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On Some Modified Backward Differentiation Formulas for Stiff ODEs

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Abstract

This paper presents a modified third derivative backward differentiation formulas (TDBDF)for the numerical integration of stiff systems in ordinary differential equations (ODEs).The sum of thefirst derivative function $\{f_{n+1}\}_{j=0}^{k}$ in theTDBDFis replaced by the hybrid term f_{n+v} ; with $v=k-\frac{1}{\tau}$, $\tau=2,3$ defining the off-step point and k is the step number of the method. A case of $v=k-\frac{1}{2}$ shows that theTDBDF are A(α)-stable for $k\le 9$ with $\alpha\in\left[0,\frac{\pi}{2}\right]$, while for $v=k-\frac{1}{3}$ the formulas are A(α)-stable for $k\le 12$.Examples of the integration processes are given in full herein.

1.0 Introduction

The interest herein is the numerical integration of the initial value problem,

$$y' = f(x,y)\ , y(a) = y_0,\ \ a \le x \le b, \tag{1.1}$$

using a modified third derivative backward differentiation formula (MTDBDF)

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_v f_{n+v} + h^2\gamma_k^{(1)} g_{n+k} + h^3\varphi_k T_{n+k},\ \ p = k+2, \tag{1.2}$$

with the hybrid predictor

$$y_{n+v} = \sum_{j=0}^{k} \alpha_j y_{n+j} + h^2\gamma_k^{(2)} g_{n+k} + h^3\omega_k T_{n+k}\ , \quad p = k+2. \tag{1.3}$$

The third derivative backward differentiation formula (TDBDF) was first considered in [1]. For further examples, see [2 – 7] and references therein. Other examples of modified multistep methods are in [8 – 26]. The approach in this paper follows that of [7 – 25]. To start the method (1.2) one need to find the second and third derivative functions in (1.2) and (1.3) from the problem (1.1); these may sometimes pose some difficulties if the right software such as MATHEMATICA or MAPLE is not readily available to automate it. One major advantage of this method is that they are highly stable at high order, readily overcoming the Dahlquist order barrier [26]. Note the special nature of the hybrid in (1.3) which avoids the need to compute $\{f_{n+1}\}_{j=0}^{k}$ and thus have no need to carry it along. This reduces the computed cost in the implementation of (1.2).In (1.2) and (1.3), $h = x_{n+1} - x_n$ is the step size, k is the step number, v =k $-\frac{1}{\tau}$, $\tau = 2, 3, \ldots$defining the off-step points. The α_j,β_j, β_v, $\gamma_k^{(1)}$, $\gamma_k^{(2)}$, φ_k and ω_k are coefficients of the methods. The function $y'(x) = f(x,y), g(x,y) = f'(x,y) = f_x + ff_y$ and $T(x,y) = y'''(x,y) = f''(x,y) = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_{yy}$ are the first, second and third derivative functions respectively, while the value of f_{n+v} in (1.2) is predicted using the hybrid predictory_{n+v}. They_{n+k} is the output solution at the point x_{n+k} of the method in (1.2).Our interest herein is to investigate the accuracy and the stability behaviourof the implicit algorithms in (1.2) and (1.3) for the values of$\tau = 2$and $\tau = 3$. For the test differential equation

$$y' = \lambda y,\ \mathrm{Re}(\lambda) < 0. \tag{1.4}$$

The stability polynomial of the MTDBDF in (1.2) is

$$\pi(w,z) = w^k - \sum_{j=0}^{k-1} \alpha_j w^j - z\beta_v\left(\sum_{j=0}^{k} \alpha_j w^j + z^2\gamma_k^{(2)} w^k + z^3\omega_k w^k\right) - z^2\gamma_k^{(1)} w^k - z^3\varphi_k w^k. \tag{1.5}$$

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The methods in (1.2) and (1.3) are implicit, see [1 – 19]. Other conditions on the coefficients of the implicit scheme in (1.2) are the requirements that it satisfy the following definitions.

Definition [c. f. [26]]: A(α)-stability: A MTDBDF (1.2) is said to be A(α)-stable for some $\alpha \in [0,\frac{\pi}{2}]$ if the wedge $s_\alpha = \{z: |\arg(-z)| < \alpha,\ z \neq 0\}$ is contained its region of absolute stability.The largest $\alpha\ that\ is\ \alpha_{max}$ is regarded as the angle of absolute stability.

The error constant and the of order p of a method in (1.2) and (1.3) is obtained from the local truncation errors [9],

$$L_1 = y(x_n + kh) - \sum_{j=0}^{k-1} \alpha_j y(x_n + jh) - h\beta_v y'(x_n + vh) - h^2\gamma_k^{(1)}\, y''(x_n + kh) - h^3\varphi_k y'''(x_n + kh)$$

$$= C_{p_1+1}h^{p_1+1}y^{(p_1+1)}(x_n), \tag{1.6}$$

$$L_2 = y(x_n + vh) - \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h^2\gamma_k^{(2)}\, y''(x_n + kh) - h^3\omega_k y'''(x_n + kh) = C_{p_2+1}h^{p_2+1}y^{(p_2+1)}(x_n). \tag{1.7}$$

Respectively, where$C_{p+1}^{(k)}$ and $C_{p+1}^{(v)}$arethe error constants. The method in (1.2) and it hybrid (1.3) are of uniform order $p = k + 2$, this eliminates the possibility of order reduction which areassociated with thehybrid LMM and Runge-Kutta methods (RKM) whenever the hybrid predictor or stage orders are far less than that of the output method.

2.0 Order conditions of the hybrid methods in (1.2) and (1.3)

Expanding the local truncation errors in (1.6) and (1.7) respectively aboutx_n and equating the powers of $h$yieldsthe order conditions of (1.2)

$$h^\sigma:\quad \frac{k^\sigma}{\sigma!} = \begin{cases} \sum_{j=0}^{k-1} \alpha_j\,; & \sigma = 0 \\ \sum_{j=1}^{k-1} j\alpha_j + \beta_v; & \sigma = 1 \\ \sum_{j=1}^{k-1} \frac{j^2\alpha_j}{2!} + v\beta_v + \gamma_k^{(1)}; & \sigma = 2 \\ \sum_{j=1}^{k-1} \frac{j^\sigma\alpha_j}{\sigma!} + \frac{v^{\sigma-1}}{(\sigma-1)!}\beta_v + \frac{k^{\sigma-2}}{(\sigma-2)!}\gamma_k^{(1)} + \frac{k^{\sigma-3}}{(\sigma-3)!}\varphi_k; & \sigma = 3, 4\cdots \end{cases} \tag{2.1}$$

For the hybrid predictor in (1.3) the order condition is

$$h^\sigma:\quad \frac{v^\sigma}{\sigma!} = \begin{cases} \sum_{j=0}^{k} \alpha_j\,; & \sigma = 0 \\ \sum_{j=1}^{k} j\alpha_j\,; & \sigma = 1 \\ \sum_{j=1}^{k} \frac{j^2\alpha_j}{2!} + \gamma_k^{(2)}; & \sigma = 2 \\ \sum_{j=1}^{k} \frac{j^\sigma\alpha_j}{\sigma!} + \frac{k^{\sigma-2}}{(\sigma-2)!}\gamma_k^{(2)} + \frac{k^{\sigma-3}}{(\sigma-3)!}\omega_k; & \sigma = 3, 4, \cdots \end{cases} \tag{2.2}$$

