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Analysis of Pressure Variation of Fluid in Bounded Circular Reservoirs: Pseudo-Steady State Case

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Abstract

In this work, we usedthe pseudo steady form of the diffusivity equation to investigate the pressure distribution in a bounded circular reservoir. The reservoir domain was discretized into ten Lagrange quadratic elements and was assembled to represent the cross-section of the reservoir. It was assumed that before the well begins production, there was uniform distribution of pressure throughout the reservoir and that the well has been producing long enough for the reservoir outer boundary to feel the disturbance created at the wellbore. Thus, this work addresses the pseudo-steady state flow when the pressure at different locations in the reservoir is declining linearly as a function of time, i.e., at a constant declining rate. The result shows that there was an increase in pressure from the wellbore to the external boundary of the reservoir. This increase was very pronounced around the vicinity of the wellbore and flattens out within the region of the external boundary. The results obtained from this analysis were compared with the results obtained from the exact differential equation method. The comparison shows that there was a strong agreement between both methods.

Keywords: Bounded circular reservoir, pseudo-steady state, diffusivity equation, and Finite element formulation

Nomenclature:

B	Formation volume factor, RB/STB
C	Constant
c	Compressibility, psia-1
h	Thickness, ft
K	Stiffness matrix
k	Permeability, md
M	Mass matrix
n	Number of elements
P	Pressure, psi
P_D	Dimensionless pressure
P_i	Initial reservoir pressure, psi
Q	Terminal flow rate
q	Volumetric flow rate, STB/Day
r	Radius, ft
r_D	Dimensionless radius
r_e	External radius, ft
r_{eD}	Dimensionless external radius
r_w	Wellbore radius, ft
t	Time, hr
t_D	Dimensionless time
V	Pore volume, bbl
w	Weight function
$\forall$	For all

Greek letters

Δt	Time increment, hr
α	Family of approximation
ϕ	Porosity, fraction
η	Diffusivity constant
μ	Viscosity, cp
π	Pi
ψ	Interpolation function

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1.0 Introduction

Challenges to reservoir engineers begin when a reservoir is opened to production and the flow of hydrocarbons begins. At this point, reservoir pressure begins to drop, fluids comprising gas, oil, and water expands, phase equilibrium is disturbed, and alterations in the physical properties of the fluid phases occur in various degrees throughout the entire reservoir. With further withdrawal of fluids, changes continue and difficult second-order differential equations are needed to describe the unsteady state flow of expandable fluids [1].

In other to solve this second-order differential equations, the fluid flow have to be divided into three basic flow regimes which describes the fluid flow behaviour and reservoir pressure distribution as a function of time. These flow regimes are the steady-state flow, unsteady-state flow and the pseudo-steady-state flow.

Steady-state flow occurs when the pressure at every location in the reservoir remains constant. In reservoirs, the steady-state flow condition occurs when the reservoir is completely recharged and supported by strong aquifer or pressure maintenance operations due to either natural water influx or the injection of some displacing fluid [2].

Unsteady-state flow occurs when the pressure at every location in the reservoir varies with time. In this case, it is assumed that a well is located in a very large reservoir and producing at a constant flow rate. This rate creates a pressure disturbance in the reservoir that travels throughout this infinite-size reservoir. During this transient flow period, reservoir boundaries have no effect on the pressure behaviour and this is often very short in length. This regime ends as soon as the pressure disturbance reaches all drainage boundaries. Pressure response in transient state for a well producing from a finite reservoir of circular, square, and rectangular drainage shapes has been studied in [3 – 12].

The pseudo-steady flow regime occurs soon after the transient state when entire reservoir pressure has been affected. The change in pressure with time at all radii in the reservoir becomes uniform. Therefore, the pressure distributions at subsequent times are parallel [13]. It is necessary at this point to impose different boundary conditions on the diffusivity equation and drive an appropriate solution to this flow regime.

In the pseudo-steady state flow regime, the change in pressure with time becomes the same throughout the drainage area. This definition means that the rate of change of pressure with respect to time at every position is constant. This condition is also referred to as semi steady-state flow and quasi steady-state flow [2]. The condition of pseudo-steady state is most appropriately applied to describe reservoirs which have been under development for some time.

