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Dynamic Stability of a Spherical Cap Pre-Loaded Statically But Trapped By a Step Load

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Abstract

In this investigation, we adopt perturbation method to determine the dynamic buckling load of an imperfect spherical cap pre-statically loaded but later trapped as a step load of infinite loading duration. The displacement is discretized into a pre-buckling symmetric mode, an axisymmetric buckling mode and a non-axisymmetric buckling mode a way of which all are discretized in such that their amplitudes are time-dependent. The relevant parameters are expanded asymptotically, while a two-timing multi-timing procedure is utilized to determine a uniformly valid asymptotic solution of the problem. The dynamic buckling load is determined analytically and is related to the corresponding static buckling load, hence showing that if either of the buckling loads is known, then the other can be evaluated without the labour of repeating the arduous process all over. Specializations of the result are mode on specific limitations of the imperfections.

1.0 Introduction

The subject of dynamic buckling of structural materials received a central stage in recent investigations into the dynamic stability of engineering and structural materials. Such materials include columns, shells and even plates. Consequently, many forms of loading situations have been proposed and investigated. Such loading conditions include step loading, impulsive loading, triangular loading, rectangular loading and even periodic loading, among others. The approach to the solution of any of this problems depends on the particular type of loading as well as on the simplicity or otherwise of the structure investigated. For some simple quadratic or cubic model structures, phase plane analysis can easily be evoked to solve the problem. (Budiansky and Hutchinson, 1966)and (Hutchinson and Budiansky, 1966). However, for more complicated structures such as columns, shells and plates, the accompanying nonlinearity makes phase plane analysis on unsuitable method of approach and so, this gives way to more vigorous analytical or numerical analysis.

Relatively, recent investigations into the subject matter are many and varied. These include investigations by Ferri et al (2006) who studied impulsively loaded prismatic cores; the investigation by Kolakowski (2010) on the static and dynamic interactive buckling regarding axial extension mode of thin-walled channels, as well as the investigation by Chitra and Priyadarsini (2013), who studied the dynamic buckling of composite cylindrical shells subjected to axial impulse. On the other hand, Priyadarsini et al (2012) had earlier investigated numerical and experimental study of buckling of advanced fibre composite cylinders under axial compression, while Belyaev et al (2013) studied the stability of traverse vibration of rod under longitudinal step-wise loading. In the same token, Kowal-Michalska investigated some important parameters in dynamic buckling analysis of plate structures subjected to pulse loading while Reda and Forbes (2012) carried out an investigation into the dynamic effects of lateral buckling of high temperature/high pressure offshore pipelines. Mention must, at this stage, be made of the study by Mcshane et al (2012) who studied the dynamic buckling of an inclined struct.While in most of the above cases, the loading histories investigated were either the static loading or dynamic loading, cases where these same structures were subjected to a combination of both static and dynamic loadings as a continuous loading system was first investigated by Simitses (1983, 1987,1989). In this type of loading, the pre-static load is first applied and after sometimes, a time dependent load is thereafter imposed on the structure. Physical example of this type of loading includes the following: (a) a submarine resting on the bottom of the ocean (static pre-loading) only to be subjected to a sudden blast

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(depth discharged), (b) airplanes under gust loading, (c) submerged pipelines under impact loading and (d) jet engine casings under rotor imbalance, among others. Since the pre-loading is meant to be strictly static, it must be applied slowly so that the inertial effect created at this stage is either negligible or completely non-existent. Here, the static pre-loading is first applied for some time, and before the structure can buckle statistically, it is trapped by a dynamic step load of infinite duration. Tanov et al (1999) had earlier studied the effect of static pre-loading on the dynamic buckling of laminated cylinders under sudden pressure, while Birman (1989) similarly investigated problems of dynamic buckling of anti-symmetric rectangular laminates in which he solved a problem on a pre-loading situation.

Except for some few cases, most problems on pre-loading have been approached by a way of numerical investigation through finite element. This underlines the intricate difficulty inherent in any attempt at investigating such problems analytically. Particularly, where the structure investigated is equally complicated. This in a nutshell, is our mission in this investigation, in which we aim at analytically determining the dynamic buckling load of a pre-statically loaded spherical cap trapped by a step load.

2.0 Derivation of Relevant Equations

A simplified derivation of the governing equations was given by Danielson (1969) in which he assumed the normal displacement $W(x,y,\bar{t})$, on the shell surface to be of the form

$W(x,y,\bar{t}) = \xi_0(\bar{t})W_0(x,y) + \xi_1(\bar{t})W_1(x,y) + \xi_2(\bar{t})W_2(x,y)$(2.1)

Where W_0 and W_1 and W_2 are the pre-buckling symmetric mode, the axisymmetric buckling mode and the non – axisymmetric buckling mode respectively, with $\xi_0(\bar{t}), \xi_1(\bar{t})$ and $\xi_2(\bar{t})$ amplitudes. The imperfection, $\bar{W}(x,y)$, was taken in the form of the buckling modes namely

$$\bar{W}(x,y) = \bar{\xi}_1 W_1(x,y) + \bar{\xi}_2 W_1(x,y), \tag{2.2}$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ are their respective amplitudes. By substituting (2.1) and (2.2) into both the compatibility equation and governing equation of motion characterizing a spherical cap, he derived the following equations:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\bar{t}^2} + \xi_0 = \lambda f(\bar{t}) \tag{2.3}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\bar{t}^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\xi_0 \tag{2.4}$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{d\bar{t}^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_1\xi_0 \tag{2.5}$$

$$\xi_i(0) = \frac{d\xi_i(0)}{d\bar{t}} = 0, \quad i = 0,1,2., \tag{2.6}$$

where $k$and k_2 are constants, such that $k_1 > 0, k_2 > 0$, and $\omega_{i,}$ $i = 0,1,2$ are circular frequencies of the associated modes, while $\bar{t}$ is the time parameter and λ is a non-dimensional load parameter, satisfying the inequality $0 < \lambda < 1$, and $f(\bar{t})$ is the actual time dependence on the load. Danielson studied the use of a step load whereby$f(\bar{t}) = 1$, $\bar{t} > 0$.