The error constants in (1.6) and (1.7) are

$$C_{p+1}^{(k)} = \frac{k^{(p_1+1)}}{(p_1+1)!} - \sum_{j=1}^{k-1} \frac{j^\sigma\alpha_j}{(p_1+1)!} + \frac{v^{\sigma-1}}{(p_1)!}\beta_v + \frac{k^{\sigma-2}}{(p_1-1)!}\gamma_k^{(1)} + \frac{k^{\sigma-3}}{(p_1-2)!}\varphi_k;\ \ p \geq 1$$

and

$$C_{p+1}^{(v)} = \frac{v^{(p_2+1)}}{(p_2+1)!} - \sum_{j=1}^{k} \frac{j^\sigma\alpha_j}{(p_2+1)!} + \frac{k^{\sigma-2}}{(p_2-1)!}\gamma_k^{(2)} + \frac{k^{\sigma-3}}{(p_2-2)!}\omega_k;\ \ p \geq 1$$

respectively.

3.0 The Derivation of the Methods in (1.2) and (1.3)

Solving the order conditions (2.1)corresponding to$k$=1, here p=3,gives

$\alpha_0 = 1,\ \ \beta_v = 1,\ \ \gamma_1^{(1)} = \frac{1}{2} - v,\qquad \varphi_1 = \frac{1}{6}(-2 + 6v - 3v^2),$

$C_4^{(1)} = -\frac{1}{24}((-1 + 4v^3\beta_v + 12\gamma_1^{(1)} + 24\varphi_1).$ (2.3)

Case 1: Substituting (2.3) into (1.2) and fixing $v = k - \frac{1}{2}$yields,

$y_{n+1} = y_n + h\, f_{n+\frac{1}{2}} + \frac{1}{24}h^3T_{n+1},\ p = 3,\ C_4^{(2)} = -\frac{1}{48}.$ (2.4)

Case 2: Inserting (2.3) with $v = k - \frac{1}{3}$ into (1.2) gives

$y_{n+1} = y_n + h\, f_{n+\frac{2}{3}} - \frac{1}{6}h^2g_{n+1} + \frac{1}{9}h^3T_{n+1}\ ,\ p = 3,\ C_4^{(3)} = -\frac{23}{648}.$ (2.5)

Again, solving the order conditions (2.1)corresponding to k=2, which p=4 in (1.2) leads to

$\alpha_0 = 1 - \dfrac{8(-6 + 12v - 6v^2 + v^3)}{-17 + 48v - 24v^2 + 4v^3},\qquad \alpha_1 = \dfrac{8(-6 + 12v - 6v^2 + v^3)}{-17 + 48v - 24v^2 + 4v^3},$

$\beta_v = 2 - \frac{8(-6+12v-6v^2+v^3)}{-17+48v-24v^2+4v^3},\ \ \gamma_2^{(1)} = \frac{2(-5+17v-12v^2+2v^3)}{-17+48v-24v^2+4v^3},$ (2.6)

$\varphi_2 = -\frac{-16+60v-51v^2+12v^3}{3(-17+48v-24v^2+4v^3)},\ \ C_5^{(2)} = -\frac{1}{120}(-32 + \alpha_1 + 5v^4\beta_v + 160\gamma_2^{(1)} + 240\varphi_2).$

Case 1: Fixing (2.6) with $v = k - \frac{1}{2}$ in (1.2) gives,

$y_{n+2} = \frac{1}{29}(-y_n + 30y_{n+1}) + \frac{28h}{29}f_{n+\frac{3}{2}} + \frac{1}{29}h^2g_{n+2} + \frac{1}{174}h^3T_{n+2},\ \ p = 4,\ C_5^{(2)} = -\frac{43}{13920}.$ (2.7)

Case 2: Putting (2.7) with $v = k - \frac{1}{3}$ into (1.2) yields

$y_{n+2} = \frac{1}{401}(-23y_n + 424y_{n+1}) + \frac{378h}{401}f_{n+\frac{5}{3}} - \frac{40}{401}h^2g_{n+2} + \frac{19}{401}h^3T_{n+2},\ p = 4\ .$ (2.8)

The error constant of (2.8) is $C_5^{(2)} = -\frac{503}{72180}$. For the hybrid predictor in (1.3) we fix$k$=1, p=3 in (1.3).Solving the order conditions (2.2) yields

$\alpha_0 = 1 - v,\ \ \alpha_1 = v,\ \ \gamma_1^{(2)} = -\frac{v}{2} + \frac{v^2}{2},\ \ \omega_1 = \frac{v}{3} - \frac{v^2}{2} + \frac{v^3}{6},\ C_4^{(v)} = \frac{1}{24}(v^4 - \alpha_1 - 12\gamma_1^{(2)} - 24\omega_1).$ (2.9)

Case 1: Setting$v = k - \frac{1}{2}$ in (1.3) leads to,

$y_{n+\frac{1}{2}} = \frac{1}{2}(y_n + y_{n+1}) - \frac{1}{8}h^2g_{n+1} + \frac{1}{16}h^3T_{n+1},\ \ p = 3,\ C_4^{(\frac{1}{2})} = \frac{-7}{384}.$ (2.10)

Case 2: Fixing$v = k - \frac{1}{3}$ into (1.3) gives

$y_{n+\frac{2}{3}} = \frac{1}{3}(y_n + 2y_{n+1}) - \frac{1}{9}h^2g_{n+1} + \frac{4}{81}h^3T_{n+1},\ \ p = 3,\ C_4^{(\frac{2}{3})} = \frac{-13}{972}.$ (2.11)

Setting k=2, p=4 in (1.3) and solving the order conditions (2.2) yields

$\alpha_0 = \frac{1}{14}(14 - 31v + 24v^2 - 8v^3 + v^4),\ \ \alpha_1 = \frac{1}{7}(24v - 24v^2 + 8v^3 - v^4),$

$\alpha_2 = -\frac{17v}{14} + \frac{12v^2}{7} - \frac{4v^3}{7} + \frac{v^4}{14},\quad \gamma_2^{(2)} = \frac{5v}{7} - \frac{17v^2}{14} + \frac{4v^3}{7} - \frac{v^4}{14},$ (2.12)

$\omega_2 = -\dfrac{8v}{21} + \dfrac{5v^2}{7} - \dfrac{17v^3}{42} + \dfrac{v^4}{14},\quad C_5^{(v)} = \dfrac{1}{120}\left(v^5 - \alpha_1 - 32\alpha_2 - 160\gamma_2^{(2)} - 240\omega_2\right).$

Case 1: Inserting$v = k - \frac{1}{2}$ into (2.12) and substituting the resulting expressions into (1.3) yields,