There are four solutions that are useful in well testing analysis,: the solution for a bounded circular reservoir; the solution for an ideal reservoir with a well are considered to be a line with zero well bore radius; the pseudo steady-state solution; and the solution that includes well bore radius for a well in an infinite reservoir.

Research in the field of reservoir engineering using Finite Element Method (FEM) is sparse. Therefore, this work presents the formulation of the finite element model to analyse the problem of a fluid in a bounded circular reservoir in the pseudo-steady state flow regime.

2.0 Theory

The second order partial differential equation to represent pressure variation in a reservoir has been modelled from the combination of the law of conservation of mass, Darcy's law and the equation of state:

$$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}=\frac{\phi\mu c}{0.000264k}\frac{\partial P}{\partial t} \qquad (1)$$

with the assumptions that compressibility, c, is small and independent of pressure, P; permeability, k, is constant and isotropic; viscosity, μ, is independent of pressure; porosity, ϕ, is constant; and that certain terms in the basic differential equation (including pressure gradients squared) are negligible. Eq. (1) is called the diffusivity equation and the term $\frac{\phi\mu c}{0.000264k}$ is the inverse of the diffusivity constant, η.

In this work, the diffusivity equation is analysed for bounded circular reservoirs, the case in which the well is assumed to be located in the centre of a cylindrical reservoir and also, the flow is assumed to be in pseudo-steady state where the pressure differential with respect to time is assumed to be constant.

3.0 Governing Equation

The governing second order partial differential equation for flow in porous media for a slightly compressible liquid is given as:

$$\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}=\frac{\phi\mu c}{0.000264k}\frac{\partial P}{\partial t} \qquad (2)$$

The significant assumptions made are:

i. Slightly compressible liquid (constant compressibility)
ii. Constant fluid viscosity
iii. Single-phase liquid flow
iv. Gravity and capillary pressure are neglected
v. Constant permeability
vi. Horizontal radial flow (no vertical flow)

If we assume that the flow rate (q) is constant, then $\frac{\partial P}{\partial t}$ is constant as well.

Using the definition of the compressibility:

$$c = -\frac{1}{V}\frac{dV}{dP} \tag{3}$$

Eq. (3) can be rearranged and differentiated with respect to time (t) to give:

$$\frac{dP}{dt} = -\frac{q}{cV} \tag{4}$$

Expressing the eq. (3) in oilfield units gives:

$$\frac{dP}{dt} = -\frac{q}{24cV} \tag{5}$$

For a radial drainage system, the pore volume (V) is given by:

$$V = \frac{\pi r_e^2 h\phi}{5.615} \tag{6}$$

Substituting eq. (6) into eq. (5) gives:

$$\frac{dP}{dt} = -\frac{0.0745q}{c_t r_e^2 \phi h} \tag{7}$$

Substituting eq. (7) into eq. (2) gives:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = -\frac{1}{r_e^2}\left(\frac{282.0515q\mu}{hk}\right) \tag{8}$$

If we assume that the rock and fluid properties $\left(\frac{282.0515q\mu}{hk}\right)$ are constant (C),

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = -\frac{C}{r_e^2} \tag{9}$$

Eq. (9) is referred to as the pseudo-steady form of the diffusivity equation. This equation cannot be solved analytically except we introduce some boundary conditions. These are shown in eq. (10).

At $r = r_w$; $P = P_w$; and $\left(\frac{\partial P}{\partial r}\right)_{r=r_e} = 0$ (10)

4.0 Finite Element Formulation

4.1 Weak Formulation

In the development of the weak form, we assumed a quadratic element mesh and placed it over the domain and applied the following steps:

Multiply eq. (9) by the weighted function and integrate the final equation over the domain.

$$\int_v w\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) + \frac{C}{r_e^2}\right]dv = 0 \tag{11}$$

From mathematics,

$$\int_v dv = \int_0^1 \int_0^{2\pi} \int_{r_w}^{r_e} r dr d\theta dz \qquad (12)$$

Substitute eq. (12) into eq. (11)

$$\int_0^1 \int_0^{2\pi} \int_{r_w}^{r_e} w\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)+\frac{C}{r_e^2}\right] r dr d\theta dz = 0 \qquad (13)$$