In our investigation, we shall assume that the magnitude of the load parameter, λ is the same during the entire loading process, i.e. during the pre-loading stage and at the stage of application of the step load. For solution, Danielson made the following assumptions:

(i) Quantities of the order of shell thickness divided by the radius can be neglected relative to unity.

(ii) Tangential inertias and boundary effects and negligible.

(iii) $\bar{\xi}_1$ can be set equal to zero, assuming that non-axisymmetric imperfections are the main cause of the buckling phenomenon.

(iv) The effects of the quadratic term $k_1\xi_1^2$ may be neglected compared to the effects of the coupling between the buckling modes for initial post buckling behavior.

(v) The ratio of subsequent frequencies, i.e. $\frac{\omega_i}{\omega_{i-1}}$, is taken as $(1-\upsilon)^{\frac{1}{2}}$, where υ is the Poisson's ratio.

Besides, Danielson used Mathieu-type of instability to determine the dynamic buckling load based on the above assumptions. In our analysis, we shall ignore assumptions (iii) – (v) based on the findings by Ette (1997,2004). We shall similarly ignore the use of Mathew- type of instability for, as noted by Budiansky (1966), Mathew-type of instability is usually associated with many cycles of oscillation, as opposed to just one shot of oscillation that normally triggers off dynamic buckling.

The organization of this work is arranged in the following fashion: We shall first determine the static deformation, in which case, we first determine the displacement and hence, the static buckling load during the pre-loading period. Next, we determine the displacement occasioned by the imposition of the step load on the pre-static load. At this stage, the displacement during the pre-loading stage is coupled to the displacement produced during the step loading regime. Lastly, we determine the dynamic buckling load, λ_D and relate it to the corresponding static buckling load λ_s, and thus showing that if one of them is given, then, the other can easily be obtained. From our result, we shall deduce the corresponding results with a strict step loading (without a previous pre-loading). The dynamic buckling load is defined as a maximum load perimeter for which the displacement remains bounded and is obtained from the maximization

$$\frac{d\lambda}{d\zeta_a} = 0 \quad (2.7)$$

where ζ_a is the maximum displacement.

3.0 Static Deformation

In this case, we ignore the inertia terms in (2.3) – (2.6), set $f(\bar{t}) = 1$, and introduce a superscript as in $\xi_i^{(0)}$ on each mode ξ_i, $i = 0,1,2$, so that the equations associated with (2.3) – (2.6) respectively become

$$\xi_0^{(0)} = \lambda = \lambda\left(\frac{\omega_1}{\omega_0}\right)^2\left(\frac{\omega_0}{\omega_1}\right)^2 = \frac{\epsilon}{Q^2}; \quad (3.1)$$

$$\xi_1^{(0)}(1 - \xi_0^{(0)}) - k_1\xi_1^{(0)^2} + k_2\xi_2^{(0)^2} = \bar{\xi}_1\xi_0^{(0)} \quad (3.2)$$

$$\xi_2^{(0)}(1 - \xi_0^{(0)}) + \xi_1^{(0)}\xi_2^{(0)} = \bar{\xi}_2\xi_0^{(0)} \quad (3.3)$$

where $\epsilon = \lambda\left(\frac{\omega_1}{\omega_0}\right)^2$, $Q = \left(\frac{\omega_1}{\omega_0}\right)$, $0 < \epsilon << 1$, $0 < \frac{\omega_1}{\omega_2} << 1$

Let

$$\xi_1^{(0)} = \sum_{i=0}^{\infty}\eta^{(i)}\epsilon^i, \quad \xi_2^{(i)} = \sum_{i=1}^{\infty}\zeta^{(i)}\epsilon^i \quad (3.4)$$

Then, for (3.2), we get (equating terms of orders of ϵ)

$$O(\epsilon): \quad \eta^{(1)} = \frac{\bar{\xi}_1}{Q^2} \quad (3.5)$$

$$O(\epsilon^2): \quad \eta^{(2)} - \frac{\eta^{(1)}}{Q^2} - k_1\eta^{(1)^2} + k_2\zeta^{(1)^2} = 0 \quad (3.6)$$

$$O(\epsilon^3): \quad \eta^{(3)} - \frac{\eta^{(2)}}{Q^2} - 2\left[k_1\eta^{(1)}\eta^{(2)} - k_2\zeta^{(1)}\zeta^{(2)}\right] = 0 \quad (3.7)$$

For (3.3), we similarly get

$$O(\epsilon): \quad \zeta^{(i)} = \frac{\bar{\xi}_2}{Q^2}$$

$$O(\epsilon^2): \quad \zeta^{(2)} - \frac{\zeta^{(1)}}{Q^2} + \eta^{(1)}\zeta^{(1)} = 0 \quad (3.9)$$

$$O(\epsilon^3): \quad \zeta^{(3)} - \frac{\zeta^{(2)}}{Q^2} + \eta^{(1)}\zeta^{(2)} + \eta^{(2)}\zeta^{(1)} = 0 \quad (3.10)$$

etc.