$y_{n+\frac{3}{2}} = y_{n+\frac{3}{2}} = \frac{1}{32}(-y_n + 18y_{n+1} + 15y_{n+2}) - \frac{3}{32}h^2g_{n+2} + \frac{1}{32}h^3T_{n+2},\ p = 4,\ C_5^{(\frac{3}{2})} = \frac{-1}{256}.$ (2.13)

Case 2: Fixing $v = k - \frac{1}{3}$ into (1.3) gives

$y_{n+\frac{5}{3}} = \frac{1}{567}(-13y_n + 215y_{n+1} + 365y_{n+2}) - \frac{50}{567}h^2g_{n+2} + \frac{5}{189}h^3T_{n+2},\ p = 4, C_5^{\left(\frac{5}{3}\right)} = \frac{-61}{20412}$ (2.14)

The stability plotsfor the algorithms in (2.4) and (2.5) are in Figures 1 and 3.The graphs in Figures 1, 3 imply that the algorithms in (2.4) and (2.5) are A(86^0) and A-stable respectively. Similarly, the boundary loci in Figures 1, 3 of the roots of the stability polynomials in(1.5) of the algorithms in (2.7) and (2.8) shows that the methods areA-stable respectively. Solvingthe above order conditions (2.1) and (2.2) we obtain the following hybrid methods, giving the two cases $\tau = 2, 3$, simultaneously as above with the changing step number k.

For $k = 3, v = k - \frac{1}{\tau},\ \tau = 2,\ p = 5$:

$y_{n+\frac{5}{2}} = \frac{1}{1088}(7y_n - 73y_{n+1} + 669y_{n+2} + 485y_{n+3}) - \frac{21}{272}h^2g_{n+3} + \frac{23}{1088}h^3T_{n+3},$ (2.15)

$y_{n+3} = \frac{1}{8605}(43y_n - 531y_{n+1} + 9093y_{n+2}) + \frac{1632}{1721}hf_{n+\frac{5}{2}} + \frac{402}{8605}h^2g_{n+3} - \frac{19}{8605}h^3T_{n+3}.$

Their error constantsare $C_6^{(3)} = \frac{-821}{1032600}$, and $C_6^{(5/2)} = \frac{-361}{261120}$. The stability region of (2.15) is given in Figure 1. This algorithm is A-stable.For $k = 3, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 5$:

$$y_{n+\frac{8}{3}} = \frac{61}{12393} y_n - \frac{208}{4131} y_{n+1} + \frac{1732}{4131} y_{n+2} + \frac{7760}{12393} y_{n+3} - \frac{104}{1377} h^2 g_{n+3} + \frac{232}{12393} h^3 T_{n+3},$$
$$y_{n+3} = \frac{503}{45620} y_n - \frac{1329}{11405} y_{n+1} + \frac{50433}{45620} y_{n+2} + \frac{4131}{4562} hf_{n+\frac{8}{3}} - \frac{1539}{22810} h^2 g_{n+3} + \frac{643}{22810} h^3 T_{n+3}. \qquad (2.16)$$

The error constants are $C_6^{(3)} = \frac{-6323}{2737200}$ and $C_6^{(8/3)} = \frac{-607}{557685}$.This method (2.16) is A-stable, see Figure 3 for the stability region.

Similarly, for $k = 4, v = k - \frac{1}{\tau}, \ \tau = 2, \ p = 6$:

$$y_{n+7/2} = \frac{-361}{169984} y_n + \frac{217}{10624} y_{n+1} - \frac{9079}{84992} y_{n+2} + \frac{7021}{10624} y_{n+3} + \frac{72695}{169984} y_{n4} - \frac{2835}{42496} h^2 g_{n+4} + \frac{343}{21248} h^3 T_{n+4}, C_7^{(7/2)} = \frac{-1591}{2549760},$$
$$y_{n+4} = \frac{-821}{679205} y_n + \frac{8768}{679205} y_{n+1} - \frac{56838}{679205} y_{n+2} + \frac{728096}{679205} y_{n+3} + \frac{127488}{135841} hf_{n+\frac{7}{2}} + \frac{7092}{135841} h^2 g_{n+4} - \frac{3416}{679205} h^3, \ C_7^{(4)} = \frac{-37189}{142633050}. \qquad (2.17)$$

These processes (2.17) is A-stable, see Figure 1. In like manner when $k = 4, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 6$:

$$y_{n+\frac{11}{3}} = -\frac{607}{363042} y_n + \frac{2893}{181521} y_{n+1} - \frac{4939}{60507} y_{n+2} + \frac{82676}{181521} y_{n+3} + \frac{222145}{363042} y_{n+4} - \frac{4070}{60507} h^2 g_{n+4} + \frac{2684}{181521} h^3 T_{n+4}, \quad C_7^{(11/3)} = \frac{-4103}{8168445},$$
$$y_{n+4} = \frac{1}{1835795}(-6323 y_n + 61304 y_{n+1} - 325914 y_{n+2}) + \frac{162056}{141215} y_{n+3} + \frac{322704}{367159} hf_{n+\frac{11}{3}} - \frac{17796}{367159} h^2 g_{n+4} + \frac{35352}{1835795} h^3 T_{n+4}, C_7^{(4)} = \frac{-1715614}{1734826275}. \qquad (2.18)$$

The algorithm in (2.18) is A-stable, see Figure 3. In what follows are the hybrid methods in (1.2) and (1.3) with $k = 5, v = k - \frac{1}{\tau}, \ \tau = 2, \ p = 7$:

$$y_{n+\frac{9}{2}} = \frac{1591}{1758208} y_n - \frac{62685}{7032832} y_{n+1} + \frac{37719}{879104} y_{n+2} - \frac{527235}{3516416} y_{n+3} + \frac{1234755}{1758208} y_{n+4} + \frac{2902851}{7032832} y_{n+5} - \frac{104355}{1758208} h^2 g_{n+5} + \frac{11565}{879104} h^3 T_{n+5}, C_8^{(9/2)} = \frac{-128577}{393838592},$$
$$y_{n+5} = \frac{148756}{395864721} y_n - \frac{530555}{131954907} y_{n+1} + \frac{974920}{43984969} y_{n+2} - \frac{40009630}{395864721} y_{n+3} + \frac{142847660}{131954907} y_{n+4} + \frac{17582080}{18850701} hf_{n+\frac{9}{2}} + \frac{2416580}{43984969} h^2 g_{n+5} - \frac{823240}{131954907} h^3 T_{n+5}, \qquad C_8^{(5)} = \frac{-1060769}{11084212188}. \qquad (2.19)$$

In Figure 3 the method is A(89.9^0)-stable.Considered next are the methods in (1.2) and (1.3) for $k = 5, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 7$. The hybrid algorithm is,

$$y_{n+\frac{14}{3}} = \frac{1}{11265237}(8206 y_n - 80405 y_{n+1} + 383405 y_{n+2} - 1310518 y_{n+3} + 5506490 y_{n+4} + 6758059 y_{n+5}) - \frac{76780}{1251693} h^2 g_{n+5} + \frac{139480}{11265237} h^3 T_{n+5}, C_8^{(14/3)} = \frac{-759869}{2838839724},$$
$$y_{n+5} = \frac{1}{1224514529}(1715614 y_n - 16992545 y_{n+1} + 82569320 y_{n+2} - 294115870 y_{n+3} + 1451338010 y_{n+4}) + \frac{150203160}{174930647} hf_{n+\frac{14}{3}} - \frac{44029980}{1224514529} h^2 g_{n+5} + \frac{17372920}{1224514529} h^3 T_{n+5}. \qquad (2.20)$$

The error constant of the output method in (2.20) is $C_8^{(5)} = \frac{-152683921}{308577661308}$. The region of absolute stability of the methods is shown in Figure 3. The method is A-stable.