Integrating eq. (13) with respect to z , then θ , and incorporate the limits, we have;

$$2\pi \int_{r_w}^{r_e} w\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)+\frac{C}{r_e^2}\right] r dr = 0 \qquad (14)$$

Simplifying eq. (14) by integrating by part,

$$w\left[r\frac{\partial P}{\partial r}\right]_{r_w}^{r_e} - \int_{r_w}^{r_e} r\frac{\partial w}{\partial r}\frac{\partial P}{\partial r}dr + \frac{C}{r_e^2}\int_{r_w}^{r_e} rw dr = 0 \qquad (15)$$

Expand eq. (15),

$$wQ_w + wQ_e - \int_{r_w}^{r_e} r\frac{\partial w}{\partial r}\frac{\partial P}{\partial r}dr + \frac{C}{r_e^2}\int_{r_w}^{r_e} rw dr = 0 \qquad (16)$$

Where $Q = r\dfrac{\partial P}{\partial r}$ (17)

4.2 Interpolation Function

The weak form in eq. (16) requires that the approximation chosen for P should be at least quadratic in r so that there are no terms in eq. (16) that are identically zero. Since the primary variable is simply the function itself, the Lagrange family of interpolation functions is admissible. We proposed that P is the approximation over a typical finite element domain by the expression:

$$P(r) = \sum_{j=1}^{n} P_j \psi_j^e(r) \text{ and } w = \psi_i^e(r) \qquad (18)$$

Substituting eq. (17) into eq. (16), we have;

$$Q_i^e - \int_{r_w}^{r_e} r\frac{d\psi_i^e}{dr}\frac{\partial}{dr}\sum_{j=1}^{n} P_j \psi_j^e(r) dr + \frac{C}{r_e^2}\int_{r_w}^{r_e} r\psi_i^e(r) dr = 0 \qquad (19)$$

Factor out $\sum_{j=1}^{n} P_j$

$$Q_i^e - \sum_{j=1}^{n} P_j \int_{r_w}^{r_e} r\frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr}dr + \frac{C}{r_e^2}\int_{r_w}^{r_e} r\psi_i^e(r) dr = 0 \qquad (20)$$

In matrix form we can represent the semi-discrete finite element model thus,

$$\left[K_{ij}^e\right]\{P\} = \left\{Q_i^e\right\} + \frac{C}{r_e^2}\left\{M_i^e\right\} \qquad (21)$$

Eq. (21) is referred to as the finite element model.

Where

$$K_{ij}^e = \int_{r_w}^{r_e} r\frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr}dr \qquad (22)$$

$$M_i^e = \int_{r_w}^{r_e} r\psi_i^e(r)dr \tag{23}$$

Where $r_e = r_w + nh$ (24)

Also, $h = \dfrac{r_e - r_w}{n}$ (25)

Where n = No of elements
Using Quadratic Lagrange Interpolation functions for a quadratic element:

$$\psi_1(r) = \frac{1}{h^2}(h + r_w - r)(h - 2r + 2r_w) \tag{26}$$

$$\psi_2(r) = \frac{4}{h^2}(r - r_w)(h + r_w - r) \tag{27}$$

$$\psi_3(r) = \frac{-1}{h^2}(r - r_w)(h - 2r + 2r_w) \tag{28}$$

The coefficient matrix can be easily derived by substituting the Lagrange interpolation functions into eq.(22) respectively. The matrices are shown below:

$$[K^e] = \frac{1}{6h}\begin{bmatrix} 3h+14r_w & -(4h+16r_w) & h+2r_w \\ -(4h+16r_w) & 16h+32r_w & -(12h+16r_w) \\ h+2r_w & -(12h+16r_w) & 11h+14r_w \end{bmatrix} \tag{29}$$

$$\{M^e\} = \frac{h}{6}\begin{Bmatrix} r_w \\ 4r_w + 2h \\ r_w + h \end{Bmatrix} \tag{30}$$

4.3 Shape Assembly

For the purpose of this work, ten quadratic elements has been used to represent the entire reservoir,

$$r_{DA} = r_{Dw} + (n-1)h \tag{31}$$

Where n = number of elements
In this analysis, we have withheld the computational details of the finite element analysis (FEA) used. However, the authors would be glad to interact with researchers who may want to refer to the computational mathematics involved in the shape assembly.

