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Series Solution for Initial Value Problems (IVPs)

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Abstract

Variants of Runge-Kutta (R-K) method or Multistep methods have provided solution to initial value problems (IVPs) over the years. All these methods have provided approximate solutions with non providing answers in closed form where they exist. However, the method of successive approximation or Piccards method has provided answer in closed form with limited success. This is why the series method is introduced as answers in closed form can be recovered whenever they exist. Moreover the method is simple to apply as it involves only primary functions and the whole process can be automated.

Keywords: Series, Initial Value Problem, and coefficients.

1.0 Introduction

The approximation of a continuous function by a polynomial on a finite segment is espoused by Weierstrass's theorem which asserts that for any function f(x) continuous on a finite interval [a, b], there exists a sequence of ordinary polynomial which converges uniformly to f(x) on [a, b] [1].

2.0 Proof of Weierstrass's Fundamental Theorem

Many of the proofs of the above theorem reduce to direct construction for each function f(x) of the sequence of polynomial

$$P_n(x) = C_0 + C_1x^2 + C_2x^2 + \ldots + C_nx^n \quad \ldots\ldots(2.1)$$

for the interval [a, b] with a = 0 and b = 1.

The sequence of polynomials $B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$ ....(2.2)

by [1] is simple and elegant in its construction. So making use of the uniform convergence of the binomial series in the closed segment [0, 1], we note that if the assertion of the theorem is true for the function f(x), it is true also for the function |f(x)|. Infact if |f(x)|≤ k then

$$|f(x)| = k\sqrt{1-\left\{1-\frac{(f(x))^2}{k^2}\right\}}$$

$$= k\left\{1-\frac{1}{2}\left[1-\frac{f^2(x)}{k^2}\right]-\sum_{n=2}^{\infty}\frac{1.3\ldots(2n-3)}{2.4\ldots 2n}\left[1-\frac{f^2(x)}{k^2}\right]^n\right\} \quad \ldots\ldots(2.3)$$

And for any Є > 0 we have for all sufficiently great values of m, $\text{Max}\|f(x| - S_m(x)| < \varepsilon$ where $S_m(x)$ is the partial sum of order m in the series (2.3). Since we are assuming that the theorem holds for f(x), there is an algebraic polynomial $P_n(x)$ of sufficiently high degree n for which $\max_{a \le X \le b}|S_m(x) - P_n(x)| < \varepsilon$. Consequently, |f(x)| is the limit of a uniformly convergent sequence of polynomial.

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3.0 Applications

From the above, any function f(x) in a finite interval [a, b] can be represented by a polynomial $f(x)=\sum_{k=0}^{n} a_n x^n$

So if y = f(x) then $y=\sum_{n=0}^{\infty} a_n x^n$ .... (3.1)

Hence if given an IVP $y' = kf(x),\ y(0)=c$...(3.2)

Then $y' = k\sum_{n=1}^{\infty} n a_n x^{n-1}$ and that $y(0)=a_0=c$...(3.3)

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = k\sum_{n=0}^{\infty} a_n x^{n-1} \qquad (3.4)$$

Equating the coefficients of the equal powers of x in equation (3.4) leads to recurrence relationship in a_n's [2-4]. This procedure is adopted in all the examples and this leads to a solution which expresses y in various forms of power series. The method was experimented on various classes of IVPs.

3.1 Class A: Single Problems

A1. Given $y' = -y,\ y(0)=1$...(3.5)

Let $y=\sum_{n=0}^{\infty} a_n x^n$..(3.6)

$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$..(3.7)

ie $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

On substituting in (3.5) yields y(0) = a_0 =1 and

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = -(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)$

