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Extrapolation-Based Implicit-Explicit Second Derivative Linear Multistep Methods

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Abstract

Reducing a stiff ordinary differential equations into a sum of two or more parts that are simple to integrate than the original has given rise to several implicit-explicit (IMEX)numerical methods, which are now emerging techniques for the numerical solutions of stiff ordinary differential equations. Some of these differential equations which are amenable to splitting arise from discretization in space of partial differential equations by method of lines. In this paper, we propose to present a variable order extrapolation-based implicit-explicit-explicit second derivative linear multistep methods.

Mathematics subject classification: 65L05, 65L06.
Keywords: IM(EX)^2 SDLMM, stability, Dahlquist test problem, Prothero-Robinson test problem, stiff, additively separable ODEs, splitting.

1.0 Introduction

Some problems in chemical reaction models gives rise to systems of stiff ordinary differential equations (ODEs). Large systems of stiff ODEs also arise from discretization in space of partial differential equations (PDEs) by method of lines (MOL). For such systems there is often natural splitting of the right hand side of the differential system into two or more parts, such system can be written in the general form

$$y^{/}(t)=\sum_{j=1}^{S}F_j(t,y(t)),\ y(t_0)=y_0,\ t\in[t_0,T] \tag{1}$$

where each of the $F_j(t,y(t));\quad j=1(1)s$ may represent a process in the model which will be resolved with its respective methods. Such methods can be written in its general form as

$$\text{IM(EX)}^{\tau};\ \tau=0,1,2,... \tag{2}$$

where τ is the number of application of extrapolation process on the implicit method. However, the interestwill be in when $s=2$ in (1). In particular, consider the three term additive splitting,

$$y^{/}(t)=f(t,y(t))+g(t,y(t))+m(t,y(t)),\ y(t_0)=y_0,\ t\in[t_0,T] \tag{3}$$

where $f(t,y(t)),\quad g(t,y(t))$ represents the non-stiff processes and suitable for explicit time integrationand $m(t,y(t))$ represents the stiff process and suitable for implicit time integration. The implicit-explicit-explicit (IM(EX)^2) integration approach discretizes the non-stiff parts $f(t,y(t)),\quad g(t,y(t))$ with explicit methods, and the stiff part $m(t,y(t))$ with an implicit stable method. The multiple splitting seeks to reduce the effect of stiffness of the ODE on the method. This strategy ensures that the computational cost of resolving the implicitness is greatly reduced as compared with applying the implicit method directly on the ODE. This is the computational advantage of IM(EX)$^{\tau}$ SDLMM. The IMEX linear multistep method was introduced by Crouzeix [1] and Varah [2] and further analysis of its stability is by Frank et al [3].

In this paper, the purpose is to develop an extrapolated-based IM(EX)^2 linear multistep method based on the second derivative backward differentiation formulas (SDBDF). The first objective is to develop IM(EX)^2 SDLMM based on the second derivative backward differentiation formula (SDBDF) up to the ninth order. The second objective is to analyze the

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basic properties of the methods in terms of its order, zero stability and region of absolute stability. The last objective is to numerically validate the proposed $IM(EX)^2$ second derivative schemes.
The organization of this paper is as follows. In section 2, a variety of k-step extrapolation-based $IM(EX)^2$ SDLMM are derived. The stability of the $IM(EX)^2$ SDLMM are analyzed and discussed in section 3. Section 4 presents the numerical experiments on the Prothero-Robinson problem amongst others and Section 5, the concluding remarks.

2.0 The $IM(EX)^2$ SDLMM

Consider the additive splitting of the ODEs (1) into three parts in (3). Given the SDLMM

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\sum_{j=0}^{k}\lambda_j f'_{n+j} \tag{4}$$

we shall derive an $IM(EX)^2$ SDLMM based on the SDBDF

$$\frac{h^2}{2} f'_{n+1} = \left(\sum_{i=1}^{k}\frac{1}{i}\right)hf_{n+1} - \sum_{i=1}^{k}\left(\sum_{j=1}^{k}\frac{1}{j}\right)\frac{\nabla^j y_{n+1}}{j}; \quad k\le 9, \qquad p = k+1 \tag{5}$$

where

$$\nabla^j y_{n+1} = \sum_{r=1}^{k}\binom{j}{r}(-1)^r\, y_{n+1-r}, \quad k = 1(1)9$$

which are A-stable for k=1(1)3 and A(α)-stable for $k = 4(1)9$. The $IM(EX)^2$ SDLMM to be derived are such that the explicit parts are obtained by extrapolation of the implicit terms $f_{n+k},\, f'_{n+k}$ of the SDLMM (5).

2.1 Extrapolation-based $IM(EX)^2$ SDLMM

The fully implicit second derivative linear k-step method is

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j\left(f(t_{n+j},y_{n+j}) + g(t_{n+j},y_{n+j}) + m(t_{n+j},y_{n+j})\right)$$

$$+h^2\sum_{j=0}^{k}\lambda_j\left(f'(t_{n+j},y_{n+j}) + g'(t_{n+j},y_{n+j}) + m'(t_{n+j},y_{n+j})\right); \qquad \alpha_k = 1 \tag{6}$$

with respect to the additive splitting in (3). An $IM(EX)^2$ SDLMM can be derived by reducing $f(t_{n+k},y_{n+k})$, $f'(t_{n+k},y_{n+k})$, and $g(t_{n+k},y_{n+k})$, $g(t_{n+k},y_{n+k})$ through extrapolation as follows,

$$\Phi(t_{n+k}) = \sum_{j=0}^{k-1}\gamma_j\Phi(t_{n+j}) + O(h^q); \quad \Phi(t) = f(t,y(t))$$

$$\Phi'(t_{n+k}) = \sum_{j=0}^{k-1}\gamma_j\Phi'(t_{n+j}) + O(h^q); \quad \Phi'(t) = f'(t,y(t)) \tag{7}$$

and

$$\psi(t_{n+k}) = \sum_{j=0}^{k-1}\gamma_j\psi(t_{n+j}) + O(h^q); \quad \psi(t) = g(t,y(t))$$

$$\psi'(t_{n+k}) = \sum_{j=0}^{k-1}\gamma_j\psi'(t_{n+j}) + O(h^q); \quad \psi'(t) = g'(t,y(t)) \tag{8}$$

This lead to the k-step $IM(EX)^2$SDLMM

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k-1}\beta^*_j f(t_{n+j},y_{n+j}) + h^2\sum_{j=0}^{k-1}\lambda^*_j f'(t_{n+j},y_{n+j}) + h\sum_{j=0}^{k-1}\beta^*_j g(t_{n+j},y_{n+j})$$

$$+h^2\sum_{j=0}^{k-1}\lambda^*_j g'(t_{n+j},y_{n+j}) + h\sum_{j=0}^{k}\beta_j m(t_{n+j},y_{n+j}) + h^2\sum_{j=0}^{k}\lambda_j m'(t_{n+j},y_{n+j}) \tag{9}$$

with

$$\beta^*{}_j = \beta_j + \beta_k \gamma_j; \quad \lambda^*{}_j = \lambda_j + \lambda_k \gamma_j$$

Theorem

Assume the implicit SDLMM (6) has order r and the extrapolation procedures (7) and (8) has order q. Then the IM(EX)^2 SDLMM (9) has order p=min(r,q) as $h \to 0, \quad \lambda h \to -\infty,$ and $\lambda h \in R_{AS}$. Here R_{AS} is the region of absolute stability of the SDLMM (6)