For $k = 6, v = k - \frac{1}{\tau}, \ \tau = 2, \ p = 8$ in (1.2) and (1.3) the arising methods are

$$y_{n+\frac{11}{2}} = -\frac{42859}{94715904} y_n + \frac{18623}{3946496} y_{n+1} - \frac{734085}{31571968} y_{n+2} + \frac{442079}{5919744} y_{n+3} - \frac{6190635}{31571968} y_{n+4} + \frac{2921391}{3946496} y_{n+5} + \frac{37899323}{94715904} y_{n+6} - \frac{26565}{493312} h^2 g_{n+6} + \frac{44055}{3946496} h^3 T_{n+6}, \quad C_9^{(11/2)} = \frac{-500819}{2652045312},$$
$$y_{n+6} = -\frac{1060769}{8026651577} y_n + \frac{11945952}{8026651577} y_{n+1} - \frac{65765655}{8026651577} y_{n+2} + \frac{251736160}{8026651577} y_{n+3} - \frac{916429455}{8026651577} y_{n+4} + \frac{8746225344}{8026651577} y_{n+5} + \frac{1065553920}{1146664511} hf_{n+\frac{11}{2}} + \frac{64609560}{1146664511} h^2 g_{n+6} - \frac{54562680}{8026651577} h^3 T_{n+6}. \qquad (2.21)$$

The error constant of the output algorithm in (2.21) is $C_9^{(6)} = \frac{-16056623}{449492488312}$. The method (2.21) is A(84^0)-stable. Again, see Figure 1. When $k = 6, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 8$ in (1.2) and (1.3) the method is,

$y_{n+\frac{17}{3}}=\frac{-759869}{2048173614}y_n+\frac{146234}{37929141}y_{n+1}-\frac{1433185}{75858282}y_{n+2}+\frac{61531415}{1024086807}y_{n+3}-\frac{11694793}{75858282}y_{n+4}+\frac{19720646}{37929141}y_{n+5}+\frac{1207514539}{2048173614}y_{n+6}-\frac{6466460}{113787423}h^2g_{n+6}+\frac{407660}{37929141}h^3T_{n+6}$, $C_9^{(17/3)}=\frac{-13529263}{86023291788}$,

$y_{n+6}=\frac{-152683921}{226891772569}y_n+\frac{94049532}{13346574857}y_{n+1}-\frac{7927431255}{226891772569}y_{n+2}+\frac{25733004320}{226891772569}y_{n+3}-\frac{69044565375}{226891772569}y_{n+4}+\frac{276684606756}{226891772569}y_{n+5}+\frac{27308981520}{32413110367}hf_{n+\frac{17}{3}}-\frac{878357160}{32413110367}h^2g_{n+6}+\frac{2486361960}{226891772569}h^3T_{n+6}$. (2.22)

The error constant of the output scheme in (2.21) is $C_9^{(6)}=\frac{-874382049}{3176484815966}$. The coupled method is A(86^0)-stable, see Figure 3.Setting $k=7,\ v=k-\frac{1}{\tau},\ \tau=2,\ p=9$ in (1.2), (1.3) andsolving the expressions in (2.1) and (2.2)yields

$y_{n+\frac{13}{2}}=\frac{1502457}{5949857792}y_n-\frac{201258421}{71398293504}y_{n+1}+\frac{87472385}{5949857792}y_{n+2}-\frac{1149782205}{23799431168}y_{n+3}+\frac{2078677601}{17849573376}y_{n+4}-\frac{5830537713}{23799431168}y_{n+5}+\frac{4614052443}{5949857792}y_{n+6}+\frac{27789473855}{71398293504}y_{n+7}-\frac{147402255}{2974928896}h^2g_{n+7}+\frac{28963935}{2974928896}h^3T_{n+7}$,

$y_{n+7}=\frac{578038428}{12126219836635}y_n-\frac{7015929067}{12126219836635}y_{n+1}+\frac{8172269532}{2425243967327}y_{n+2}-\frac{31303014561}{2425243967327}y_{n+3}+\frac{95031019300}{2425243967327}y_{n+4}-\frac{1493587560213}{12126219836635}y_{n+5}+\frac{13266743916132}{12126219836635}y_{n+6}+\frac{2249046245376}{2425243967327}hf_{n+\frac{13}{2}}+\frac{138339871992}{2425243967327}h^2g_{n+7}-\frac{17105402664}{2425243967327}h^3T_{n+7}$.

(2.23)

The error constants are $C_{10}^{(13/2)}=\frac{-335572523}{2855931740160}$ and$C_{10}^{(7)}=\frac{-2837539213}{242524396732700}$.This method (2.23) is A(77^0)-stable, see Figure 1.Again, when$k=7, v=k-\frac{1}{\tau},\ \tau=3,\ p=9$ in (1.2) and (1.3) gives

$y_{n+\frac{20}{3}}=\frac{27058526}{128662043247}y_n-\frac{2711640470}{1157958389223}y_{n+1}+\frac{521913260}{42887347749}y_{n+2}-\frac{5116101550}{128662043247}y_{n+3}+\frac{109863748225}{1157958389223}y_{n+4}-\frac{8358622118}{42887347749}y_{n+5}+\frac{70671888980}{128662043247}y_{n+6}+\frac{672151807030}{1157958389223}y_{n+7}-\frac{6850782400}{128662043247}h^2g_{n+7}+\frac{15184400}{1588420287}h^3T_{n+7}$, $C_{10}^{(20/3)}=\frac{-689289667}{6947750335338}$,

$y_{n+7}=\frac{7869438441}{21719592858625}y_n-\frac{3525194071}{868783714345}y_{n+1}+\frac{3694015962}{173756742869}y_{n+2}-\frac{12223548351}{173756742869}y_{n+3}+\frac{29813214487}{173756742869}y_{n+4}-\frac{8029955146941}{21719592858625}y_{n+5}+\frac{1086773926266}{868783714345}y_{n+6}+\frac{3602537210916}{4343918571725}hf_{n+\frac{20}{3}}-\frac{3562269984}{173756742869}h^2g_{n+7}+\frac{1518478920}{173756742869}h^3T_{n+7}$, $C_{10}^{(7)}=\frac{-25868854937}{156381068582100}$.