Equating the coefficients of the equal powers of x leads to

$$\left.\begin{array}{l} a_1 = a_0 = 1 \\ 2a_2 = -a_1 \Rightarrow 2a_2 = 1 \quad \therefore\ a_2 = 1/2 = \dfrac{1}{2!} \\ 3a_3 = -a_2 \Rightarrow 3a_3 = -\dfrac{1}{2} \ \therefore a_3 = -\dfrac{1}{3.2} = \dfrac{1}{3!} \\ 4a_4 = -a_3 \Rightarrow 4a_4 = \dfrac{1}{3.2} \ \therefore a_4 = \dfrac{1}{4.3.2} = \dfrac{1}{4!} \end{array}\right\} \qquad (3.8)$$

Hence $y=\sum_{n=0}^{\infty} a_n x^n$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad = e^{-x}$$

$\therefore$ $y = e^{-x}$...(3.9)

A2. Given $y' = -\frac{y^3}{2}, \ y(0) = 1$ .... (3.10)

again writing $y = \sum_{n=0}^{\infty} a_n x^n$ .... (3.11)

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (3.12)$$

substituting (3.11) and (3.12) into (3.9) gives

$$\left.\begin{aligned}
&a_0 = 1 \\
&a_1 = -\frac{a_0^3}{2} \Rightarrow a_1 = 1/2 = \frac{1}{2!} \\
&2a_2 = -\frac{3}{2}a_0^2 a_1 \Rightarrow a_2 = \frac{3}{8} \\
&a_3 = -\frac{15}{48} \\
&\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&\quad = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15x^3}{48} + \ldots \quad = \frac{1}{\sqrt{1+x}}
\end{aligned}\right\} \quad (3.13)$$

A3. Given $y' = y\cos x, \ y(0) = 1$ .... (3.14)

$$\text{let } y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y = a_0 + a_1 x + a_2 x^2 \ldots \quad (3.15)$$

$$\text{and } y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \ldots \quad (3.16)$$

substituting (3.15) and (3.16) into (3.14) we have

$$y(0) = a_0 = 1$$

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!}\right) \quad (3.17)$$

Again equating the coefficient of the power x in (3.17), we have

$$\left.\begin{aligned}
&a_0 = a_1 \Rightarrow a_1 = 1 \\
&2a_2 = a_1 \Rightarrow 2a_2 = 1 \quad \therefore a_2 = \frac{1}{2} \\
&3a_3 = -\left(\frac{a_0}{2} - a_2\right) = -\left(\frac{1}{2} - \frac{1}{2}\right) \Rightarrow a_3 = 0 \\
&4a_4 = -\frac{a_1}{2!} + a_3 \Rightarrow \quad a_4 = -\frac{1}{8} \\
&5a_5 = \frac{a_0}{4!} - \frac{a_2}{2!} = \frac{1}{24} - \frac{1}{4} = -\frac{5}{24} \therefore a_5 = -\frac{1}{24} \\
&\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\
&\quad = 1 + x + \frac{1}{2}x^2 - \frac{x^4}{8} - \frac{x^5}{24} \ldots \quad = e^{\sin x}
\end{aligned}\right\} \quad (3.18)$$

A4. Consider $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right), \quad y(0) = 1$ ....(3.19)

again let $y = \sum_{n=0}^{\infty} a_n x^n$ ie $y = a_0 + a_1 x + a_2 x^2 \ldots$ ....(3.20)

and $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \ldots$ ....(3.21)

substituting (3.20) and (3.21) into (3.19) we have

$$y(0) = a_0 = 1$$

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = \frac{1}{4}\sum_{n=0}^{\infty} a_n x^n \left(1 - \frac{1}{20}\sum_{n=0}^{\infty} a_n x^n\right)$$

$$= \frac{1}{4}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right)\left(1 - \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots}{20}\right) \ldots (3.22)$$