2.2 Derivation of the IM(EX)^2 SDLMM (9) based on the SDBDF (5)

As an application of this theorem, the derivation of the IM(EX)^2 SDLMM based on the SDBDF (5) for the case of k=1,2 appears trivial, but consider for example when k=3, the SDBDF for k=3 is

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{66}{85} hg_{n+3} - \frac{18}{85} h^2 g'_{n+3}, \qquad r = 4 \tag{10}$$

According to (6), we have

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{66}{85} h(f_{n+3} + g_{n+3} + m_{n+3}) - \frac{18}{85} h^2 (f'_{n+3} + g'_{n+3} + m'_{n+3})$$

and the explicit term for $f_{n+3}, \ f'_{n+3}, \ g_{n+3}, \ g'_{n+3}$ are obtained from the extrapolation procedure (7). The extrapolated explicit methods of (10) becomes

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{198}{85} hf_{n+2} - \frac{198}{85} hf_{n+1} + \frac{66}{85} hf_n - \frac{54}{85} h^2 f'_{n+2} + \frac{54}{85} h^2 f'_{n+1} - \frac{18}{85} h^2 f'_n$$

$$+ \frac{198}{85} hg_{n+2} - \frac{198}{85} hg_{n+1} + \frac{66}{85} hg_n - \frac{54}{85} h^2 g'_{n+2} + \frac{54}{85} h^2 g'_{n+1} - \frac{18}{85} h^2 g'_n,$$

$q = 3$

The IM(EX)^2 method from SDBDF (5) for k=3 according to (9) becomes

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = \frac{198}{85} hf_{n+2} - \frac{198}{85} hf_{n+1} + \frac{66}{85} hf_n - \frac{54}{85} h^2 f'_{n+2} +$$

$$\frac{54}{85} h^2 f'_{n+1} - \frac{18}{85} h^2 f'_n + \frac{198}{85} hg_{n+2} - \frac{198}{85} hg_{n+1} + \frac{66}{85} hg_n - \frac{54}{85} h^2 g'_{n+2} +$$

$$\frac{54}{85} h^2 g'_{n+1} - \frac{18}{85} h^2 g'_n + \frac{66}{85} hm_{n+3} - \frac{18}{85} h^2 m'_{n+3}, \qquad p = 3$$

The IM(EX)^2 SDBDF for k=1(1)9based on the SDBDF (5) are obtained using (6) and following the extrapolation process in (7). The resultant IM(EX)^2 SDBDF methods are now listed below.They will be referred to as IM(EX)^2 SDBDFk subsequently. The k indicates the step number of the SDBDF in (5) from which it was obtained.

IM(EX)^2 SDBDF1, p=1

$$y_{n+1} - y_n = h\left(f_n + g_n + m_{n+1}\right) + h^2\left(-\frac{1}{2} f'_n - \frac{1}{2} g'_n - \frac{1}{2} m'_{n+1}\right) \tag{11}$$

IM(EX)^2 SDBDF2,p=2

$$y_{n+2} - \frac{8}{7} y_{n+1} + \frac{1}{7} y_n = h\left(\frac{12}{7} f_{n+1} - \frac{6}{7} f_n + \frac{12}{7} g_{n+1} - \frac{6}{7} g_n + \frac{6}{7} m_{n+2}\right) + h^2\left(-\frac{4}{7} f'_{n+1} + \frac{2}{7} f'_n - \frac{4}{7} g'_{n+1}\right.$$

$$\left. + \frac{2}{7} g'_n - \frac{2}{7} m'_{n+2}\right), \tag{12}$$

IM(EX)^2 SDBDF3, p=3

$$y_{n+3} - \frac{108}{85} y_{n+2} + \frac{27}{85} y_{n+1} - \frac{4}{85} y_n = h\left(\frac{198}{85} f_{n+2} - \frac{198}{85} f_{n+1} + \frac{66}{85} f_n + \frac{198}{85} g_{n+2} - \frac{198}{85} g_{n+1} + \frac{66}{85} g_n\right.$$

$$\left. + \frac{66}{85} m_{n+3}\right) + h^2\left(-\frac{54}{85} f'_{n+2} + \frac{54}{85} f'_{n+1} - \frac{18}{85} f'_n - \frac{54}{85} g'_{n+2} + \frac{54}{85} g'_{n+1} - \frac{18}{85} g'_n - \frac{18}{85} m'_{n+3}\right) \tag{13}$$

IM(EX)^2 SDBDF4,p=4

$$y_{n+4}-\frac{576}{415}y_{n+3}+\frac{216}{415}y_{n+2}-\frac{64}{415}y_{n+1}+\frac{9}{415}y_n=h\left(\frac{240}{83}f_{n+3}-\frac{360}{83}f_{n+2}+\frac{240}{83}f_{n+1}-\frac{60}{83}f_n\right.$$
$$+\frac{240}{83}g_{n+3}-\frac{360}{83}g_{n+2}+\frac{240}{83}g_{n+1}-\frac{60}{83}g_n+\left.\frac{60}{83}m_{n+4}\right)+h^2\left(-\frac{288}{415}f'_{n+3}+\frac{432}{415}f'_{n+2}-\frac{288}{415}f'_{n+1}\right.$$
$$\left.+\frac{72}{415}f'_n-\frac{288}{415}g'_{n+3}+\frac{432}{415}g'_{n+2}-\frac{288}{415}g'_{n+1}+\frac{72}{415}g'_n-\frac{72}{415}m'_{n+4}\right) \quad (14)$$

IM(EX)^2 SDBDF5,p=5

$$y_{n+5}-\frac{18000}{12019}y_{n+4}+\frac{9000}{12019}y_{n+3}-\frac{4000}{12019}y_{n+2}+\frac{1125}{12019}y_{n+1}-\frac{144}{12019}y_n=h\left(\frac{41100}{12019}f_{n+4}-\right.$$
$$\frac{82200}{12019}f_{n+3}+\frac{82200}{12019}f_{n+2}-\frac{41100}{12019}f_{n+1}+\frac{8220}{12019}f_n+\frac{41100}{12019}g_{n+4}-\frac{82200}{12019}g_{n+3}+\frac{82200}{12019}g_{n+2}$$
$$-\frac{41100}{12019}g_{n+1}+\frac{8220}{12019}g_n+\left.\frac{8220}{12019}m_{n+5}\right)+h^2\left(-\frac{9000}{12019}f'_{n+4}+\frac{18000}{12019}f'_{n+3}-\frac{18000}{12019}f'_{n+2}+\right.$$
$$\frac{9000}{12019}f'_{n+1}-\frac{1800}{12019}f'_n-\frac{9000}{12019}g'_{n+4}+\frac{18000}{12019}g'_{n+3}-\frac{18000}{12019}g'_{n+2}+\frac{9000}{12019}g'_{n+1}-\frac{1800}{12019}g'_n$$
$$\left.-\frac{1800}{12019}m'_{n+5}\right) \quad (15)$$

