtained for change in dimensionless pressure against the change in dimensionless radius for both the finite element method and the exact differential equation method. It can be seen that the variation in pressure within the vicinity of the wellbore was very pronounced and later becomes almost uniform outside the region of the wellbore radius to the reservoir external boundary.

Table 2: Numerical results using FEM

r_D	1	3	5	7	9	11	13	15	17	19	21
P_D	0.5000	1.7798	2.4198	2.8315	3.1402	3.3862	3.5912	3.7667	3.9202	4.0566	4.1794

Table 2: Contd

r_D	23	25	27	29	31	33	35	37	39	41
P_D	4.2910	4.3932	4.4876	4.5753	4.6571	4.7338	4.8059	4.8741	4.9387	5.0000

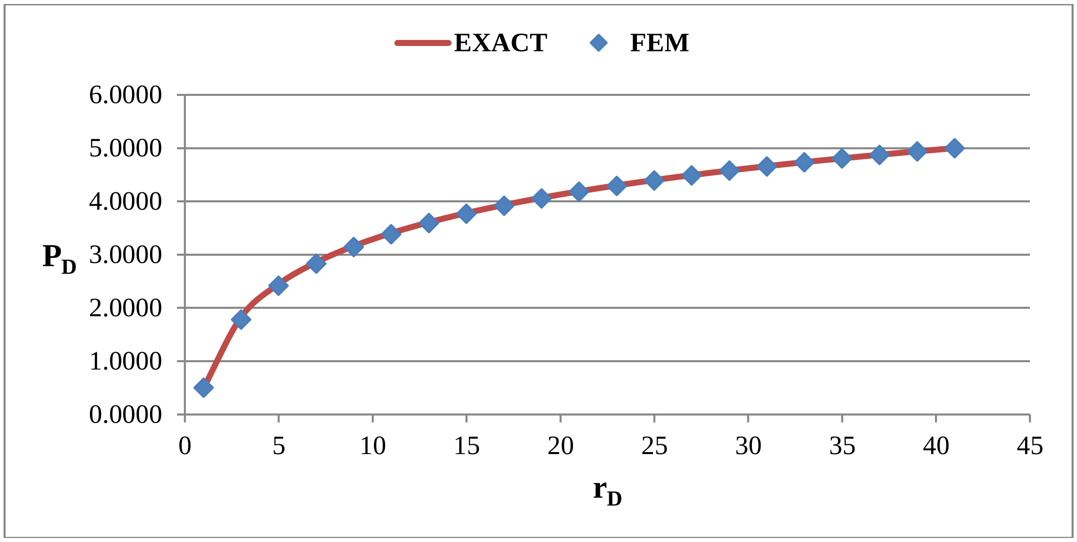

Fig. 1: A graph of dimensionless pressure against radial displacement

This is maintained throughout the flow of the fluid due to the fact that the reservoir is been recharged by a very strong aquifer. The sudden change in the pressure within the region of the wellbore radius is due to the fact that, as the fluid is been withdrawn from the reservoir through the wellbore, there is adirect reduction in the pressure within the region of the wellbore. This disturbance as a result of the change in pressure quickly corrected by replenishing the lost fluid from the region outside the wellbore radius and the overall effect spread through the entire reservoir. When this pressure disturbance gets to the external reservoir boundary, it fades out. This is because the pressure drop as a result of withdrawing fluid from the reservoirhas been replaced by the strong aquifer thereby maintaining the pressure within the reservoir all through the steady state flow regime.

InFig.1, the plot of the two results almost completely merged due to the high accuracy of the results.To ascertain the accuracy of this analysis, the same problem was analysed using the exact differential equation method. It was realized that the two results obtained converged. To test for the degree of convergence, the percentage error between the two methods was calculated. It was observed that the two results converge but with high percentage error at the region close to the wellbore. This was due to the fact the there was much pressure variation within the region around the wellbore radius. The percentage error around the reservoir external boundary was minimal because there was no much sudden change in the pressure within that region. Table 3 shows the dimensionless radius and their corresponding percentage error between the results obtained from the finite element method and the exact differential equation method.

Table 3: Comparison between the FEM and the exact method in eq. (31) using percentage error

r_D	1	3	5	7	9	11	13	15	17
% ERROR	0.0000	2.8081	1.2450	0.9285	0.7051	0.5720	0.4690	0.3927	0.3299

Table 3: contd

r_D	19	21	23	25	27	29	31	33	35
% ERROR	0.2790	0.2354	0.1983	0.1657	0.1371	0.1115	0.0885	0.0676	0.0485

Table 3: contd

r_D	37	39	41
% ERROR	0.0310	0.0149	0.0000

6.0 Conclusion

In this work, we have formulated the Finite Element Model and the model was used to analyse thediffusivity equation for bounded circular reservoir which is rarely seen in literature. The result obtained from the analysis was used to describe the pressure distribution across a reservoir from its wellbore to the external boundary in the steady state flow regime. It was seen that thedimensionless pressurein the reservoir increases from the wellbore to the external boundary. The results obtained from this work were validated by comparingit with the results obtained from the exact differential equation method.The accuracy of the results obtained shows high degree of correlation between the formulated finite element model and the exact differential equation method.

7.0 References

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