Equating the coefficient of the power of x we have

$$\left.\begin{array}{l} a_1 = \frac{a_0}{4} - \frac{a_0}{40} \Rightarrow a_1 = \frac{10a_0 - a_0}{40} = \frac{9}{40} \\ 2a_2 = \frac{a_1}{4} - \frac{2a_0 a_1}{80} = \frac{a_1}{4} - \frac{a_1}{40} = \frac{10a_1 - a_1}{40} = \frac{9a_1}{40} \Rightarrow a_2 = \frac{81}{3200} \\ 3a_3 = \frac{a_2}{4} - \frac{1}{80}(2a_0 a_2 + a_1^2) \Rightarrow a_3 = \frac{945}{512000} \\ \therefore y = 1 + \frac{9}{40}x + \frac{81}{3200}x^2 + \frac{945}{512000}x^3 + \ldots \quad = \frac{20}{1 + 19e^{-x^4}} \end{array}\right\} \ldots (3.23)$$

A5. $y' = \frac{y - x}{y + x}, \quad y(0) = 4$ ....(3.24)

again let $y = \sum_{n=0}^{\infty} a_n x^n$ ie $y = a_0 + a_1 x + a_2 x^2 \ldots$ ....(3.25)

and $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \ldots$ ....(3.26)

substituting (3.25) and (3.26) into (3.24) and equating coefficients of x we have

$$\left.\begin{array}{l} a_0 = 4,\ a_1 = 1,\ a_2 = \frac{-1}{4}, \text{ and } a_3 = \frac{1}{24} \\ \therefore y = 4 + x - \frac{x^2}{4} + \frac{x^3}{24} + \ldots \quad = x \\ \text{As } y' = \frac{y - x}{y + x}, \quad y(0) = 4 \text{ in polar coordinates } r = 4e^{\left(\frac{\pi}{2} - \theta\right)} \end{array}\right\} \ldots (3.27)$$

3.2 Class B: Small System

B1. Consider $y_1' = 2(y_1 - y_1y_2), \quad y_1(0) = 1$ ....(3.28)

$$y_2' = -(y_2 - y_1y_2), \quad y_2(0) = 3 \qquad (3.29)$$

$$\text{Let } y_1 = \sum_{n=0}^{\infty} a_n x^n \text{ ie } y = a_0 + a_1x + a_2x^2 \ldots \qquad (3.30)$$

$$y_2 = \sum_{n=0}^{\infty} b_n x^n \text{ ie } y = b_0 + b_1x + b_2x^2 \ldots \qquad (3.31)$$

$$\Rightarrow y_1' = \sum_{n=1}^{\infty} na_n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 \ldots \qquad (3.32)$$

$$\text{and } y_2' = \sum_{n=1}^{\infty} nb_n x^{n-1} = b_1 + 2b_2x + 3b_3x^2 \ldots \qquad (3.33)$$

substituting (3.30), (3.31), (3.32) and (3.33) into (3.28) and (3.29) respectively we have

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots = 2\{(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots) - (a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots)\} \ldots (3.34)$$

$$= 2\left\{(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots) - \begin{pmatrix} a_0b_0 + a_0b_1x + a_0b_2x^2 + a_0b_3x^3 + \ldots \\ + a_1b_0x + a_1b_1x^2 + a_1b_2x^3 + a_1b_3x^4 + \ldots \\ + a_2b_0x^2 + a_2b_1x^3 + a_2b_2x^4 + a_2b_3x^4 + \ldots \\ + a_3b_0x^3 + a_3b_1x^4 + \ldots + a_4b_0x^4 + \end{pmatrix}\right\} \ldots (3.35)$$

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots = (2a_0 + 2a_1x + 2a_2x^2 + 2a_3x^3 + 2a_4x^4 + \ldots) - 2a_0b_0 - 2(a_0b_1 + a_1b_0)x - 2(a_0b_2 + a_1b_1 + a_2b_0)x^2 - 2(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 - 2(a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0)x^4 - \ldots \quad (3.36)$$

$$= 2(a_0 - a_0b_0) + 2(a_1 - a_0b_1 - a_1b_0)x + 2(a_2 - a_0b_2 - a_1b_1 - a_2b_0)x^2 + 2(a_3 - a_0b_3 - a_1b_2 - a_2b_1 - a_3b_0)x^3 + 2(a_4 - a_0b_4 - a_1b_3 - a_2b_2 - a_3b_1 - a_4b_0)x^4 \ldots (3.37)$$