IM(EX)^2 SDBDF6,p=6

$$y_{n+6}-\frac{21600}{13489}y_{n+5}+\frac{13500}{13489}y_{n+4}-\frac{8000}{13489}y_{n+3}+\frac{3375}{13489}y_{n+2}-\frac{864}{13489}y_{n+1}+\frac{100}{13489}y_n=$$
$$h\left(\frac{7560}{1927}f_{n+5}-\frac{18900}{1927}f_{n+4}+\frac{25200}{1927}f_{n+3}-\frac{18900}{1927}f_{n+2}+\frac{7560}{1927}f_{n+1}-\frac{1260}{1927}f_n+\frac{7560}{1927}g_{n+5}\right.$$
$$-\frac{18900}{1927}g_{n+4}+\frac{25200}{1927}g_{n+3}-\frac{18900}{1927}g_{n+2}+\frac{7560}{1927}g_{n+1}-\frac{1260}{1927}g_n+\left.\frac{1260}{1927}m_{n+6}\right)+h^2\left(-\frac{10800}{13489}f'_{n+5}\right.$$
$$+\frac{27000}{13489}f'_{n+4}-\frac{36000}{13489}f'_{n+3}+\frac{27000}{13489}f'_{n+2}-\frac{10800}{13489}f'_{n+1}+\frac{1800}{13489}f'_n-\frac{10800}{13489}g'_{n+5}+\frac{27000}{13489}g'_{n+4}$$
$$\left.-\frac{36000}{13489}g'_{n+3}+\frac{27000}{13489}g'_{n+2}-\frac{10800}{13489}g'_{n+1}+\frac{1800}{13489}g'_n-\frac{1800}{13489}m'_{n+6}\right) \quad (16)$$

IM(EX)^{2}SDBDF7, p=7

$$y_{n+7}-\frac{1234800}{726301}y_{n+6}+\frac{926100}{726301}y_{n+5}-\frac{686000}{726301}y_{n+4}+\frac{385875}{726301}y_{n+3}-\frac{148176}{726301}y_{n+2}+\frac{34300}{726301}y_{n+1}$$
$$-\frac{3600}{726301}y_n=h\left(\frac{3201660}{726301}f_{n+6}-\frac{9604980}{726301}f_{n+5}+\frac{16008300}{726301}f_{n+4}-\frac{16008300}{726301}f_{n+3}+\frac{9604980}{726301}f_{n+2}\right.$$
$$-\frac{3201660}{726301}f_{n+1}+\frac{457380}{726301}f_n+\frac{3201660}{726301}g_{n+6}-\frac{9604980}{726301}g_{n+5}+\frac{16008300}{726301}g_{n+4}-\frac{16008300}{726301}g_{n+3}+\frac{9604980}{726301}g_{n+2}$$
$$-\frac{3201660}{726301}g_{n+1}+\frac{457380}{726301}g_n+\left.\frac{457380}{726301}m_{n+7}\right)+h^2\left(-\frac{617400}{726301}f'_{6+n}+\frac{1852200}{726301}f'_{5+n}-\frac{3087000}{726301}f'_{4+n}\right.$$
$$+\frac{3087000}{726301}f'_{n+3}-\frac{1852200}{726301}f'_{n+2}+\frac{617400}{726301}f'_{n+1}-\frac{88200}{726301}f'_n-\frac{617400}{726301}g'_{n+6}+\frac{1852200}{726301}g'_{n+5}-$$
$$\left.\frac{3087000}{726301}g'_{n+4}+\frac{3087000}{726301}g'_{n+3}-\frac{1852200}{726301}g'_{n+2}+\frac{617400}{726301}g'_{n+1}-\frac{88200}{726301}g'_n-\frac{88200}{726301}m'_{n+7}\right) \quad (17)$$

IM(EX)^2 SDBDF8,p=8

$$y_{n+8}-\frac{5644800}{3144919}y_{n+7}+\frac{4939200}{3144919}y_{n+6}-\frac{4390400}{3144919}y_{n+5}+\frac{3087000}{3144919}y_{n+4}-\frac{1580544}{3144919}y_{n+3}+\frac{548800}{3144919}y_{n+2}$$
$$-\frac{115200}{3144919}y_{n+1}+\frac{11025}{3144919}y_n=h\left(\frac{15341760}{3144919}f_{n+7}-\frac{53696160}{3144919}f_{n+6}+\frac{107392320}{3144919}f_{n+5}-\frac{134240400}{3144919}f_{n+4}\right.$$
$$+\frac{107392320}{3144919}f_{n+3}-\frac{53696160}{3144919}f_{n+2}+\frac{15341760}{3144919}f_{n+1}-\frac{1917720}{3144919}f_n+\frac{15341760}{3144919}g_{n+7}-\frac{53696160}{3144919}g_{n+6}+$$
$$\frac{107392320}{3144919}g_{n+5}-\frac{134240400}{3144919}g_{n+4}+\frac{107392320}{3144919}g_{n+3}-\frac{53696160}{3144919}g_{n+2}+\frac{15341760}{3144919}g_{n+1}-\frac{1917720}{3144919}g_n$$
$$\left.+\frac{1917720}{3144919}m_{n+8}\right)+h^2\left(-\frac{2822400}{3144919}f'_{n+7}+\frac{9878400}{3144919}f'_{n+6}-\frac{19756800}{3144919}f'_{n+5}+\frac{24696000}{3144919}f'_{n+4}-\right.$$
$$\frac{19756800}{3144919}f'_{n+3}+\frac{9878400}{3144919}f'_{n+2}-\frac{2822400}{3144919}f'_{n+1}+\frac{352800}{3144919}f'_n-\frac{2822400}{3144919}g'_{n+7}+\frac{9878400}{3144919}g'_{n+6}$$
$$-\frac{19756800}{3144919}g'_{n+5}+\frac{24696000}{3144919}g'_{n+4}-\frac{19756800}{3144919}g'_{n+3}+\frac{9878400}{3144919}g'_{n+2}-\frac{2822400}{3144919}g'_{n+1}+\frac{352800}{3144919}g'_n$$
$$\left.-\frac{352800}{3144919}m'_{n+8}\right) \qquad (18)$$