5.0 Results and Discussion

The pseudo-steady state form of the diffusivity equation as shown in eq. (9) can also be analysed using the exact differential equation method. This can be done by integrating twice and then imposing the boundary conditions stated in eq. (10) [2]. The result of the analysis is shown in eq. (32).

$$P - P_w = \frac{C}{2}\left[r_e^2 \ln\frac{r}{r_w} - \frac{1}{2}\left(r^2 - r_w^2\right)\right] \tag{32}$$

Table 1: Parameters for Analysis

P	r_e	q	μ	h	κ
4500	10000	300	0.4	40	100

The initial condition in the reservoir states that the pressure is uniform i.e., the pressure from the wellbore to the external boundary is the same at time t=0. In this pseudo state flow regime, the reservoir has been producing for a sufficient period of time so that the effect of the outer boundary has been felt. In this case, the influence of the reservoir boundaries or the shape of the drainage area has effect on the rate at which the pressure disturbance spreads in the reservoir. It is therefore considered that the well acts as if it is surrounded at its outer boundary by a solid "brick wall" which prevents the flow of fluids into the radial cell. This “brick wall” can either be in the form of a fault bringing about variation in the permeability of the walls of the reservoir or a high degree of anisotropy.

Table 2: Numerical Results Using FEM

r	1	501	1001	1501	2001	2501	3001	3501	4001	4501	5001
P	4500.0	4509.5	4512.6	4514.3	4515.5	4516.4	4517.1	4517.7	4518.2	4518.6	4518.9

Table 2: Contd

r	5501	6001	6501	7001	7501	8001	8501	9001	9501	10001
P	4519.2	4519.5	4519.7	4519.9	4520.0	4520.1	4520.2	4520.2	4520.3	4520.3

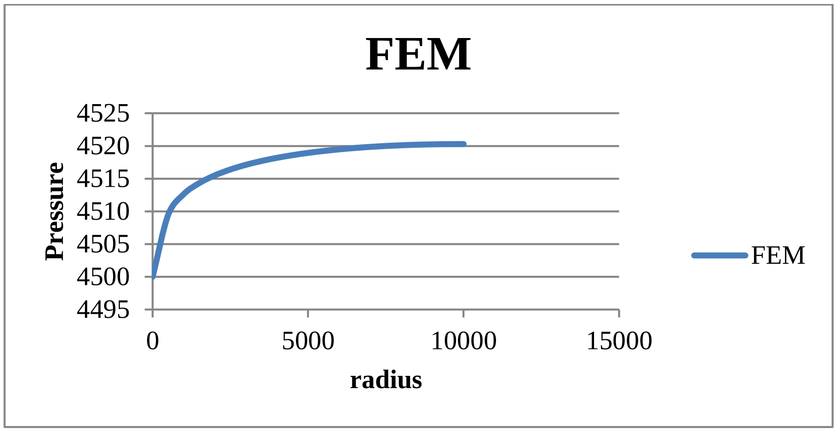

Fig. 1: A Graph of Pressure Against Radius

Fig.1 shows the results obtained for change in pressure against the change in radius for the finite element method. It can be seen that the variation in pressure within the vicinity of the wellbore was very pronounced and later becomes almost uniform outside the region of the wellbore radius to the presumed "brick wall" which might not necessarily be the reservoir external boundary. Thus, for the same boundary conditions P_e and P_w , the constant well flow in the pseudo-steady state flow regime is slightly higher than the one obtained in the case of steady state flow regime and the difference is explained by the fact that most of the produced fluid travels a shorter distance in the pseudo-steady case than in the steady state flow regime..

The abrupt change in the pressure within the region of the wellbore radius is due to the fact that, fluid is been withdrawn from the reservoir through the wellbore, thereby increasing the pressure within the region of the wellbore. Therefore, Table 2 shows the numerical values of the pressure at different points within the reservoir formation using the formulated finite element model.

To test for the accuracy of this analysis, the same problem was analysed using the exact differential equation method. It was realized that the two results obtained converged. To test for the degree of convergence, the percentage error between the two methods was calculated. It was observed that the two results converged. Table 3 shows the radius and their corresponding percentage error between the results obtained from the finite element method and the exact differential equation method.