(2.24)

The method (2.24) is A(85^0)-stable; see Figure 3.Fixing$k=8, v=k-\frac{1}{\tau},\ \tau=2,\ p=10$ in (1.2), (1.3) and using the order conditions in (2.1) and (2.2) yields

$y_{n+\frac{15}{2}}=-\frac{1006717569}{6595373170688}y_n+\frac{94223745}{51526352896}y_{n+1}-\frac{4207812245}{412210823168}y_{n+2}+\frac{1829222395}{51526352896}y_{n+3}-\frac{288626770575}{3297686585344}y_{n+4}+\frac{8701812119}{51526352896}y_{n+5}-\frac{122196559485}{412210823168}y_{n+6}+\frac{41659153965}{51526352896}y_{n+7}+\frac{2503698612415}{6595373170688}y_{n+8}-\frac{18991647675}{412210823168}h^2g_{n+8}+\frac{892566675}{103052705792}h^3T_{n+8}$, $C_{11}^{(15/2)}=\frac{-127435867}{1648843292672}$,

$y_{n+8}=-\frac{274600569}{18084244855705}y_n+\frac{105619968}{516692710163}y_{n+1}-\frac{678461456}{516692710163}y_{n+2}+\frac{2808499968}{516692710163}y_{n+3}-\frac{8678206170}{516692710163}y_{n+4}+\frac{114607419392}{2583463550815}y_{n+5}-\frac{66225111888}{516692710163}y_{n+6}+\frac{3965127121152}{3616848971141}y_{n+7}+\frac{478696439808}{516692710163}hf_{n+\frac{15}{2}}+\frac{29623480560}{516692710163}h^2g_{n+8}-\frac{3696027840}{516692710163}h^3T_{n+8}$, $C_{11}^{(8)}=\frac{-316229614}{198926693412755}$.

(2.25)

This method in (2.25) is A(65^0)-stable. The plot is in Figure 1.Setting$k=8, v=k-\frac{1}{\tau},\ \tau=3, p=10$in (1.2), (1.3) and using order conditions in (2.1) and (2.2) gives

$y_{n+\frac{23}{3}}=-\frac{3446448335}{26741422372464}y_n+\frac{2575705462}{1671338898279}y_{n+1}-\frac{129071002445}{15042050084511}y_{n+2}+\frac{49690571260}{1671338898279}y_{n+3}-\frac{974361667825}{13370711186232}y_{n+4}+\frac{2092984296865}{15042050084511}y_{n+5}-\frac{398352310433}{1671338898279}y_{n+6}+\frac{964624753180}{1671338898279}y_{n+7}+\frac{137750141093785}{240672801352176}y_{n+8}-\frac{84088098325}{1671338898279}h^2g_{n+8}+\frac{4820130700}{557112966093}h^3T_{n+8}$,

$y_{n+8}=-\frac{232819694433}{1098067997207105}y_n+\frac{699914929040352}{27451699930177625}y_{n+1}-\frac{15666603163568}{1098067997207105}y_{n+2}+\frac{10951562405952}{219613599441421}y_{n+3}-\frac{27205542449190}{219613599441421}y_{n+4}+\frac{265857370541728}{1098067997207105}y_{n+5}-\frac{11975897181194352}{27451699930177625}y_{n+6}+\frac{1405619239825728}{1098067997207105}y_{n+7}+\frac{4492558958573952}{5490339986035525}hf_{n+\frac{23}{3}}-\frac{3382115596272}{219613599441421}h^2g_{n+8}+\frac{1565766514368}{219613599441421}h^3T_{n+8}$. (2.26)

The error constants are $C_{11}^{(23/3)} = \frac{-5962155289}{90252300507066}$ and $C_{11}^{(8)} = \frac{-19102209856648}{181181219539172325}$.The method is A(81^0)-stable. See Figure 3. For methods with $k = 9, v = k - \frac{1}{\tau}, \ \tau = 2, \ p = 11$ in (1.2), (1.3) and using order conditions in (2.1) and (2.2) gives

$$y_{n+\frac{17}{2}} = \frac{637179335}{6498871934976} y_n - \frac{14552670327}{11553550106624} y_{n+1} + \frac{1362188835}{180524220416} y_{n+2} - \frac{60840114335}{2166290644992} y_{n+3} + \frac{26453382865}{361048440832} y_{n+4} - \frac{835038697545}{5776775053312} y_{n+5} + \frac{125941565477}{541572661248} y_{n+6} - \frac{252931884465}{722096881664} y_{n+7} + \frac{606309565095}{722096881664} y_{n+8} + \frac{38583390550705}{103981950959616} y_{n+9} - \frac{31189956525}{722096881664} h^2 g_{n+9} + \frac{1411908225}{180524220416} h^3 T_{n+9},$$

$$C_{12}^{(17/2)} = \frac{-20250705371}{381267153518592}.$$

$$y_{n+9} = \frac{78424944272}{38896431312622499} y_n - \frac{7379769533391}{194482156563112495} y_{n+1} + \frac{1788953812416}{5556633044660357} y_{n+2} - \frac{9337855353648}{5556633044660357} y_{n+3} + \frac{34513641944352}{5556633044660357} y_{n+4} - \frac{99827496798966}{5556633044660357} y_{n+5} + \frac{1268674059799488}{27783165223301785} y_{n+6} - \frac{5027584059168912}{38896431312622499} y_{n+7} + \frac{42659308511805456}{38896431312622499} y_{n+8} + \frac{467918779318272}{505148458605487} hf_{n+\frac{17}{2}} + \frac{318850707223440}{5556633044660357} h^2 g_{n+9} - \frac{39836944722240}{5556633044660357} h^3 T_{n+9}, \ C_{12}^{(9)} = \frac{5717864041422}{2139303722194237445}. \qquad (2.27)$$

The composite method in (2.27) is A(55^0)-stable, see Figure 1.For $k = 9, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 11$ in (1.2), (1.3) the method is

$$y_{n+\frac{26}{3}} = \frac{29810776445}{355727039336703} y_n - \frac{37769506645}{35133534749304} y_{n+1} + \frac{28228531994}{4391691843663} y_{n+2} - \frac{2829314281430}{118575679778901} y_{n+3} + \frac{272338956370}{4391691843663} y_{n+4} - \frac{2136399056155}{17566767374652} y_{n+5} + \frac{22951607843255}{118575679778901} y_{n+6} - \frac{1248802531106}{4391691843663} y_{n+7} + \frac{2651621949665}{4391691843663} y_{n+8} + \frac{1608004044447515}{2845816314693624} y_{n+9} - \frac{210146723150}{4391691843663} h^2 g_{n+9} + \frac{104529861800}{13175075530989} h^3 T_{n+9},$$

$$y_{n+9} = \frac{19102209856648}{144491144554368205} y_n - \frac{49190105095209}{28898228910873641} y_{n+1} + \frac{36940508595228528}{361227861385920512 5} y_{n+2} - \frac{5520744779554704}{144491144554368205} y_{n+3} + \frac{2896117699441680}{28898228910873641} y_{n+4} - \frac{28803611713173318}{144491144554368205} y_{n+5} + \frac{9396814116821712}{28898228910873641} y_{n+6} - \frac{18198944186854085 28}{3612278613859205125} y_{n+7} + \frac{188895846685005864}{144491144554368205} y_{n+8} + \frac{53121904540947648}{65677792979258275} hf_{n+\frac{26}{3}} - \frac{327720346229808}{28898228910873641} h^2 g_{n+9} + \frac{170921171300544}{28898228910873641} h^3 T_{n+9}. \qquad (2.28)$$