IM(EX)^2 SDBDF9,p=9

$$y_{n+9}-\frac{57153600}{30300391}y_{n+8}+\frac{57153600}{30300391}y_{n+7}-\frac{59270400}{30300391}y_{n+6}+\frac{50009400}{30300391}y_{n+5}-\frac{32006016}{30300391}y_{n+4}$$
$$+\frac{14817600}{30300391}y_{n+3}-\frac{4665125}{30300391}y_{n+2}+\frac{893025}{30300391}y_{n+1}-\frac{78400}{30300391}y_n=h\left(\frac{161685720}{30300391}f_{n+8}-\right.$$
$$\frac{646742880}{30300391}f_{n+7}+\frac{1509066720}{30300391}f_{n+6}-\frac{2263600080}{30300391}f_{n+5}+\frac{2263600080}{30300391}f_{n+4}-\frac{1509066720}{30300391}f_{n+3}$$
$$+\frac{646742880}{30300391}f_{n+2}-\frac{161685720}{30300391}f_{n+1}+\frac{17965080}{30300391}f_n+\frac{161685720}{30300391}g_{n+8}-\frac{646742880}{30300391}g_{n+7}+$$
$$\frac{1509066720}{30300391}g_{n+6}-\frac{2263600080}{30300391}g_{n+5}+\frac{2263600080}{30300391}g_{n+4}-\frac{1509066720}{30300391}g_{n+3}+\frac{646742880}{30300391}g_{n+2}$$
$$\left.-\frac{161685720}{30300391}g_{n+1}+\frac{17965080}{30300391}g_n+\frac{17965080}{30300391}m_{n+9}\right)+h^2\left(-\frac{28576800}{30300391}f'_{n+8}+\frac{114307200}{30300391}f'_{n+7}\right.$$
$$-\frac{266716800}{30300391}f'_{n+6}+\frac{400075200}{30300391}f'_{n+5}-\frac{400075200}{30300391}f'_{n+4}+\frac{266716800}{30300391}f'_{n+3}-\frac{114307200}{30300391}f'_{n+2}$$
$$+\frac{28576800}{30300391}f'_{n+1}-\frac{3175200}{30300391}f'_n-\frac{28576800}{30300391}g'_{n+8}+\frac{114307200}{30300391}g'_{n+7}-\frac{266716800}{30300391}g'_{n+6}$$
$$+\frac{400075200}{30300391}g'_{n+5}-\frac{400075200}{30300391}g'_{n+4}+\frac{266716800}{30300391}g'_{n+3}-\frac{114307200}{30300391}g'_{n+2}+\frac{28576800}{30300391}g'_{n+1}$$
$$\left.-\frac{3175200}{30300391}g'_n-\frac{3175200}{30300391}m'_{n+9}\right) \qquad (19)$$

3.0 Stability of the IM(EX)^2 SDLMMin (9)

Consider applying the IM(EX)^2 SDLMM (9) to the ODE (3) where $f(t, y(t)),\ \ g(t, y(t))$ are the non-stiff parts and $m(t, y(t))$ is the stiff part of the system. Following [3] the scalar test problem to determine the stability of IM(EX)^2 methods is $y'(t)=\lambda y(t)+\mu y(t)+\eta y(t),$ To determine the conditions under which λh, μh and ηh lie in the region of stability of their respective methods in other to determine sufficient conditions for the IM(EX)^2 method to be asymptotically stable is very difficult and complicated. More so, the independent stability of the explicit methods and implicit method does not imply the stability of the IM(EX)^2 scheme. Determining the stability of a multiple part splitting for a LMM with step number $k \geq 2$ is even far more difficult. However the approach of Jorgenson [4] is simpler and have been adopted in this work, but with extension to the more generalized Dahlquist test problem,

$$y^{/}(t) = \left(\sum_{j=1}^{s} e_j\right)\lambda y(t) - \lambda \sum_{j=1}^{s-1}\left(e_j y(t)\right) \tag{20}$$

as suggested by the additive splitting (1). In the consideration herein $e_s = v$ will represent the stiff part of the ODE (1), while the rest $\{e_j\}_{j=1(1)s-1}$ are the non-stiff parts. In our case s=3 in (6). We shall use this to study the stability of the IM(EX)^2 methods derived in (11-100), where

$$\begin{cases} \mathrm{Re}(\lambda) < 0, & \lambda \in \not\subset \\ v = e_s > 0, \quad 0 < e_j < v, & e_j \in R; \quad j = 1(1)s \end{cases}$$

In particular when s=3 as in (3),the Cauchy test problem for this is

$$\begin{aligned} y^{/}(t) &= [m]+[f]+[g] = [(w+e+v)\lambda y(t)] + [-e\lambda y(t)] + [-w\lambda y(t)] \\ y^{//}(t) &= [m]+[f]+[g] = \left[[(w+e+v)^2 - 2e(w+e+v) - 2w(w+e+v)]\lambda^2 y(t)\right] \\ &\quad + \left[(ew+e^2)\lambda^2 y(t)\right] + \left[(ew+w^2)\lambda^2 y(t)\right] \\ &\quad y(0)=1, \quad \mathrm{Re}(\lambda)<0, \quad \lambda \in \not\subset; \quad e,w,v>0 \end{aligned} \tag{21}$$

Notice that this reduces to the Dahlquist test problem $y^{/}(t) = v\lambda y(t)$, with exact solution $y(t) = e^{v\lambda t}$. The stability polynomial $\prod(r,z;e,w,v)$ is obtained by applying the IM(EX)^2 method (9) on the generalized Dahlquist test problem (20), thus

$$\begin{aligned} &\prod(r,z;e,w,v) = \\ &\sum_{j=0}^{k}\alpha_j r^j + \left(\frac{ez}{v}\right)\sum_{j=0}^{k-1}\beta_j^* r^j + \frac{z^2}{v^2}(ew+e^2) \\ &\sum_{j=0}^{k-1}\lambda_j^* r^j + \left(\frac{wz}{v}\right)\sum_{j=0}^{k-1}\beta_j^* r^j + \frac{z^2}{v^2}(ew+w^2)\sum_{j=0}^{k-1}\lambda_j^* r^j - (e+w+v)\frac{z}{v}\sum_{j=0}^{k}\beta_j r^j \\ &+[(e+w+v)^2 - 2e(e+w+v) - 2w(e+w+v)]\frac{z^2}{v^2}\sum_{j=0}^{k}\lambda_j r^j \end{aligned} \tag{22}$$

Where $z = h\lambda v$. From this polynomial $\prod(r,z;e,w,v)$, the stability of the IM(EX)^2 SDLMM (9) is therefore parameterized by the stability variables e, w and v. In fact, these variables suggests the separable additive splitting in (3) into a stiff part $m(t, y(t))$ by v and non-stiff parts $f(t, y(t))$ and $g(t, y(t))$ by e and w respectively. If $e \to 0, \quad w \to 0$ and $v \to 1$ then the stability of the IM(EX)^2 SDLMM (9) approaches the stability of the underlying implicit second derivative method (5). It is indeed exactly so when e=0, w=0 and v=1. The A-stability region of a method (9) can be illustrated by plotting its boundary locus curve of the stability polynomial $\prod(r,z;e,w,v)$ with respect to the values of $z = h\lambda v$ corresponding to the boundary roots

$$\prod(r,z;e,w,v) = 0; \; r = e^{i\theta}, \; 0 < \theta \leq 2\pi, \; z = h\lambda v, \; i = \sqrt{-1} \tag{23}$$

This gives the roots of the stability polynomial $\prod(r,z;e,w,v)$ lying within or on the unit circle. The collection of z for which (23) holds defines the stability region of an IM(EX)^2 method in (9). Following Jorgenson [3], the essence here is to explore IM(EX)^2 methods which, when applied to the Dahlquist test problem (21), display stable behavior. We shall be considering IM(EX)^2 methods (9) which apply the implicit scheme to the first part of (21) and its explicit schemes to the second and third part. A priori it is not obvious which mixed method will exhibit stable behavior, and if so, whether the stability properties of the implicit or explicit parts will dominate in the IM(EX)^2 method. Consider the IM(EX)^2 SDBDF (11), to see the stability plot of this IM(EX)^2 SDLMM in terms of step-size and roots of its stability polynomial (22), take $r^n = y_n, z = h\lambda v, z^2 = h^2\lambda^2 v^2$ and applying to the Dahlquist test problem (21). To illustrate the following ideas, take for example the IM(EX)^2 SDLMM,

$$y_{n+1} - y_n = hf_n - \frac{1}{2}h^2 f'_n + hg_n - \frac{1}{2}h^2 g'_n + hm_{n+1} - \frac{1}{2}h^2 m'_{n+1} \qquad (24)$$