On substituting (3.5) and (3.8) into (3.6), we get

$$\eta^{(2)} = \frac{\bar{\xi}_1}{Q^4} + k_1\left(\frac{\bar{\xi}_1}{Q^4}\right)^2 - k_2\left(\frac{\bar{\xi}_2}{Q^2}\right)^2$$

$$= \frac{1}{Q^4}\left\{\bar{\xi}_1 + k_1\bar{\xi}_1^2 - k_1\bar{\xi}_2^2\right\} \quad (3.11)$$

We next substitute (3.8) and (3.5) into (3.9), and simplify to get

$$\zeta^{(2)} = \frac{1}{Q^4}\left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right) \quad (3.12)$$

0n substituting (3.5),(3.11),(3.8) and (3.12) into (3.7), and simplifying we get

$$\eta^{(3)} = \frac{1}{Q^6}\left(\bar{\xi}_1 + k_1\bar{\xi}_1{}^2 - k_2\bar{\xi}_2{}^2\right)$$

$$+2\left[\frac{k_1\bar{\xi}_1}{Q^6}\left(\bar{\xi}_1 + k_1\bar{\xi}_1{}^2 - k_2\bar{\xi}_2{}^2\right) - \frac{k_2\bar{\xi}_2}{Q^6}\left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right)\right] \quad (3.13)$$

Similarly, on substituting (3.5), (3.8),(3.11) and (3.12) into (3.10), we get

$$\zeta^{(3)} = \left(\frac{\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2}{Q^6}\right) - \frac{\bar{\xi}_1}{Q^6}\left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right) - \frac{\bar{\xi}_2}{Q^6}\left(\bar{\xi}_1 + k_1\bar{\xi}_1{}^2 - k_2\bar{\xi}_2{}^2\right). \quad (3.14)$$

So far, we write

$$\xi_0^{(0)} = \frac{\epsilon}{Q^2} \quad (3.15a)$$

$$\xi_1^{(0)} = \eta^{(1)}\epsilon + \eta^{(2)}\epsilon^2 + \eta^{(3)}\epsilon^3 + \ldots \quad (3.15b)$$

$$\xi_2^{(0)} = \zeta^{(1)}\epsilon + \zeta^{(2)}\epsilon^2 + \zeta^{(3)}\epsilon^3 + \cdots \quad (3.15c)$$

The net displacement, ζ, is

$$\zeta = \xi_0^{(0)} + \xi_1^{(0)} + \xi_2^{(0)} = c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^2 + \cdots \quad (3.16a)$$

where

$$c_1 = \left(\frac{1 + \bar{\xi}_1 + \bar{\xi}_2}{Q^2}\right), \quad (3.16b)$$

$$c_2 = \frac{1}{Q^4}\left(\bar{\xi}_1 + k_1\bar{\xi}_1{}^2 - k_2\bar{\xi}_2{}^2\right) \quad (3.16c)$$

$$c_3 = \frac{1}{Q^6}\left\{\left(\bar{\xi}_1 + k_1\bar{\xi}_1^{\ 2} - k_2\bar{\xi}_2^{\ 2}\right) + \left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right) - \bar{\xi}_1\left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right)\right\}$$
$$+\frac{1}{Q^6}\left[2\left\{k_1\bar{\xi}_1\left(\bar{\xi}_1 + k_1\bar{\xi}_1^{\ 2} - k_2\bar{\xi}_2^{\ 2}\right) - k_2\left(\bar{\xi}_2 - \bar{\xi}_1\bar{\xi}_2\right)\right\}\right.$$
$$\left.-\bar{\xi}_2\left(\bar{\xi}_1 + k_1\bar{\xi}_1^{\ 2} - k_2\bar{\xi}_2^{\ 2}\right)\right] \qquad (3.16d)$$

To determine the static buckling loadλ_S, we (Ette, 1997, 2004) have to reverse the series (3.16a) in the form

$$\epsilon = d_1\zeta + d_2\zeta^2 + d_3\zeta^3 + \cdots \qquad (3.17)$$

If we substitute in (3.17) and equate the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{c_1}, \quad d_2 = -\frac{c_2}{c_1^3}, \quad d_3 = \frac{2c_2^2 - c_1c_3}{c_1^5} \qquad (3.18)$$

To determine the static buckling load the requisite maximization equivalent to (2.7) is used and takes the form

$$\frac{d\lambda}{d\zeta} = 0 \qquad (3.19)$$

This yields

$$d_1 + 2d_2\zeta_s + 3d_3\zeta_s^2 = 0, \qquad (3.20)$$

where ζ_s is the value ofζ at static buckling.

If we retain only the first two terms in (3.20), we get

$$d_1 + 2d_2\zeta_s = 0$$
$$\therefore \zeta_s = -\frac{d_1}{2d_2} = \frac{c_1^2}{2c_2} \qquad (3.21)$$

On evaluating (3.17) at $\zeta = \zeta_s$ and retaining only the first two terms on the right hand side, we get

$$\epsilon = d_1 + 2d_2\zeta_s = \frac{1}{4}\left(\frac{c_1}{c_2}\right) \qquad (3.22)$$

On substituting c_1 and c_2 in (3.22), we get

$$\lambda_S\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{Q^2}{4}\left[\frac{1 + \bar{\xi}_1 + \bar{\xi}_2}{\bar{\xi}_1 + k_1\bar{\xi}_1^2 - k_2\bar{\xi}_2^2 - \bar{\xi}_1\bar{\xi}_2}\right]$$

Or

$$\lambda_S = \frac{1}{4}\left[\frac{1 + \bar{\xi}_1 + \bar{\xi}_2}{\bar{\xi}_1 + k_1\bar{\xi}_1^2 - k_2\bar{\xi}_2^2 - \bar{\xi}_1\bar{\xi}_2}\right] \qquad (3.23)$$

This gives the static buckling load λ_S, which is independent of the circular frequencies ω_i.