Table 3: Comparison between the FEM and the exact method in eq. (32)using percentage error

r (ft)	1	501	1001	1501	2001	2501	3001	3501	4001	4501	5001
% Error	0.0000	0.3722	0.3667	0.3667	0.3665	0.3665	0.3664	0.3664	0.3663	0.3663	0.3663

Table 3: Contd

r (ft)	5501	6001	6501	7001	7501	8001	8501	9001	9501	10001
% Error	0.3663	0.3662	0.3662	0.3662	0.3662	0.3662	0.3662	0.3662	0.3662	0.3662

Examination of the expression in eq. (8) reveals the following important characteristics of the behaviour of the pressure decline rate during the pseudo-steady state flow:

Pressure variation with fluid viscosity

The reservoir pressure increases with an increase in the fluids viscosity and vice versa. This is shown in Fig. 2. The range of values of viscosity used varies from 0.1 to 0.9.

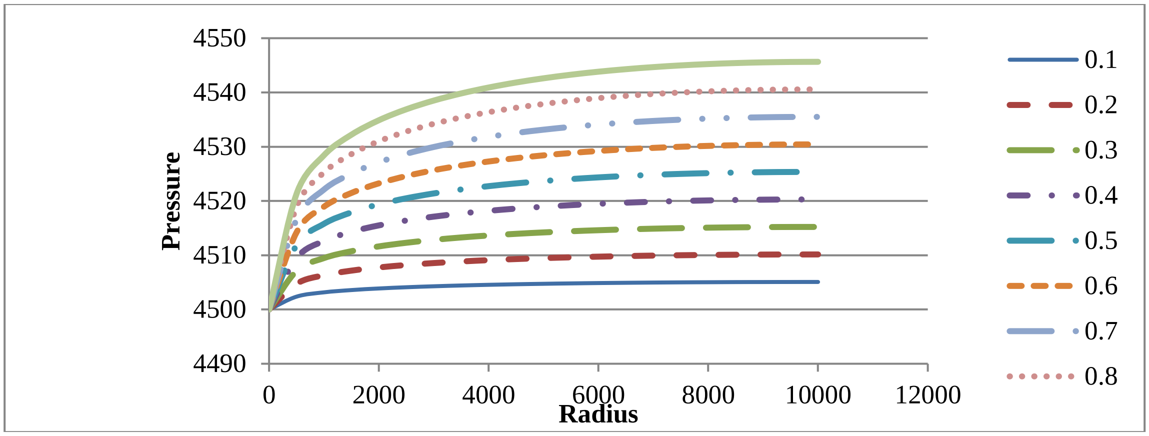

Fig. 2: A Graph of Pressure Against Radius for Increase in Viscosity

Table 4: Parameters for Analysis for Increase in Fluid Viscosity

P	r_e	q	h	κ
4500	10000	300	40	100

Pressure variation with fluid height

The reservoir pressure increases with a decrease in the height of the fluid in the reservoir and vice versa. This is shown in Fig. 3. In the course of the analysis the range of values of the height of the fluid varies from 20 to 100.