of order p=1. This implies,

$$\prod(r,z;e,w,v) = r - 1 - \left(z\left(-\frac{e}{v}\right) - z^2 \frac{1}{2}\left(\frac{ew}{v^2} + \frac{e^2}{v^2}\right) + z\left(-\frac{e}{v}\right) - z^2 \frac{1}{2}\left(\frac{ew}{v^2} + \frac{w^2}{v^2}\right) + z\left(\frac{w+e+v}{v}\right) r \right.$$

$$\left. + z^2\left(-\frac{1}{2}\right)\left[\frac{(w+e+v)^2}{v^2} - \frac{2e(w+e+v)}{v^2} - \frac{2e(w+e+v)}{v^2}\right] r \right)$$

as in (22). So that,

$$r = \frac{1 + z\left(\frac{-e}{v}\right) - \frac{z^2}{2}\left(\frac{ew+e^2}{v^2}\right) + z\left(\frac{-w}{v}\right) - \frac{z^2}{2}\left(\frac{ew+w^2}{v^2}\right)}{1 - z\left(\frac{e+w+v}{v}\right) + \frac{z^2}{2}\left(\frac{(e+w+v)^2}{v^2} - \frac{2e(e+w+v)}{v^2} - \frac{2w(e+w+v)}{v^2}\right)}$$

This $\prod(r,z;e,w,v)$ is the stability polynomial in r plotted for the parameter z according to (23). The stability plot for the method (24) for v=0.1 with eand wallowed to vary is shown in Figs (1.0a,b),

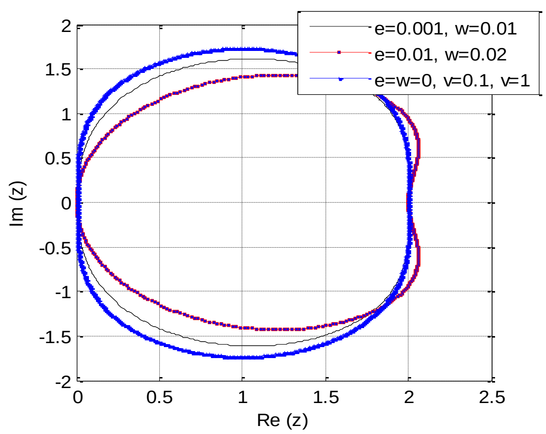


Fig 1.0a

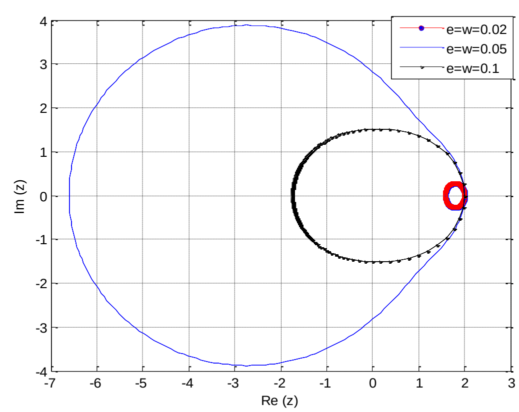
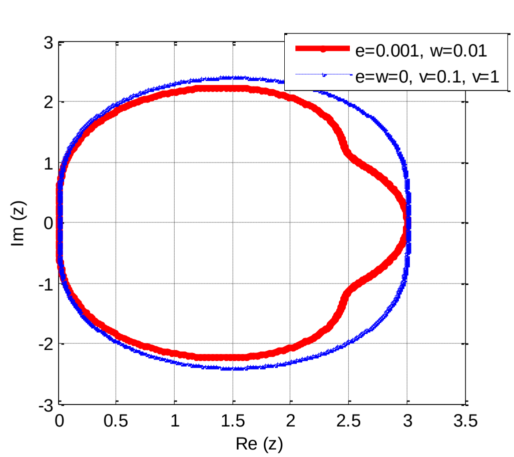


Fig 1.0b

Fig(1.0a,b): Boundary Locus of IM(EX)^2 SDBDF1 with various e and w, v=0.1

The plots in Fig (1.0a,b) shows that the stability region of IM(EX)^2 SDBDF1 is growing with e and w; that is, the region of absolute stability grows as the scaling of the explicit parts of the method approaches that of the implicit. But notice that if e=0, w=0 for v=1, we recover the stability plot of the implicit SDBDF1. The stability plots for the IM(EX)^2 SDBDF from k=2(1)9 are shown in Figs (1.1)-(1.8) with v=0.1 and for various values of e and w, the first graphs (a) will be showing the stability plots for which the IM(EX)^2 SDBDF method is stable and the second graphs (b) will be showing the stability plots for which the IM(EX)^2 SDBDF method is exhibiting instability. It is to be noted that for a stable explicit method the interior of the curve is the stability region and for the implicit method that is A/A(α)-stable, the exterior of the closed boundary curve is the region of stability.

IM(EX)^2 SDBDF2



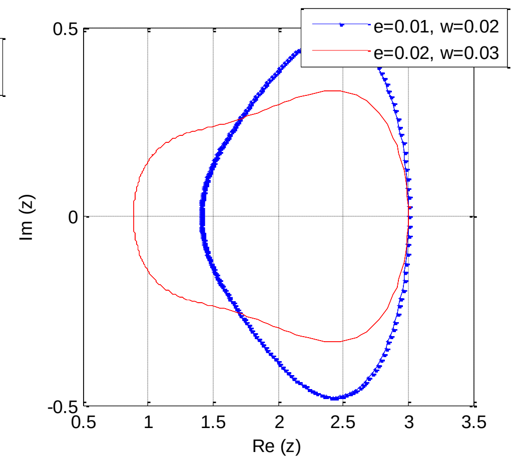
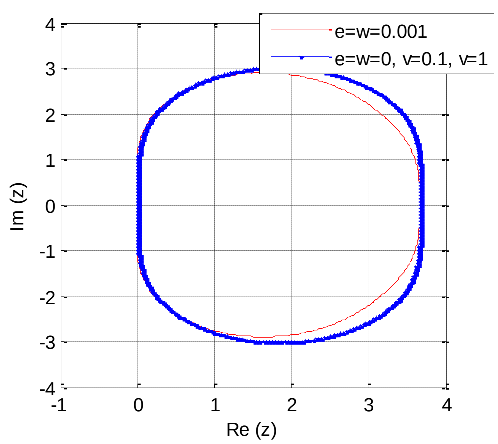