4.0 Dynamic Deformation Analysis

We shall let$\xi_i^{(l)}$, $i = 0,1,2$be the displacement strictly due to step loads. We realize that if the loading were strictly that of a step load, the equations (2.3) – (2.5), would be

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0^{(l)}}{d\bar{t}^2} + \xi_0^{(l)} = \lambda, \quad (f(\bar{t}) \equiv 1) \qquad (4.1)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1^{(l)}}{d\bar{t}^2} + \xi_1^{(l)}\left(1 - \xi_0^{(l)}\right) - k_1\xi_1^{(l)^2} + k_2\xi_2^{(l)^2} = \bar{\xi}_1\xi_0^{(l)} \qquad (4.2)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2^{(l)}}{d\bar{t}^2} + \xi_2^{(l)}\left(1 - \xi_0^{(l)}\right) + \xi_1^{(l)}\xi_2^{(l)} = \bar{\xi}_2\xi_0^{(l)} \qquad (4.3)$$

$$\xi_i^{(l)} = \frac{d\xi_i^{(l)}}{d\bar{t}} = 0, \quad i = 0,1,2 \qquad (4.4)$$

At the imposition of the step load on the erstwhile pre-static load the combined displacement $z_i(\bar{t})$ is

$$z_i = \xi_i^{(0)} + \xi_i^{(l)}, \quad i = 0,1,2 \qquad (4.5)$$

The displacements $z_i(\bar{t})$satisfy the following motion (similar to (2.3) –(2.5))

$$\frac{1}{\omega_0^2}\frac{d^2z_0}{d\bar{t}^2} + z_0 = 2\lambda f(\bar{t}) \qquad (4.6)$$

$$\frac{1}{\omega_1^2}\frac{d^2z_1}{d\bar{t}^2} + z_1(1 - z_0) - k_1z_1^2 + k_1z_1^2 = \bar{\xi}_2z_0 \qquad (4.7)$$

$$\frac{1}{\omega_2^2}\frac{d^2z_2}{d\bar{t}^2} + z_2(1 - z_0) + z_1z_2 = \bar{\xi}_2z_0 \qquad (4.8)$$

If we substitute in (4.6) –(4.8) from (4.5) we get

$$\frac{1}{\omega_0^2}\frac{d^2}{d\bar{t}^2}\left(\xi_0^{(0)} + \xi_0^{(l)}\right) + \left(\xi_0^{(0)} + \xi_0^{(l)}\right) = 2\lambda f(\bar{t}) \qquad (4.9)$$

$$\frac{1}{\omega_1^2}\frac{d^2}{d\bar{t}^2}\left(\xi_1^{(0)} + \xi_1^{(l)}\right) + \left(\xi_1^{(0)} + \xi_1^{(l)}\right)\left[1 - \left(\xi_0^{(0)} + \xi_0^{(l)}\right)\right]$$

$$-k_1\left(\xi_1^{(0)}+\xi_1^{(l)}\right)^2+k_2\left(\xi_1^{(0)}+\xi_1^{(l)}\right)^2=\bar{\xi}_1\left(\xi_0^{(0)}+\xi_0^{(l)}\right) \quad (4.10)$$

$$\frac{1}{\omega_2^2}\frac{d^2}{d\bar{t}^2}\left(\xi_2^{(0)}+\xi_2^{(l)}\right)+\left(\xi_2^{(0)}+\xi_2^{(l)}\right)\left[1-\left(\xi_0^{(0)}+\xi_0^{(l)}\right)\right]$$
$$+\left(\xi_1^{(0)}+\xi_1^{(l)}\right)\left(\xi_2^{(0)}+\xi_2^{(l)}\right)=\bar{\xi}_2\left(\xi_0^{(0)}+\xi_0^{(l)}\right) \quad (4.11)$$

We however recall that the displacements $\xi_i^{(0)}$ do not depend on the time variable $\bar{t}$. The equations satisfied by the step load alone are the following differences: (4.9) – (3.1), (4.10) – (3.2) and (4.11) – (3.3).
These give the following:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0^{(l)}}{d\bar{t}^2}+\xi_0^{(l)}=\lambda f(\bar{t}) \quad (4.12)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1^{(l)}}{d\bar{t}^2}+\xi_1^{(l)}\left(1-\xi_0^{(l)}-\beta\xi_0^{(0)}\right)-k_1\left[2\beta\xi_1^{(0)}\xi_1^{(l)}+\xi_1^{(l)^2}\right]$$
$$+k_2\left[2\beta\xi_2^{(0)}\xi_2^{(l)}+\xi_2^{(l)^2}\right]=\bar{\xi}_1\xi_0^{(l)} \quad (4.13)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2^{(l)}}{d\bar{t}^2}+\xi_2^{(l)}\left(1-\beta\xi_0^{(0)}-\xi_0^{(l)}\right)$$
$$+\left[\beta\left(\xi_1^{(0)}\xi_2^{(l)}+\xi_1^{(l)}\xi_2^{(0)}\right)+\xi_1^{(l)}\xi_2^{(l)}\right]=\bar{\xi}_2\xi_0^{(l)} \quad (4.14)$$

$$\xi_{i(0)}^{i}=\frac{d\xi_i^{(l)}}{d\bar{t}}(0)=0,\ \ i=0,1,2, \quad (4.15)$$

where$\beta=o$, or $\beta=1$.When$\beta=1$, we get the equations of motion characteristic of a step load superposed on a pre-static load whereas when $\beta=0$, we get the equations of motion charactensing a strictly step loading situation (i.e without a pre-load). Infact, when $\beta=o$, we recover equations (4.1) to (4.3). We now set$f(\bar{t})\equiv 1$ and let$t=\omega_0\bar{t}$and get

$$\frac{d\xi_\alpha}{d\bar{t}}=\omega_0\frac{d\xi_\alpha}{dt},\quad \frac{d^2\xi_\alpha}{d\bar{t}^2}=\omega_0^2\frac{d^2\xi_\alpha}{dt^2},\quad \alpha=0,1,2. \quad (4.16)$$