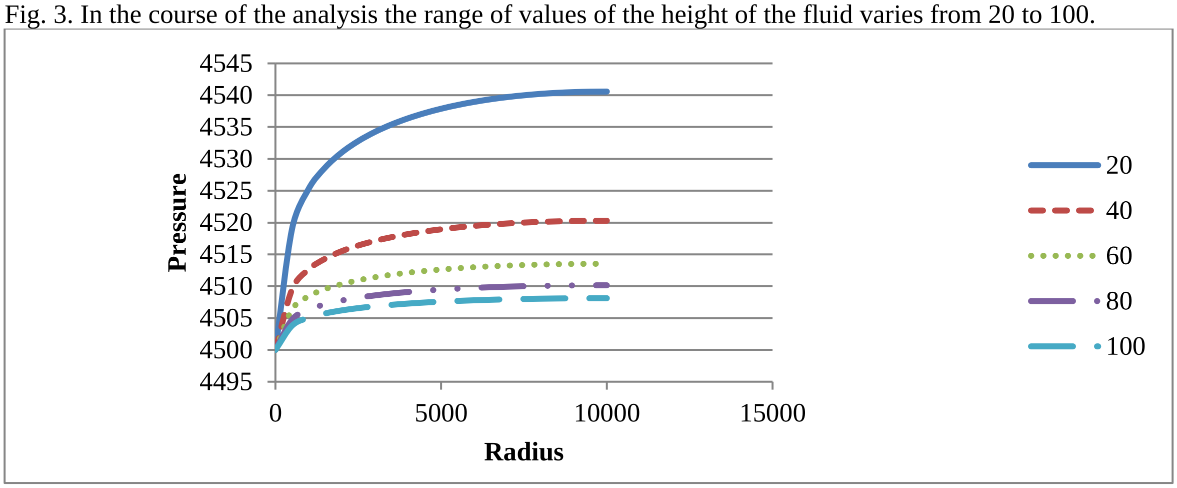

Fig.3. A graph of Pressure against radius for decrease in height

Table 5: Parameters for Analysis for Variation in Height of the Fluid

P	r_e	q	μ	κ
4500	10000	300	0.4	100

Pressure Variation with Rock Permeability

Permeability is a property of the porous medium that measures the capacity and ability of the formation to transmit fluids. The rock permeability, k, is a very important rock propertybecause it controls the directional movement and the flow rate of the reservoir fluids in the formation. From this analysis, it was observed that the reservoir pressure decreases with an increase in permeability and vice versa. This is shown in Fig. 4.

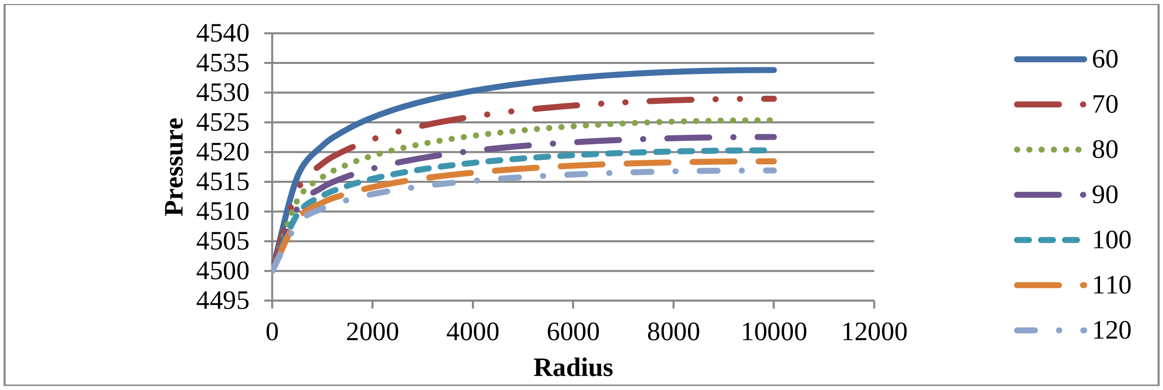

Fig. 4: A graph of Pressure against radius for increase in permeability

Table 6: Parameters for Analysis for Increase in Permeability

P	r_e	q	μ	h
4500	10000	300	0.4	40

6.0 Conclusion

In this work, we have developed a Finite Element Model to analyse the diffusivity equation in bounded circular reservoir. The results obtained were used to describe the pressure distribution across a reservoir from its wellbore to the external boundary in the pseudo-steady state flow regime. It was seen that the pressure in the reservoir increases from the wellbore to the external boundary. It was observed also that the reservoir pressure increases with an increase in the fluids viscosity while the reservoir pressure increases with a decrease in the height of the fluid in the reservoir and the rock permeability and vice versa.

The results obtained from this work were validated by comparing it with the results obtained from the exact differential equation method. The accuracy of the results obtained shows high degree of correlation between both methods. Therefore, the Finite element method can be used in analysing and evaluating well pressure distribution for pseudo-steady state flow in bounded circular reservoirs.

The beauty of the developed finite element model is that it tells us the pressure history at different points in the entire reservoir formation from the well bore to the external boundary at a glance as against other numerical methods that involve continuous iterations.

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