Fig 1.1a **Fig 1.1b**

Fig(1.1a,b): Boundary Locus of IM(EX)^2 SDBDF2 with various e and w, v=0.1

IM(EX)^2 SDBDF3



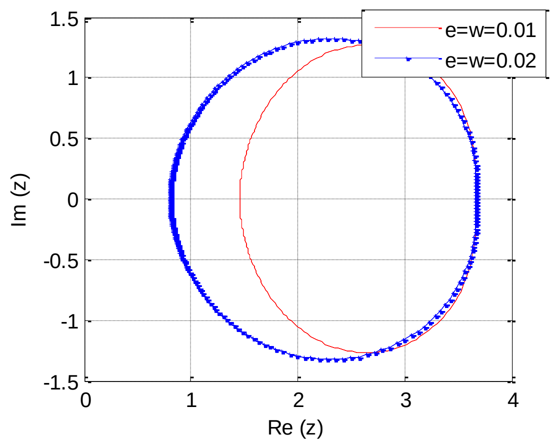
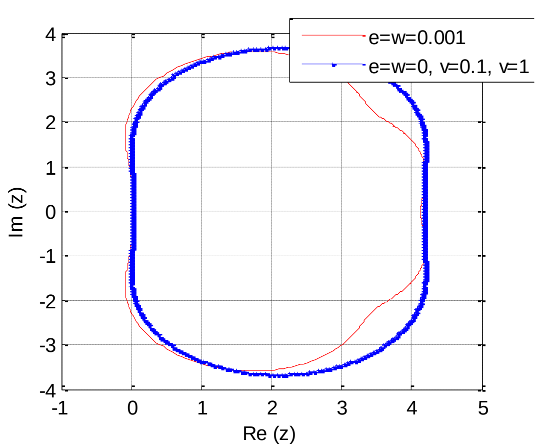


Fig 1.2a **Fig 1.2b**

Fig (3.2a,b): Boundary Locus of IM(EX)^2 SDBDF3 with various e and w, v=0.1

IM(EX)^2 SDBDF4



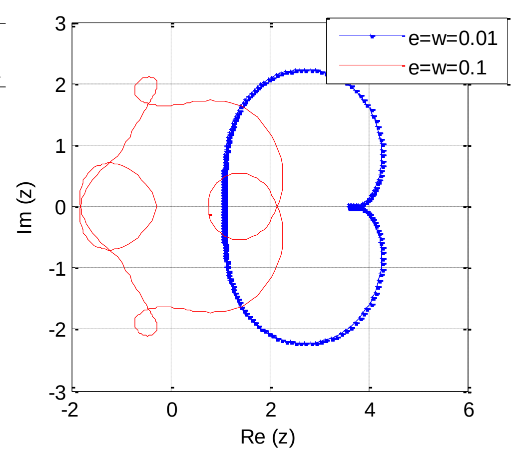


Fig 1.3a **Fig 1.3b**

Fig(1.3a,b): Boundary Locus of IM(EX)^2 SDBDF4 with various e and w, v=0.1

IM(EX)^2 SDBDF5

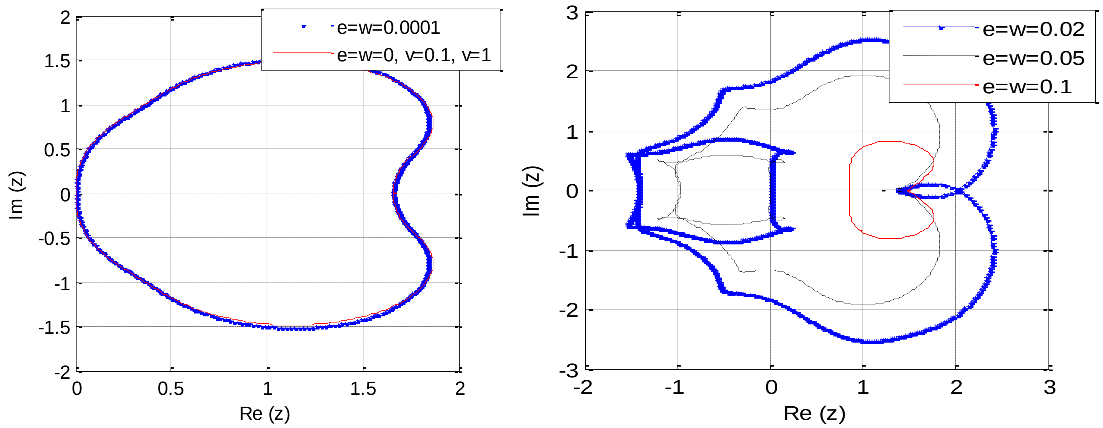


Fig 1.4a **Fig 1.4b**

Fig(1.4a,b): Boundary Locus of IM(EX)^2 SDBDF5 with various e and w, v=0.1

IM(EX)^2 SDBDF6

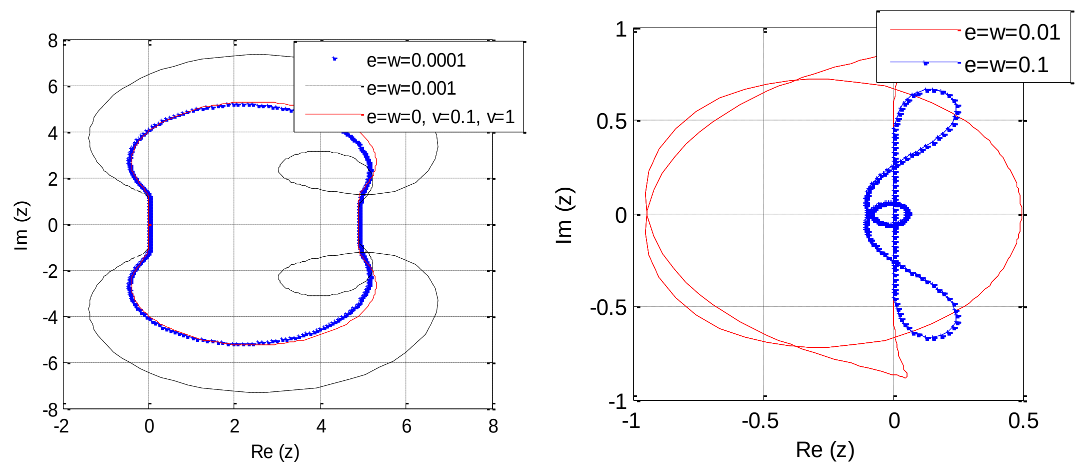


Fig 1.5a **Fig 1.5b**

Fig(1.5a,b): Boundary Locus of IM(EX)^2 SDBDF6 with various e and w, v=0.1

IM(EX)^2 SDBDF7

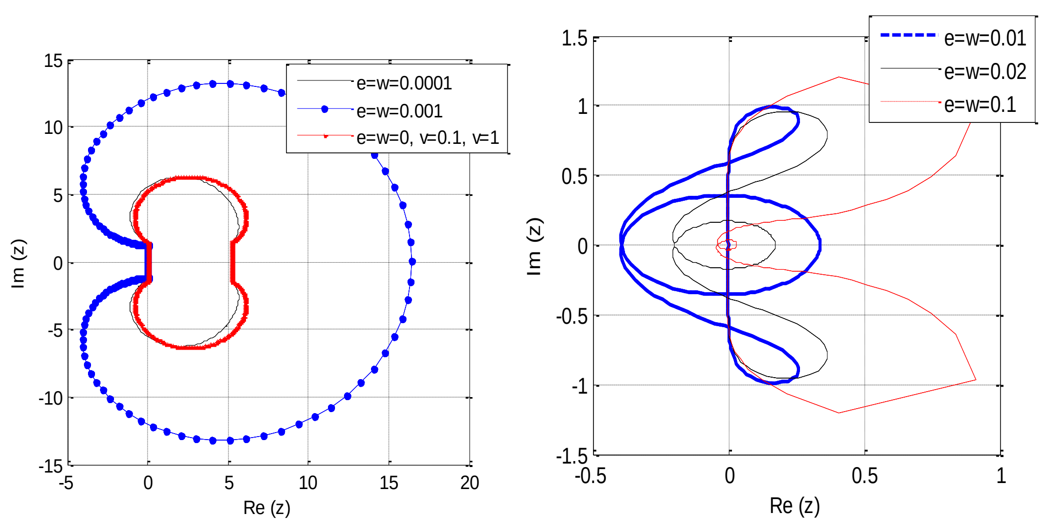
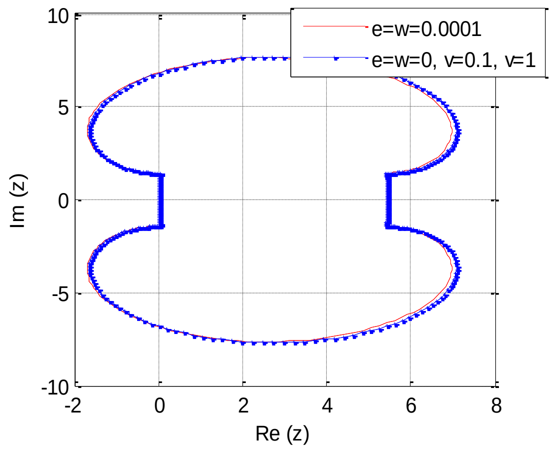