Equating the coefficient of the powers of x in equation (3.37) we have

$$\left.\begin{aligned} a_1 &= 2(a_0 - a_0b_0) \\ 2a_2 &= 2(a_1 - a_0b_1 - a_1b_0) \\ 3a_3 &= 2(a_2 - a_0b_2 - a_1b_1 - a_2b_0) \\ 4a_4 &= 2(a_3 - a_0b_3 - a_1b_2 - a_2b_1 - a_3b_0) \end{aligned}\right\} \qquad (3.38)$$

$$b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 + \ldots = -(b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots) + (a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots) \qquad (3.39)$$

$$= -(b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots) + \begin{pmatrix} a_0b_0 + a_0b_1x + a_0b_2x^2 + a_0b_3x^3 + \ldots \\ + a_1b_0x + a_1b_1x^2 + a_1b_2x^3 + a_1b_3x^4 + \ldots \\ + a_2b_0x^2 + a_2b_1x^3 + a_2b_2x^4 + a_2b_3x^4 + \ldots \\ + a_3b_0x^3 + a_3b_1x^4 + \ldots + a_4b_0x^4 + \end{pmatrix} \qquad (3.40)$$

$$b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 + \ldots = (a_0b_0 - b_0) + (a_0b_1 + a_1b_0 - b_1)x + (a_0b_2 + a_1b_1 + a_2b_0 - b_2)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_1b_0 - b_3)x^3 + (a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0 - b_4)x^4 \ldots (3.41)$$

Equating the coefficient of the powers of x in equation (3.40) we have.

$$\left.\begin{aligned}
b_1 &= (a_0 b_0 - b_0) \\
2b_2 &= (a_0 b_1 + a_1 b_0 - b_1) \\
3b_3 &= (a_0 b_2 + a_1 b_1 + a_2 b_0 - b_2) \\
4b_4 &= (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_1 b_0 - b_3) \\
5b_5 &= (a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 - b_4)
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3.42)$$

Alternating substitutions in (3.38) and (3.42), we have

$$\left.\begin{aligned}
& a_1 = 2(1-3) \ \Rightarrow a_1 = -4 \\
& b_1 = a_0 b_0 - b_0 = 1 \text{ x } 3 - 3 = 0 \\
& a_2 = a_1 - a_0 b_1 - a_1 b_0 = 1 - 3 = -2 \\
& b_2 = \frac{(a_0 b_1 + a_1 b_0 - b_1)}{2} = \frac{-12}{2} = -6 \\
& a_3 = \frac{2(a_2 - a_0 b_2 - a_1 b_1 - a_2 b_0)}{2} = \frac{20}{3} \\
& b_3 = \frac{(a_0 b_2 + a_1 b_1 + a_2 b_0 - b_2)}{3} = -6
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3.43)$$

$$\left.\begin{aligned}
& y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots \\
& \text{and } \ y_2 = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots\ldots\ldots\ldots
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3.44)$$

Substituting the values of a's and b's into (3.44), we have

$$\left.\begin{aligned}
& y_1 = 1 - 4x - 2x^2 + \frac{20}{3} x^3 + \ldots\ldots\ldots\ldots \\
& y_2 = 3 - 6x^2 - 6x^3 + \ldots\ldots\ldots\ldots
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3.45)$$