Thus, equations (4.12) to (4.15) become respectively

$$\frac{d^2\xi_0^{(l)}}{dt^2}+\xi_0^{(l)}=\frac{\epsilon}{Q^2} \quad (4.17)$$

$$\frac{d^2\xi_1^{(l)}}{dt^2}+Q^2\xi_1^{(l)}-\xi_1^{(l)}Q^2\left(\xi_0^{(l)}+\beta\xi_0^{(0)}\right)-Q^2k_1\left\{2\beta\xi_1^{(0)}\xi_1^{(l)}+\left(\xi_1^{(l)}\right)^2\right\}$$
$$+Q^2k_2\left\{2\beta\xi_2^{(0)}\xi_2^{(l)}+\xi_2^{(l)^2}\right\}=Q^2\bar{\xi}_1\xi_0^{(l)} \quad (4.18)$$

$$\frac{d^2\xi_2^{(l)}}{dt^2}+R^2\xi_2^{(l)}-R^2\xi_2^{(l)}\left(\xi_0^{(l)}+\beta\xi_0^{(0)}\right)$$
$$+R^2\left[\beta\left\{\xi_1^{(0)}\xi_2^{(l)}+\xi_1^{(l)}\xi_2^{(0)}\right\}+\xi_1^{(l)}\xi_2^{(l)}\right]=\left(\frac{\omega_2}{\omega_0}\right)^2\bar{\xi}_1\xi_0^{(l)} \quad (4.19)$$

where

$$Q=\left(\frac{\omega_1}{\omega_0}\right),\quad R=\left(\frac{\omega_2}{\omega_0}\right) \quad (4.20)$$

$$\xi_i^{(l)}(0)=\frac{d\xi_i^{(l)}(0)}{dt}=0,\quad i=0,1,2$$

Solving (4.17), we get

$$\xi_o^{(l)}=\frac{\epsilon}{Q^2}(1-cost) \quad (4.21)$$

We now let $\tau=\epsilon t$ (4.22)

$$\therefore\frac{d\xi_\alpha}{dt}=\xi_{\alpha,t}+\epsilon\xi_{\alpha,\tau}$$
$$\frac{d^2\xi_\alpha}{dt^2}=\xi_{\alpha,tt}+2\epsilon\xi_{\alpha,t\tau}+\epsilon^2\xi_{\alpha,\tau\tau} \quad (4.23)$$

where a subscript, following a comma indicates partial differentiation.
We substitute (4.21) and (4.23) into (4.18) and note that in this case the r.h.s. of (4.18) becomes $\epsilon\bar{\xi}_1(1-cost)$,while the r.h.s. of (4.19) is $\epsilon S\bar{\xi}_2(1-cost)$, where

$$S=\left(\frac{\omega_2}{\omega_1}\right)^2 \quad (4.24)$$

Let

$$\xi_1^{(l)}=\sum_{i=1}^{\infty}A^{(i)}(t,\tau)\,\epsilon^i,\qquad \xi_2^{(l)}=\sum_{i=1}^{\infty}L^{(i)}\,\epsilon^i \quad (4.25)$$

By substituting, (4.23) and (4.25) into (4.18) and (4.19), and equating coefficients of powers of ϵ, we get first, for (4.18)

$$O(\epsilon): A_{,tt}^{(1)}+Q^2A^{(1)}=\bar{\xi}_1(1-\cos t) \quad (4.26)$$

$O(\epsilon^2)$: $A_{,tt}^{(2)} + Q^2 A^{(2)} = -2A_{,t\tau}^{(1)} + (1-\cos t)\left(A^{(1)} + \beta\right)$
$+Q^2 k_1 \left[2\beta\left(\eta^{(1)}A^{(1)}\right) + {A^{(1)}}^2\right] - Q^2 k_2 \left[2\beta\zeta^{(1)}L^{(1)} + {L^{(1)}}^2\right]$(4.27)
$O(\epsilon^3)$: $A_{,tt}^{(3)} + Q^2 A^{(3)} = -2A_{,t\tau}^{(2)} - A_{,\tau t}^{(1)} - (1-\cos t)A^{(2)}$
$+Q^2 k_1\left[2\beta\left(\eta^{(1)}A^{(2)} + \eta^{(2)}A^{(1)}\right) + 2A^{(1)}A^{(2)}\right]$
$-Q^2 k_2\left[2\beta\left(\zeta^{(1)}L^{(2)} + \zeta^{(2)}L^{(1)}\right) + 2L^{(1)}L^{(2)}\right]$ (4.28) etc.

For (4.19), we get

$O(\epsilon)$: $L_{,tt}^{(1)} + R^2 L^{(1)} = S\bar{\xi}_2(1-\cos t)$ (4.29)

$O(\epsilon^2)$: $L_{,tt}^{(2)} + R^2 L^{(2)} = -2L_{,t\tau}^{(1)} + S\left[\beta L^{(1)} + (1-\cos t)L^{(1)}\right]$
$-R^2\left[\beta\eta^{(1)}L^{(1)} + A^{(1)}\zeta^{(1)} + A^{(1)}L^{(1)}\right]$ (4.30)

$O(\epsilon^3)$: $L_{,tt}^{(3)} + R^2 L^{(2)} = -2L_{,t\tau}^{(2)} - L_{,\tau\tau}^{(1)} + S(1-\cos t)L^{(2)}$
$-R^2[\ \beta\left(\eta^{(1)}L^{(2)} + \eta^{(2)}L^{(1)}\right) + \left(A^{(1)}\zeta^{(2)} + A^{(2)}\zeta^{(1)}\right)$
$+\left(A^{(1)}L^{(2)} + A^{(2)}L^{(1)}\right)]$ (4.31)

etc.