Here$C_{12}^{(26/3)} = \frac{-718478141173}{15651989730814932}$ and $C_{12}^{(9)} = \frac{-559354624522098}{7947012950490251275}$. The algorithm is A(77^0)-stable as shown in Figure 3. For $k = 10, v = k - \frac{1}{\tau}, \ \tau = 2, \ p = 12$, no stable possess is found, see Figure 2. When$k = 10, \ v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 12$ in (1.2) and (1.3) we have

$$y_{n+\frac{29}{3}} = -\frac{718478141173}{12585453592623300} y_n + \frac{887786267965}{1132690823336097} y_{n+1} - \frac{1124843173415}{223741397202192} y_{n+2} + \frac{840734940058}{41951511975411} y_{n+3} - \frac{42135597313355}{755127215557398} y_{n+4} + \frac{40561718562586}{349595933128425} y_{n+5} - \frac{63647041420235}{335612095803288} y_{n+6} + \frac{97704250988305}{377563607778699} y_{n+7} - \frac{18615838843121}{55935349300548} y_{n+8} + \frac{79204023897505}{125854535926233} y_{n+9} + \frac{252990534574388897}{453076329334438800} y_{n+10} - \frac{639739157057}{13983837325137} h^2 g_{n+10} + \frac{308337118180}{41951511975411} h^3 T_{n+10}, \quad C_{13}^{(29/3)} = \frac{-1645451762057}{49838396226788268},$$

$$y_{n+10} = -\frac{3356127747132588}{387315952238710207 13} y_n + \frac{462197750819900000}{38731595223871020713} y_{n+1} - \frac{2976184104190076 25}{38731595223871020713} y_{n+2} + \frac{11923672056835262 40}{38731595223871020713} y_{n+3} - \frac{3342532870868667000}{38731595223871020713} y_{n+4} + \frac{7017682771767550176}{38731595223871020713} y_{n+5} - \frac{11642254656774288750}{38731595223871020713} y_{n+6} + \frac{16300946543162532000}{38731595223871020713} y_{n+7} - \frac{22164660832476103740}{38731595223871020713} y_{n+8} + \frac{51624801826460622000}{38731595223871020713} y_{n+9} + \frac{2819141604747619200}{3521054111261001883} hf_{n+\frac{29}{3}} - \frac{28290342575854800}{3521054111261001883} h^2 g_{n+10} + \frac{192430218935304000}{38731595223871020713} h^3 T_{n+10}, \quad C_{13}^{(10)} = \frac{-810615882671348900}{166158543510406678858 77}. \qquad (2.29)$$

The scheme in (2.29) is A(68^0)-stable, see Figure 3. For $k = 11, v = k - \frac{1}{\tau}, \ \tau = 3, \ p = 13$, we obtain

$$y_{n+\frac{32}{3}} = \frac{1645451762057}{40752269343029133} y_n - \frac{602717587517848}{1018806733575728325} y_{n+1} + \frac{1489532068130480}{366770424087262197} y_{n+2} - \frac{235916021005610}{13584089781009711} y_{n+3} + \frac{705346819603888}{13584089781009711} y_{n+4} - \frac{14140906699847312}{122256808029087399} y_{n+5} + \frac{68068985879995232}{339602244525242775} y_{n+6} - \frac{3815105610452140}{13584089781009711} y_{n+7} + \frac{41004570948267740}{122256808029087399} y_{n+8} - \frac{15632555765942456}{40752269343029133} y_{n+9} + \frac{26649475851697072}{40752269343029133} y_{n+10} + \frac{50639625734346556 18}{9169260602181554925} y_{n+11} - \frac{198892667824928}{4528029927003237} h^2 g_{n+11} + \frac{93245165256320}{13584089781009711} h^3 T_{n+11}, C_{14}^{(32/3)} = \frac{-815471527547108}{33376108591940859927},$$

$$y_{n+11} = \frac{145910858880842802}{246765817724556150 8845} y_n - \frac{428691571531305588}{4935316354491123017 69} y_{n+1} + \frac{2952419243774943272}{4935316354491123017 69} y_{n+2} - \frac{12676846315333179177}{4935316354491123017 69} y_{n+3} + \frac{13607681540140037671 2}{17626129837468296491 75} y_{n+4} - \frac{61056229567858964472}{35252259674936592983 5} y_{n+5} + \frac{21375823215365022096}{70504519349873185967} y_{n+6} - \frac{21294159975138926878 2}{49353163544911230176 9} y_{n+7} + \frac{26123214361159081894 2}{49353163544911230176 9} y_{n+8} - \frac{79152909046440276084 84}{12338290886227807544225} y_{n+9} + \frac{33486155741052025949 52}{24676581772455615088 45} y_{n+10} + \frac{21517198213119382224}{27117122826874302295} hf_{n+\frac{32}{3}} - \frac{373114872366881040}{70504519349873185967} h^2 g_{n+11} + \frac{297021600901333440}{70504519349873185967} h^3 T_{n+11}, C_{14}^{(11)} = \frac{-7832391107405867078}{224556894129346097304895} \qquad (2.30)$$

The algorithm (2.30) is A(54^0)-stable, see Figure 3.Finally, for $k = 12, v = k - \frac{1}{\tau},\ \tau = 3,\ p = 14$:

$$y_{n+\frac{35}{3}} = -\frac{203867881886777}{6921646279577159166}y_n + \frac{58853521011145}{128178634806984429}y_{n+1} - \frac{2155804656362588}{640893174034922145}y_{n+2} + \frac{53278934453082800}{3460823139788579583}y_{n+3} - \frac{4219352999632925}{85452423204656286}y_{n+4} + \frac{5046237185802736}{42726211602328143}y_{n+5} - \frac{252932419303801160}{1153607713262859861}y_{n+6} + \frac{69577713731247392}{213631058011640715}y_{n+7} - \frac{34125898724434475}{85452423204656286}y_{n+8} + \frac{1467417970548583900}{3460823139788579583}y_{n+9} - \frac{55966992955820716}{128178634806984429}y_{n+10} + \frac{86868755195368720}{128178634806984429}y_{n+11} + \frac{18918086665770210373}{34608231397885795830}y_{n+12} - \frac{5424906329402560}{128178634806984429}h^2 g_{n+12} + \frac{827189656400800}{128178634806984429}h^3 T_{n+12}, \quad (2.31)$$

$$y_{n+12} = -\frac{979048888842573338475}{234721637342307633897 4973}y_n + \frac{3121842305399954 9400}{479023749678178844 68877}y_{n+1} - \frac{22933251121138449 7752}{479023749678178844 68877}y_{n+2} + \frac{1053119599977862868000}{479023749678178844 68877}y_{n+3} - \frac{339203437656246940 8375}{479023749678178844 68877}y_{n+4} + \frac{815821018472093779 7280}{479023749678178844 68877}y_{n+5} - \frac{1525744867603136306 2800}{479023749678178844 68877}y_{n+6} + \frac{1122300150556643931381696}{2347216373423076338974973}y_{n+7} - \frac{2854213252084882995 8625}{479023749678178844 68877}y_{n+8} + \frac{3116391760783091206 9000}{479023749678178844 68877}y_{n+9} - \frac{340826062015539795 53880}{479023749678178844 68877}y_{n+10} + \frac{660973767921605382 98400}{479023749678178844 68877}y_{n+11} + \frac{2030349575342633355 3600}{257935865211327070 21703}hf_{n+\frac{35}{3}} - \frac{10986082150758422400}{368479807444752957 4529}h^2 g_{n+12} + \frac{17231827083270861 6000}{479023749678178844 68877}h^3 T_{n+12}.$$