Fig 1.6a **Fig 1.6b**

Fig(1.6a,b): Boundary Locus of IM(EX)^2 SDBDF7 with various e and w, v=0.1

IM(EX)^2 SDBDF8



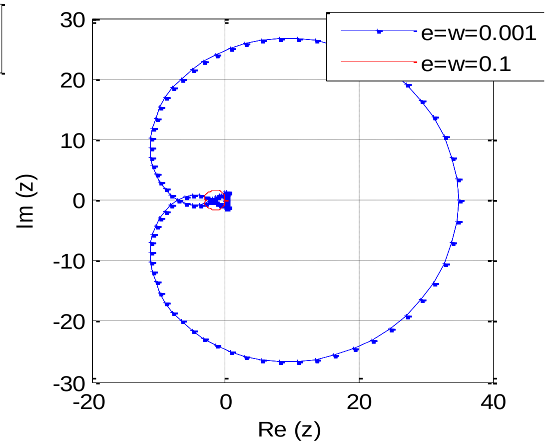
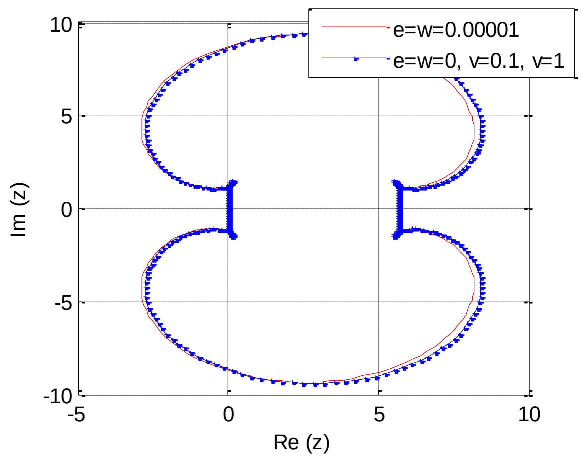


Fig 1.7a **Fig 1.7b**

Fig(1.7a,b): Boundary Locus of IM(EX)^2 SDBDF8 with various e and w, v=0.1

IM(EX)^2 SDBDF9



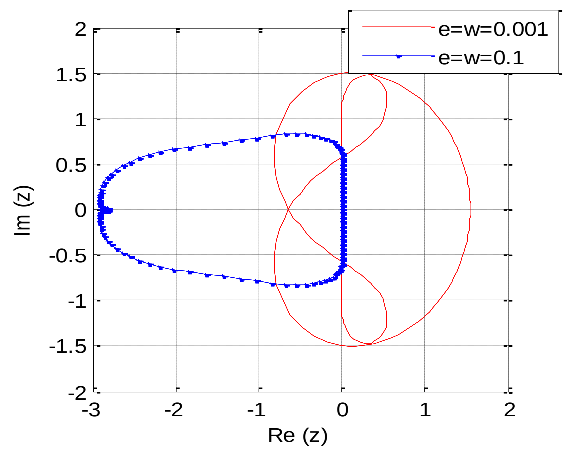


Fig 1.8a **Fig 1.8b**

Fig(1.8a,b): Boundary Locus of IM(EX)^2 SDBDF9 with various e and w, v=0.1

For further insight see [5]-[20].

4.0 Numerical experiments and applications of the IM(EX)^2 SDLMM

Consider some numerical experiments on the following initial value problems with different additive splitting.

Problem (1): Dahlquist test problem

$$y'(t) = -11y(t) = [m]+[f]+[g] = [-10y(t)] - \left[\frac{1}{2}y(t)\right] - \left[\frac{1}{2}y(t)\right]; \quad y(0)=1 \qquad (25)$$

with its exact solution as $y(t) = e^{-11t}$

Problem (2): Prothero-Robinson test problem, see [21], [22] with the additive splitting (3) as

$$y'(t) = \lambda(y(t)-q(t)) + q'(t) = [m]+[f]+[g] = [\lambda y(t)] + [-\lambda q(t)] + [q'(t)]$$
$$t \geq 0,\ y(0) = q(0),\ \lambda < 0 \qquad (26)$$

The exact solution is given by $y(t) = q(t)$, here choose that

$$q(t) = \sin\left(\frac{\pi}{4}+t\right) \qquad and \quad \lambda = -10^2$$

The application of the method (9) to problems 1 and 2 respectively leads to solving an implicit equation for the solution component y_{n+k} which is resolved by applying the Newton-Raphson scheme on (9) to get,

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - \left(J\left(y_{n+k}^{[s]}\right)\right)^{-1} F\left(y_{n+k}^{[s]}\right), \quad s = 0,1,2,\ldots,w \qquad (27)$$

where $J\left(y_{n+k}^{[s]}\right)$ is the Jacobian matrix from

$$F\left(y_{n+k}^{[s]}\right) = y_{n+k}^{[s]} - [\sum_{j=0}^{k-1}\alpha_j y_{n+j} + h\sum_{j=0}^{k-1}\beta^*{}_j f(t_{n+j}, y_{n+j}) + h^2\sum_{j=0}^{k-1}\lambda^*{}_j f^{/}(t_{n+j}, y_{n+j}) + h\sum_{j=0}^{k-1}\beta^*{}_j g(t_{n+j}, y_{n+j})$$

$$+h^2\sum_{j=0}^{k-1}\lambda^*{}_j g^{/}(t_{n+j}, y_{n+j}) + h\sum_{j=0}^{k-1}\beta_j m(t_{n+j}, y_{n+j}) + h^2\sum_{j=0}^{k-1}\lambda_j m^{/}(t_{n+j}, y_{n+j})] + h\beta_k m(t_{n+k}, y_{n+k}^{[s]})$$

$$+h^2\lambda_k m^{/}(t_{n+k}, y_{n+k}^{[s]}), \qquad s = 0,1,2,\ldots,w.$$

The solution y_{n+k} is thus given by

$$y_{n+k} = -\sum_{j=0}^{k-1}\alpha_j y_{n+j} + h\sum_{j=0}^{k-1}\beta^*{}_j f(t_{n+j}, y_{n+j}) + h^2\sum_{j=0}^{k-1}\lambda^*{}_j f^{/}(t_{n+j}, y_{n+j}) + h\sum_{j=0}^{k-1}\beta_j g(t_{n+j}, y_{n+j})$$

$$+h^2\sum_{j=0}^{k-1}\lambda_j g^{/}(t_{n+j}, y_{n+j}) + h\sum_{j=0}^{k-1}\beta_j m(t_{n+j}, y_{n+j}) + h^2\sum_{j=0}^{k-1}\lambda_j m^{/}(t_{n+j}, y_{n+j})] + h\beta_k m(t_{n+k}, y_{n+k}^{[w]})$$

$$+h^2\lambda_k m^{/}(t_{n+k}, y_{n+k}^{[w]})$$

from (9). The starting values for (27) is from the explicit SDBDF

$$y_{n+1}^{[0]} = y_n + hf_n + \frac{h^2}{2} f^{/}{}_n \qquad p = 2, \quad C_3 = \frac{1}{6}$$