The initial conditions are:

$A^{(i)}(0,0) = 0 = L^{(i)}(0,0) \quad i = 1,2,3,\ldots$ (4.32)

$A_{,t}^{(1)}(0,0) = L_{,t}^{(1)}(0,0) = 0$ (4.33)

$A_{,t}^{(j)}(0,0) + A_{,\tau}^{(j-1)}(0,0) = 0, \quad j = 2,3,4\ldots$ (4.34)

$L_{,t}^{(j)}(0,0) + L_{,\tau}^{(j-1)}(0,0) = 0, \quad j = 2,3,4\ldots$ (4.35)

Now solving (4.26), we get

$A^{(1)}(t,\tau) = \alpha_1(\tau)\cos Qt + \gamma_1(\tau) sinQt + \bar{\xi}_1\left(\frac{1}{Q^2} + \frac{cost}{Q^2-1}\right)$ (4.36a)

$\alpha_1(0) = -\bar{\xi}_1\left(\frac{1}{Q^2} + \frac{1}{Q^2-1}\right) = \frac{\bar{\xi}_1}{Q^2(Q^2-1)}$ (4.36b)

$\gamma_1(0) = 0, \quad Q \neq 1$ (4.36c)

Solving (4.29), we get

$L^{(1)}(t,\tau) = \theta_1(\tau)cosRt + \Omega_1(\tau)sinRt + S\bar{\xi}_2\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)$ (4.37a)

$\theta_1(0) = \frac{S\bar{\xi}_2}{R^2(R^2-1)}, \quad \Omega_1(0) = 0, \quad R \neq 1$ (4.37b)

In the substitution which soon follows, we shall need the following simplification

$(1-cost)A^{(1)} = \left[\left\{\frac{\bar{\xi}_1}{Q^2} + \frac{\bar{\xi}}{2(Q^2-1)}\right\} + \alpha_1 cosQt + \gamma_1 sinQt\right.$
$-\bar{\xi}_1\left(\frac{1}{Q^2} + \frac{cost}{Q^2-1}\right)cost - \frac{\alpha_1}{2}cos(1+Q)t$
$\left.-\frac{\gamma_1}{2}cos(Q-1)t - \frac{\gamma_1}{2}\sin(1-Q)\,t + \frac{\bar{\xi}_1 cos2t}{2(Q^2-1)}\right]$ (4.38)

$(1-cost)\beta = \beta(1-cost)$

$2\beta\eta^{(1)}A^{(1)} = \frac{2\beta\bar{\xi}_1}{Q^2}\left[\alpha_1 cosQt + \gamma_1 sinQt + \bar{\xi}_1\left(\frac{1}{Q^2} - \frac{cost}{Q^2-1}\right)\right]$

${A^{(1)}}^2 = \left\{\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + \bar{\xi}_1^2\left(\frac{1}{Q^4} + \frac{1}{2(Q^2-1)^2}\right)\right\} + \frac{2\alpha_1\bar{\xi}_1 cosQt}{Q^2} + \frac{2\gamma_1\bar{\xi}_1 sinQt}{Q^2}$
$+\left(\frac{\alpha_1^2}{2} - \frac{\gamma_1^2}{2}\right)cos2Qt + \alpha_1\gamma_1 - \frac{2\bar{\xi}_1^2 cost}{Q^2(Q^2-1)} + \frac{\bar{\xi}_1^2 cos2t}{2(Q^2-1)} - \frac{\alpha_1\bar{\xi}_1\cos(1+Q)\,t}{Q^2-1}$
$-\frac{\gamma_1\bar{\xi}_1\sin(1+Q)t}{Q^2-1} - \frac{\alpha_1\bar{\xi}_1\cos(1-Q)t}{Q^2-1} - \frac{\gamma_1\bar{\xi}_1\sin(1-Q)t}{Q^2-1}$ (4.39)

$L^{(1)2} = \left\{\frac{\theta_1^2}{2} + \frac{\Omega_1^2}{2} + \frac{(\bar{\xi}_2)^2}{R^4} + \frac{\varsigma^2\xi_2^2}{2(R^2-1)^2}\right\} + \frac{2\Omega_1 S\bar{\xi}_2 sinRt}{R^2} + \frac{2S\bar{\xi}_2\theta_1 cosRt}{R^2}$
$+\frac{1}{2}(\theta_1^2 - \Omega_1^2)cosRt + \theta_1\Omega_1 sin2Rt - \frac{2S^2\xi_2^2 cost}{R^2(R^2-1)} + \frac{S^2\xi_2^2 cos2t}{2(R^2-1)^2}$
$-\frac{S\bar{\xi}_2\theta_1\cos(1-R)\,t}{R^2-1} + \frac{\Omega_1 S\bar{\xi}_2\sin(1-R)\,t}{R^2-1} - \frac{S\bar{\xi}_2\theta_1\cos(1+R)\,t}{R^2-1}$
$-\frac{\Omega_1 S\bar{\xi}_2\sin(1+R)t}{R^2-1}$ (4.40)

$$2\beta\zeta^{(1)}L^{(1)} = \frac{2\beta\bar{\xi}_2}{Q^2}\left[\theta_1 cosRt + \Omega_1 sinRt + S\bar{\xi}_1\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)\right] \quad (4.41)$$