B2. Consider the system for a linear chemical reaction

$$y_1' = -y_1 + y_2, \qquad y_1(0) = 2 \ldots\ldots\ldots\ldots\ldots\ldots(3.46)$$

$$y_2' = y_1 - 2y_2 + y_3 \qquad y_2(0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3.47)$$

$$y_3' = y_2 - y_3, \qquad y_3(0) = 1 \ldots\ldots\ldots\ldots\ldots\ldots(3.48)$$

again Let $y_1 = \sum_{n=0}^{\infty} a_n x^n$ ie $y = a_0 + a_1 x + a_2 x^2 \ldots$ $\ldots\ldots\ldots\ldots\ldots\ldots(3.49)$

$$y_2 = \sum_{n=0}^{\infty} b_n x^n \text{ ie } y = b_0 + b_1 x + b_2 x^2 \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.50)$$

$$y_3 = \sum_{n=0}^{\infty} c_n x^n \text{ ie } y = c_0 + c_1 x + c_2 x^2 \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.51)$$

from (3.49), (3.50) and (3.51) we have

$$\Rightarrow y_1' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.52)$$

$$y_2' = \sum_{n=1}^{\infty} n b_n x^{n-1} = b_1 + 2b_2 x + 3b_3 x^2 \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.53)$$

$$y_3' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.54)$$

substituting (3.49), (3.50), (3.51), (3.52), (3.53) and (3.54) int o (3.46), (3.47) and (3.48) respectively we have

$$a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots=-\left(a_0+a_1x+a_2x^2+a_3x^3+\ldots\right)+\left(b_0+b_1x+b_2x^2+\ldots\right)\ldots\ldots(3.55)$$

$$b_1+2b_2x+3b_3x^2+4b_4x^3+\ldots=\left(a_0+a_1x+a_2x^2+\ldots\right)-2\left(b_0+b_1x+b_2x^2+b_3x^3+\ldots\right)+\left(c_0+c_1x+c_2x^2+\ldots\right)\ldots\ldots(3.56)$$

$$c_1+2c_2x+3c_3x^2+4c_4x^3+\ldots=\left(b_0+b_1x+b_2x^2+b_3x^3+\ldots\right)-\left(c_0+c_1x+c_2x^2+\ldots\right)\ldots\ldots(3.57)$$

Simplifying (3.55), (3.56) and (3.57), we have

$$a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots=(b_0-a_0)+(a_1-b_1)x+(b_2-a_2)x^2+(b_3-a_3)x^3+(b_4-a_4)x^4\ldots\ldots(3.58)$$

$$b_1+2b_2x+3b_3x^2+4b_4x^3+\ldots=a_0+c_0-2b_0+(a_1+c_1-2b_1)x+(a_2+c_2-2b_2)x^2+(a_3+c_3-2b_3)x^3+(a_4+c_4-2b_4)x^4\ldots(3.59)$$

$$c_1+2c_2x+3c_3x^2+4c_4x^3+\ldots=b_0-c_0+(b_1-c_1)x+(b_2-c_2)x^2+(b_3-c_3)x^3+(b_4-c_4)x^4\ldots\ldots(3.60)$$

Equating coefficients of the powers of x in (3.58), (3.59) and (3.60), we have

$$\left.\begin{array}{lll}
a_1=b_0-a_0, & b_1=a_0+c_0-2b_0, & c_1=b_0-c_0\\
2a_2=b_1-a_1, & 2b_2=a_1+c_1-2b_1, & 2c_2=b_1-c_1\\
3a_3=b_2-a_2, & 3b_3=a_2+c_2-2b_2, & 3c_3=b_2-c_2\\
4a_4=b_3-a_3, & 4b_4=a_3+c_3-2b_3, & 4c_4=b_3-c_3\\
\multicolumn{3}{l}{\text{Substituting rowwise for a's, b's and c's, we have}}\\
a_1=0-2=-2, & b_1=2-1-0=1, & c_1=0-1=-1\\
a_2=\dfrac{b_1-a_1}{2}=\dfrac{3}{2}, & b_2=\dfrac{a_1+c_1-2b_1}{2}=\dfrac{-5}{2}, & c_2=\dfrac{b_1-c_1}{2}=1\\
a_3=\dfrac{b_2-a_2,}{3}=\dfrac{-4}{3} & b_3=\dfrac{a_2+c_2-2b_2}{3}=\dfrac{-5}{6}, & c_3=\dfrac{b_2-c_2}{3}=\dfrac{-7}{6}
\end{array}\right\}\ldots\ldots(3.61)$$

$$\left.\begin{array}{l}
\therefore\ y_1=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\\
y_2=b_0+b_1x+b_2x^2+b_3x^3+\ldots\ldots\\
y_3=c_0+c_1x+c_2x^2+c_3x^3+\ldots\ldots
\end{array}\right\}\ldots\ldots(3.62)$$