Problem 1 and 2 will be solved with the IM(EX)^2 method (11) in the interval [0,1] with step size h=0.001 and h=0.0125 respectively. The numerical solutions $\bar{y}(1)$ of the IM(EX)^2 methods (11-14) for each problem and its corresponding absolute error $\left|\bar{y}(1) - y(1)\right|$ with output at t=1 will be shown in the tables 1 and 2 .

Table 1: Problem (1) $y(1) = 2.75364493497472\text{e-}005; \qquad h = 0.001$

Methods k	$\left\|\bar{y}(1) - y(1)\right\| = error$	
	IM(EX)^2 SDBDFk	SDBDFk
1	26461e-007	2.8239e-007
2	5.6699e-007	5.6698e-007
3	8.4868e-007	8.4867e-007
4	1.1274e-006	1.6189e-006

Table 2: Problem(2) $y(1) = 0.977061263899476; \qquad h = 0.0125; \quad \lambda = -100$

Methods k	$\left\|\bar{y}(1) - y(1)\right\| = error$	
	IM(EX)^2 SDBDFk	SDBDFk
1	1.1976e-003	1.1976e-003
2	3.2090e-003	3.5307e-003
3	3.6149e-003	3.9024e-003
4	3.4346e-003	3.6693e-003

The IM(EX)^2 SDBDFk methods in section (6) resolves the implicitness in the numerical solution of (1) in a more cost effective way when compared with the SDBDFk (5). And yet on the problems solved, the IM(EX)^2 SDBDFk and SDBDFk (5)gives the same numerical order of accuracy as the tables 1,2 will show.

5.0 Concluding Remarks

A family of variable order extrapolation-based IM(EX)^2 SDLMM (11)-(100) for the direct solution of IVPs in ODEs is considered for additively separable ODEs (1). The methods are based on the SDBDF (5). The boundary loci in Figs (1.0) – Fig (1.8) respectively shows that the proposed schemes from (9) based on the SDBDF (5) for step length k=1(1)9 for some values of e and w are stable on the test problem (20). Furthermore, the numerical results in Tables 1,2 respectively shows that the IM(EX)^2 SDBDFk algorithm in section (6) compares favorably with the exact solutions of each problem. In the light of this, one can consider the more general IM(EX) τ , $\tau \geq 1$ method. Splitting in (21) seeks to reduce the effect of stiffness of the ODE on the method, as mentioned in section 1, this strategy ensures that the computational cost of resolving the implicitness is greatly reduced as compared with applying the implicit method directly on the ODE. This is the computational advantage and the essence behind IM(EX) τ SDLMM.

6.0 Acknowledgment

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7.0 References

[1.] Couzeix M, (1980): An implicit-explicit multistep method for the approximation of parabolic equations, Numer. Math 35 257-276

[2.] Varah J M, (1980): Stability restrictions on second order, three finite difference schemes for parabolic equations. SIAM J. Numer. Anal,17: 300-309

[3.] Frank J, Hundsdorfer W, Verwer JG, (1997): On the stability of implicit-explicit linear multistep methods. Appl. Numer. Math, 193-205

[4.] Jorgenson A D, (2014): Unconditional stability of a Crank-Nicolson Adams-Bashforth2 Numerical Method. International Journal of Numerical Analysis and Modelling-Series B, Volume 5, pp. 171-187

[5.] Cardone A, Jackiewicz Z, Sandu A, and Zhang H, (2013): Extrapolation-based implicit-explicit general linear methods URL:http://arXiv.org/abs/13026276v1[Cs.NA] 8 April 2013

[6.] Ascher U M, Ruuth SJ, Wetton BTR, (1995): Implicit-explicit methods for time-dependent partial differential equations. SIAMJ.Numer.Anal. 32(3), 797-823

[7.] Hundsdorfer W, Ruuth S J, (2007): IM(EX)2 extensions of linear multistep methods with general monotonicity and boundedness properties. J Comput Phys 225:2016-2042

[8.] Koto T, (2008): Stability of IM(EX)2 linear multistep methods for ordinary and delay differential equations. J Computational Appl. Math

[9.] Pareschi L, Russo G, (2001): Implicit-Explicit Runge-Kutta schemes for stiff systems of differential equations. In: Recent trends in numerical analysis, Adv. Theory Comput. Math., vol. 3, pp. 269-288. Nova Sci. Publ., Huntington, NY.

[10.] Pareschi L,Russo G, (2005): Implicit-Explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation. J. Sci. Comput. 25(1-2), 129-155

[11.] Hundsdorfer W and Verwer J G, (1995): A note on splitting errors for advection-reaction equations. Appl. Num. Math., 18:191-199,

[12.] Zharovski E, Sandu A: A class of implicit-explicit two-step Runge-Kutta methods. Tech. Rep. TR-12-08, Computer Science, Virginia Tech., http://eprints.cs.vt.edu/archive/00001183(2012)

[13.] Enright W H, (1974): second derivative multistep method for stiff ordinary differential equation. SIAMJ. Numer.Anal.

[14.] Zhang H, Sandu A, (2012): A second-order diagonally-implicit-explicit multi-stage Integration method. Procedia CS 9, 1039-1046

[15.] Zhang H, Sandu A, (2013): Partitioned and implicit-explicit general linear method for ordinary differential equations arXiv 1302.2689

[16.] Koto T (2008): IMEX Runge-Kutta schemes for reaction-diffusion equations. J Comput Appl Math, 215: 182-195

[17.] Koto T (2008): Stability of IMEX Runge-Kutta methods for delay differential equations. J Comput. Appl Math, 211: 201-212.

[18.] Cash J R (1983): Split Linear Multistep Methods for the Numerical Integration of Stiff Differential Systems. Numer. Math. 42, 299-310

[19.] IBEZUTE C E, (2016): IMEX Second Derivative Linear Multistep Methods for Additively Separable Stiff Ordinary Differential Equations. M.Sc thesis, Department of Mathematics, University of Benin, Benin City, Nigeria.

[20.] Hundsdorfer W, Verwer J, (2003): Numerical Solution of Time–Dependent Advection-Diffusion–Reaction Equations. Berlin: Springer-Verlag,

[21.] Prothero A, Robinson A, (1974): On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations. Math. Comput. 28(125), 145-162.

[22.] Rang J. (2014): An Analysis of the Prothero-Robinson Example for constructing New Adaptive ESDIRK Methods of order 3 and 4. Institute of Scientific Computing, TU Braunschweig, Germany.