We now substitute relevant terms into (4.27) and get

$$A_{,tt}^{(2)} + Q^2A^2 = -2Q[\alpha_1' sinQt - \gamma_1' cosQt]$$
$$+\left[\left\{\frac{\bar{\xi}_1}{Q^2} + \frac{\bar{\xi}_1}{2(Q^2-1)}\right\} + \alpha_1 cosQt + \gamma_1 sinQt - \bar{\xi}_1\left(\frac{1}{Q^2} + \frac{1}{Q^2-1}\right)cost\right.$$
$$-\frac{\alpha_1}{2}cos(1+Q)t - \frac{\gamma_1}{2}\sin(1+Q)t - \frac{\alpha_1}{2}\cos(1-Q)t + \frac{\bar{\xi}_1 cos2t}{2(Q^2-1)}$$
$$\left.-\frac{\gamma_1\sin(1-Q)\,t}{2}\right] + Q^2k_1\left[\frac{2\beta\bar{\xi}_1}{Q^2}\left\{\alpha_1 cosQt + \gamma_1 sinQt + \bar{\xi}_1\left(\frac{1}{Q^2} - \frac{cost}{Q^2-1}\right)\right\}\right.$$
$$+\left\{\left\{\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + \xi_1^2\left(\frac{1}{Q^4} + \frac{1}{2(Q^2-1)^2}\right)\right\} + \frac{2\alpha_1\bar{\xi}_1 cosQt}{Q^2} + \frac{2\gamma_1\bar{\xi}_1 sinQt}{Q^2}\right.$$
$$+\frac{1}{2}(\alpha_1^2 - \gamma_1^2)cos2Qt + \alpha_1\gamma_1 sin2Qt - \frac{2\bar{\xi}_1 cost}{Q^2(Q^2-1)} + \frac{\bar{\xi}_1^2 cos2t}{2(Q^2-1)^2}$$
$$\left.\left.-\frac{\alpha_1\bar{\xi}_1\cos(1+Q)\,t}{Q^2-1} - \frac{\gamma_1\bar{\xi}_1\sin(1+Q)\,t}{Q^2-1} - \frac{\alpha_1\bar{\xi}_1\cos(1-Q)\,t}{Q^2-1} - \frac{\gamma_1\bar{\xi}_1\sin(1-Q)t}{Q^2-1}\right\}\right]$$
$$-Q^2k_2\left[\frac{2\beta\bar{\xi}_2}{Q^2}\left\{\theta_1 cosR + \Omega_1 sinRt + S\bar{\xi}_2\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)\right\}\right.$$
$$+\left\{\frac{\theta_1^2}{2} + \frac{\Omega_1^2}{2} + \left(\frac{S\bar{\xi}_2}{R^2}\right)^2\right\} + \frac{2\Omega_1 S\bar{\xi}_2 sinRt}{R^2} + \frac{2S\bar{\xi}_2\theta_1 cosRt}{R^2(R^2-1)} + \frac{(\theta_1^2 - \Omega_1^2)}{2}cos2Rt$$
$$+\theta_1\Omega_1\sin 2Rt - \frac{2\left(S\bar{\xi}_2\right)^2 cost}{2(R^2-1)^2} + \frac{(S\bar{\xi}_2)^2 cos2t}{2(R^2-1)^2} - \frac{S\bar{\xi}_2\theta_1\cos(1-R)\,t}{R^2-1}$$
$$\left.+\frac{\Omega_1 S\bar{\xi}_2\sin(1-R)t}{R^2-1} - \frac{S\bar{\xi}_2\theta_1\cos(1+R)t}{R^2-1} - \frac{\Omega_1 S\bar{\xi}_2\sin(1+R)t}{R^2-1}\right] \quad (4.42)$$

To ensure a uniformly valid solution in t, we equate to zero the coefficients of $cosQt$ andsin Qt in (4.42) and get

$$\gamma_1' - \alpha_1\omega = 0, \quad \alpha_1{}' - \gamma_1\omega = 0 \quad (4.43)$$

Where$(') = \frac{d()}{d\tau}$ and

$$\omega = \frac{1}{2Q}\left[1 + \bar{\xi}_1 k_1(1+\beta)\right] \quad (4.44)$$

From (4.43) and (4.44), we get

$$\gamma_1{}'(0) = \alpha_1(0)\omega = \frac{-\bar{\xi}_1\omega}{Q^2(Q^2-1)}, \quad \alpha_1{}'(0) = 0 \quad (4.45)$$

If we differentiate the first of (4.43) and substitute for$\alpha_1{}'$from the second of (4.43), we get

$$\gamma_1{}'' + \omega^2\gamma_1 = 0 \quad (4.46a)$$

$$\gamma_1(0) = 0, \gamma_1{}'(0) = \frac{-\bar{\xi}_1\omega}{Q^2(Q^2-1)} \quad (4.46b)$$

The solution of (4.46a,b) is

$$\gamma_1(\tau) = b_1 sin\omega\tau, \quad b_1 = \alpha_1(0) = \frac{\bar{\xi}_1\omega}{Q^2(Q^2-1)} \quad (4.47)$$

Similarly, differentiating the second of (4.43) and substituting for $\gamma_1{}'$ from the first of (4.43), we get

$$\alpha_1'' + \omega^2\alpha_1 = 0, \quad \alpha_1(0) = \frac{\bar{\xi}_1}{Q^2(Q^2-1)}, \quad \alpha_1'(0) = \theta_1 \quad (4.48)$$

The solution of (4.48) is

$$\alpha_1(\tau) = a_2 cos\omega\tau, \quad a_2 = \alpha_1(0) = \frac{\bar{\xi}_1}{Q^2(Q^2-1)} \quad (4.49)$$

Hence, we have determined (4.36a) in full. The remaining equation in (4.42) can now be recollected as

$$A_{,tt}^{(2)} + Q^2A^{(2)} = P_1 + P_2\cos\ t + P_3 cos2t + P_4 cos2Qt + P_5 sin2Qt$$
$$+P_6 cos(Q+1)t + P_7 sin(1+Q)t + P_8 cos(Q-1)t$$
$$+P_9 sin(1-Q)t + P_{10} cosRt + P_{11} sinRt + P_{12} sin2Rt$$
$$+P_{13} cos2Rt + P_{14} cos(1-R)t + P_{15} sin(1-R)t$$
$$+P_{16} cos(1+R)t + P_{17}\sin(1+R)t \quad (4.50a)$$

$$A^{(2)}(0,0) = 0, \quad A_{,t}^{(2)}(0,0) + A_{,\tau}^{(1)}(0,0) = 0 \quad (4.50b)$$