Here $C_{15}^{(35/3)} = \frac{-833197020184268}{44990700817251534579}$ and $C_{15}^{(12)} = \frac{-111573364105092167160}{4359116122071427486667807}$. The scheme in (2.31) is A(43^0)-stable, see Figure 3. When $k = 13, v = k - \frac{1}{\tau},\ \tau = 3,\ p = 15$ instability sets in, see Figure 4.

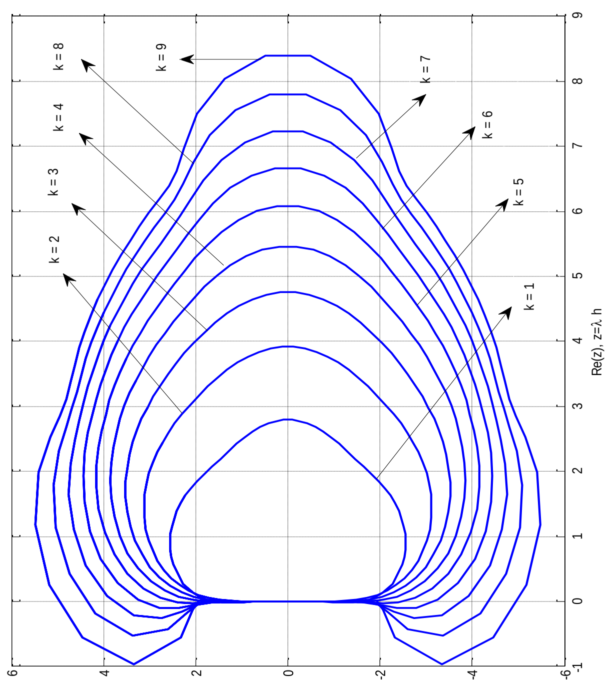

Figure 1: The region of absolutestability (exterior of the closed curve) of the method in (1.2) for step number $k \leq 9,\ v = k - \frac{1}{2}$.

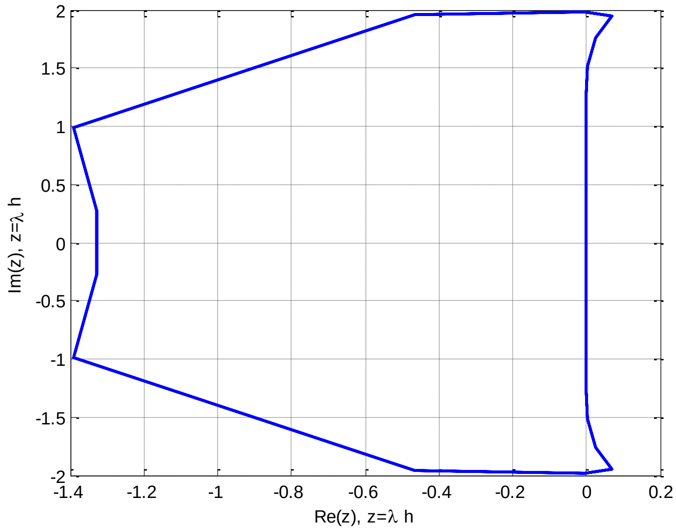

Figure 2: The interval of stability (-1.3, 0) of the method in (1.2) for step number $k = 10, v = k - \frac{1}{2}$.

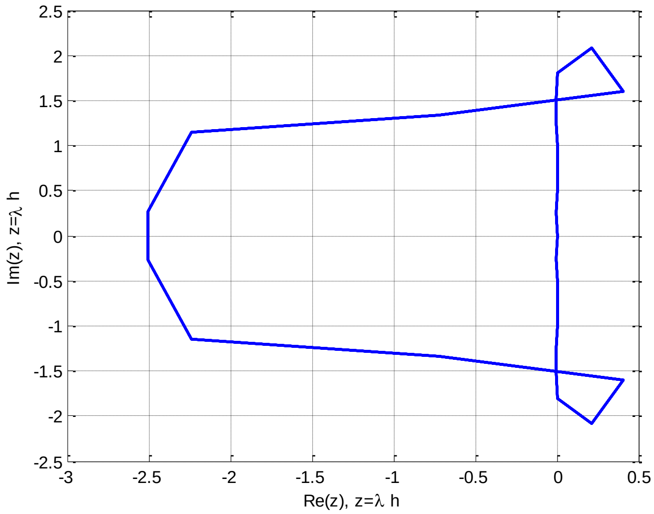

Figure 4: The interval of stability (-2.5, 0) of the method in (1.2) for step number $k = 13, v = k - \frac{1}{3}$.

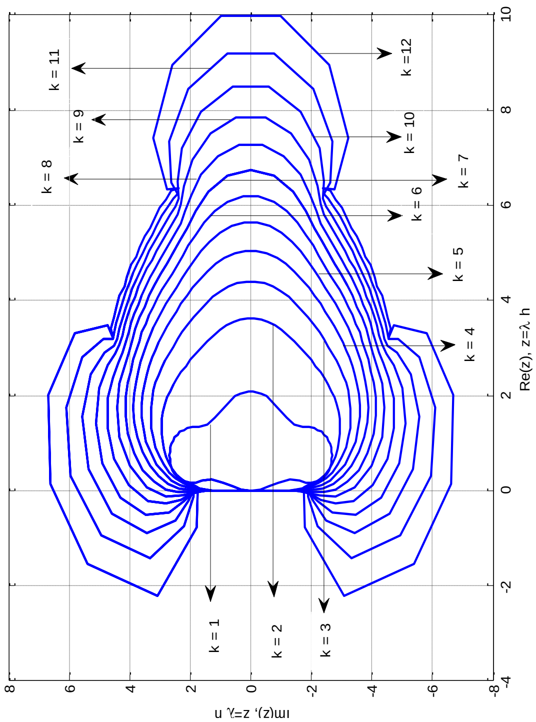

Figure 3: The region of absolutestability (exterior of the closed curve) of the method in (1.2) for step number $k \le 12, v = k - \frac{1}{3}$.