Substituting the values of a’s, b’s and c’s into (3.62), we have

$$\left.\begin{array}{l}
\therefore\ y_1=2-x+x^2-\dfrac{4}{3}x^3+\ldots\ldots\\
y_2=x-\dfrac{5}{2}x^2-\dfrac{5}{6}x^3+\ldots\ldots\\
y_3=1-x+x^2-\dfrac{7}{6}x^3+\ldots\ldots
\end{array}\right\}\ldots\ldots(3.63)$$

Summary of Results For Class A Problems (Single Problems)

CLASS A	SINGLE EQUATIONS	ANALYTIC SOLUTIONS	SERIES SOLUTIONS
A1	$y' = -y,\ y(0) = 1$	e^{-x}	$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
A2	$y' = -\frac{y^3}{2},\ y(0) = 1$	$\frac{1}{\sqrt{1+x}}$	$1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15x^3}{48} + \ldots$
A3	$y' = y\cos x,\ y(0) = 1$	$e^{\sin x}$	$1 + x + \frac{1}{2}x^2 - \frac{x^4}{8} - \frac{x^5}{24}\ldots$
A4	$y' = \frac{y}{4}\left(1 - \frac{y}{20}\right),\ y(0) = 1$	$\frac{20}{1 + 19e^{-x^4}}$	$1 + \frac{9}{40}x + \frac{81}{3200}x^2 + \frac{945}{512000}x^3 + \ldots.$
A5	$y' = \frac{y - x}{y + x},\ y(0) = 1$	$r = e^{\left(\frac{\pi}{2} - \theta\right)}$ solution in polar coordinate	$4 + x - \frac{x^2}{4} + \frac{x^3}{24} + \ldots$

Summary of Results For Class B Problems (Small Systems)

CLASS B	TITLE OF SYSTEM	SYSTEM OF EQUATIONS	SERIES SOLUTION
B1	The growth of two conflicting population	$y_1' = 2(y_1 - y_1y_2),\quad y_1(0) = 1$ $y_2' = -(y_2 - y_1y_2),\quad y_2(0) = 3$	$y_1 = 1 - 4x - 2x^2 + \frac{20}{3}x^3 + \ldots$ $y_2 = 3 - 6x^2 - 6x^3 + \ldots$
B2	A linear chemical reaction	$y_1' = -y_1 + y_2,\quad y_1(0) = 2$ $y_2' = y_1 - 2y_2 + y_3\quad y_2(0) = 0$ $y_3' = y_2 - y_3,\quad y_3(0) = 1.$	$y_1 = 2 - x + x^2 - \frac{4}{3}x^3 + \ldots\ldots\ldots\ldots\ldots$ $y_2 = x - \frac{5}{2}x^2 - \frac{5}{6}x^3 + \ldots\ldots\ldots\ldots\ldots$ $y_3 = 1 - x + x^2 - \frac{7}{6}x^3 + \ldots\ldots\ldots\ldots\ldots$

The tables above shows the series solutions and analytic solutions (where it exist) for single problems and small systems as discussed in sections 3.1 and 3.2.

4.0 Conclusion

From the above it is clear that analytic results can be recovered when they exist by this series approach. Analytic solutions for the two systems (B1 and B2) examples are unknown and hence their omission here. However, this method is simple and a general routine can be written for all the cases treated. So it can be automated. The accuracy is unlimited unlike all other methods. Apart from this , it has a more general application than Piccards method which is difficult and limited in use.

5.0 References

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