4.0 Application of the Methods and Numerical Experiments

Consider the methods

$$y_{n+\frac{1}{2}} = \frac{1}{2}(y_n + y_{n+1}) - \frac{1}{8}h^2 g_{n+1} + \frac{1}{16}h^3 T_{n+1}, \;\; p = 3, \;\; C_4 = -\frac{7}{384},$$

$$y_{n+1} = y_n + h\, f_{n+\frac{1}{2}} + \frac{1}{24}h^3 T_{n+1}, \qquad p = 3, \qquad C_4 = -\frac{1}{48}. \qquad (3.1)$$

In (2.4) and that in (2.5),

$$y_{n+\frac{2}{3}} = \frac{1}{3}(y_n + 2y_{n+1}) - \frac{1}{9}h^2 g_{n+1} + \frac{4}{81}h^3 T_{n+1}, \;\; p = 3, \qquad C_4 = -\frac{13}{972},$$

$$y_{n+1} = y_n + h\, f_{n+\frac{2}{3}} - \frac{1}{6}h^2 g_{n+1} + \frac{1}{9}h^3 T_{n+1}\,, \qquad p = 3, \qquad C_4 = -\frac{23}{648}. \qquad (3.2)$$

These methods are applied to the following initial value problems:

Problem 1: A stiff system of equations [13]

$$\begin{array}{l} y_1' = -8y_1 + 7y_2, \\ y_2' = 42y_1 - 43y_2 \end{array} \quad y(0) = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \; x \in [0, 15], \; \begin{cases} y_1(x) = 2e^{-x} - e^{-50x}, \\ y_2(x) = 2e^{-x} + 6e^{-50x}. \end{cases}$$

Problem 2: A stiff system of equations [9]

$$\begin{array}{l} y_1' = -0.1y_1, \\ y_2' = -10y_2 \end{array} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; x \in [0, 10], \; \begin{cases} y_1(x) = e^{-0.1x}, \\ y_2(x) = e^{-10x}. \end{cases}$$

For problems 1-2, the fixed step size usedis $h = 0.0001$. The implicitness that arises in methods (3.1) and (3.2) when applied to the stiff IVPs 1, 2 is resolved using the Newton Raphson iterative scheme,

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - (G'(y_{n+k}^{[s]}))^{-1} G(y_{n+k}^{[s]}), \quad s = 0, 1, \ldots \qquad (3.3)$$

where $G'(y_{n+k}^{[s]})$ is the Jacobian matrix. For (3.1)

$$G(y_{n+1}^{[s]}) = \; y_{n+1}^{[s]} - y_n - hf\left(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}\right) - \frac{h^3}{24} T\left(x_{n+1}, y_{n+1}^{[s]}\right),$$

with the hybrid predictor,

$$y_{n+\frac{1}{2}}=\frac{1}{2}\left(y_n+y_{n+1}^{[s]}\right)-\frac{h^2}{8}g\left(x_{n+1},y_{n+1}^{[s]}\right)+\frac{h^3}{16}T\left(x_{n+1},y_{n+1}^{[s]}\right).$$

and for the method in (3.2)

$$G(y_{n+1}^{[s]})= \quad y_{n+1}^{[s]}-y_n-hf\left(x_{n+\frac{2}{3}},y_{n+\frac{2}{3}}\right)-\frac{h^2}{6}g\left(x_{n+1},y_{n+1}^{[s]}\right)+\frac{h^3}{9}T\left(x_{n+1},y_{n+1}^{[s]}\right),$$

$$y_{n+\frac{2}{3}}=\frac{1}{3}\left(y_n+2y_{n+1}^{[s]}\right)-\frac{h^2}{9}g\left(x_{n+1},y_{n+1}^{[s]}\right)+\frac{4h^3}{81}T\left(x_{n+1},y_{n+1}^{[s]}\right).$$

The starting value $y_{n+1}^{[0]}$ for (3.3) is obtained from an explicit one-step formula:

$$y_{n+1}^{[0]}=y_n+\frac{h}{2}\left(f_{n-1}+f_n\right).$$

Table 1-2 gives the numerical solution of our experiment of the MTDBDF (3.1) and (3.2).

Table 1: Results for **Problem** 1

x	y_i	*MTDBDF*($v=k-\frac{1}{2}$) (3.1) y_n	*MTDBDF*($v=k-\frac{1}{3}$) (3.2) y_n	*Exact solution* $y(x_n)$
5.0	y_1	1.34772416549482(-2)	1.347724165494748(-2)	1.347724165495247(-2)
	y_2	1.34772416549482(-2)	1.347724165494748(-2)	1.347724165495247(-2)
10.0	y_1	9.080893996488039(-5)	9.080893996486734(-5)	9.080893996493662(-5)
	y_2	9.080893996488039(-5)	9.080893996486735(-5)	9.080893996493662(-5)
15.0	y_1	6.118658245263073(-7)	6.118658245261824(-7)	6.118658245268770(-7)
	y_2	6.118658245263075(-7)	6.118658245261824(-7)	6.118658245268770(-7)

$a(-q)=a(10)^{-q}$

Continuation of **Table 1**

x	*Error* (3.1) $\|y(x_n)-y_n\|_\infty$	*Error* (3.2) $\|y(x_n)-y_n\|_\infty$
5.0	4.2292(-15)	4.9890(-15)
10.0	5.6229(-17)	6.9280(-17)
15.0	5.6962(-19)	6.9456(-19)

Table 2: Results for **Problem 2**

x	y_i	*MTDBDF*($v=k-\frac{1}{2}$) (3.1) y_n	*MTDBDF*($v=k-\frac{1}{3}$) (3.2) y_n	*Exact solution* $y(x_n)$
5	y_1	6.06536725050820(-1)	6.06536725048560(-1)	6.06536725049557(-1)
	y_2	1.9306795564678(-22)	1.93067955519761(-22)	1.9306795625083(-22)
10	y_1	3.678831199857768(-1)	3.678831199827841(-1)	3.678831199842481(-1)
	y_2	3.723797889353580(-44)	3.723797884453487(-44)	3.723797912655030(-44)
15	y_1	2.231323914625826(-1)	2.231323914596593(-1)	2.231323914611878(-1)
	y_2	7.182274590467591(-66)	7.182274576291025(-66)	7.182274657881744(-66)

Continuation of **Table 2**

x	*Error* (3.1) $\|y(x_n)-y_n\|_\infty$	*Error* (3.2) $\|y(x_n)-y_n\|_\infty$
5.0	1.2632(-12)	9.9653(-13)
10.0	1.5286(-12)	1.4639(-12)
15.0	1.3948(-12)	1.5284(-12)

The numerical results in Tables 1, 2 shows that theMTDBDF (1.2)for $k=1,\ v=k-\frac{1}{2}$ and $k=1,\ v=k-\frac{1}{3}$ are of comparable accuracy andperformed respectively well on the solved stiff ODEs.

Finally, this paper considers a modifiedTDBDF for the numerical solution of stiff systems. Two cases have been investigated by varying the off-step point*v* parameter. For case 1 ($v=k-\frac{1}{2}$), the new methods are A(α)-stable for $k\leq 9$ and unstable when $k\geq 10$, while in case 2 ($v=k-\frac{1}{3}$) they are A(α)-stable for $k\leq 12$ and unstable when $k\geq 13$.

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