Where

$$P_1=\bar{\xi}_1\left(\frac{1}{Q^2}+\frac{1}{2(Q^2-1)}\right)+\frac{2\beta\bar{\xi}_1Q^2k_1}{Q^4}+Q^2k_1\left\{\frac{1}{2}(\alpha_1^2+\gamma_1^2)+\bar{\xi}_1^2\left(\frac{1}{Q^4}+\frac{1}{2(Q^2-1)^2}\right)\right\}$$
$$-Q^2k_2\left\{\frac{2S\bar{\xi}_2^2\beta}{R^2Q^2}+\frac{\theta_1^2}{2}+\frac{\Omega_1^2}{2}\left(\frac{\bar{\xi}_2}{R^2}\right)^2+\frac{1}{2}\left(\frac{S\bar{\xi}_2}{R^2-1}\right)^2\right\} \quad (4.51a)$$
$$P_2=\bar{\xi}_1\left(\frac{1}{Q^2}+\frac{1}{Q^{2-1}}\right)-Q^2k_1\left\{\frac{2\beta\bar{\xi}_1^2}{Q^2(Q^2-1)}+\frac{2\bar{\xi}_1^2}{Q^2(Q^2-1)}\right\}$$
$$+k_2Q^2\left\{\frac{2\beta\bar{\xi}_2^2}{Q^2(R^2-1)}+\frac{2(S\bar{\xi}_2)^2}{R^2(R^2-1)}\right\} \quad (4.51b)$$
$$P_3=\frac{\bar{\xi}_1}{2(Q^2-1)}+\frac{Q^2k_1\bar{\xi}_2^2}{2(Q^2-1)^2}-\frac{k_2Q^2}{2}\left(\frac{S\bar{\xi}_1}{R^2-1}\right)^2 \quad (4.51c)$$
$$P_4=-\frac{Q^2k_1}{2}(\alpha_1^2-\gamma_1^2),P_5(\tau)=k_1Q^2\alpha_1\gamma_1 \quad (4.51d)$$
$$P_6=-\left[\frac{\alpha_1}{2}+\frac{\alpha_1\bar{\xi}_1k_1Q^2}{Q^2-1}\right],\qquad P_7=-\left[\frac{\gamma_1}{2}+\frac{k_1Q^2\gamma_1\bar{\xi}_1}{Q^2-1}\right] \quad (4.51e)$$
$$P_8=-\left[\frac{\alpha_1}{2}+\frac{Q^2k_1\alpha_1\bar{\xi}_1}{Q^2-1}\right],P_9=-\left[\frac{\gamma_1}{2}+\frac{k_1Q^2\gamma_1\bar{\xi}_1}{Q^2-1}\right] \quad (4.51f)$$
$$P_{10}=\frac{-2Q^2k_2\beta\bar{\xi}_2\theta_1}{Q^2}\ ,P_{11}=\frac{-2k_2Q^2\beta\bar{\xi}_2\Omega_1}{Q^2} \quad (4.51g)$$
$$P_{12}=-k_2\theta_1\Omega_1Q^2,P_{13}=\frac{-k_2Q^2}{2}(\theta_1^2-\Omega_1^2) \quad (4.51h)$$
$$P_{14}=\frac{k_2Q^2S\bar{\xi}_2\theta_1}{R^2-1}\ ,\qquad P_{15}=\frac{-k_2Q^2\Omega_1S\bar{\xi}_2}{R^2-1} \quad (4.51i)$$
$$P_{16}=\frac{k_2Q^2S\bar{\xi}_2\theta_1}{R^2-1},\qquad P_{17}=\frac{k_2Q^2\Omega_1S\bar{\xi}_2}{R^2-1} \quad (4.51j)$$

Similarly, we get

$$P_1(0)=M_{11}\bar{\xi}_1+M_{12}\bar{\xi}_2+M_{13}k_1Q^2\bar{\xi}_1^2-M_{14}k_2Q^2S^2\bar{\xi}_2^2 \quad (4.52a)$$

where

$$M_{11}=\frac{1}{Q^2}+\frac{1}{2(Q^2-1)}+\frac{2k_1\beta}{Q^2} \quad (4.52b)$$
$$M_{12}=\frac{-2k_2S\beta}{R^2}\ ,\quad M_{13}=\frac{1}{Q^4}+\frac{1}{2\{Q^2(Q^2-1)\}^2} \quad (4.52c)$$
$$M_{14}=\left[\frac{1}{2\{Q^2(Q^2-1)\}^2}+\frac{1}{2(R^2-)^2}\right] \quad (4.52d)$$
$$P_2(0)=M_{21}\bar{\xi}_1+M_{22}\bar{\xi}_2+M_{23}k_1Q^2\bar{\xi}_1^2+M_{24}k_2(QS\bar{\xi}_2)^2 \quad (4.52e)$$

where

$$M_{21}=\frac{1}{Q^2}+\frac{1}{Q^2-1}\ ,\quad M_{22}=\frac{2K_2Q^2\beta\varsigma}{Q^2(R^2-1)} \quad (4.52f)$$
$$M_{23}=-\left[\frac{2\beta}{Q^2(Q^2-1)}+\frac{2}{Q^2(Q^2-1)}\right],\quad M_{24}=\frac{2\beta}{SQ^2(R^2-1)} \quad (4.52g)$$
$$P_3(0)=M_{31}\bar{\xi}_1+M_{33}k_1\left(Q\bar{\xi}_1\right)^2+M_{34}k_2(QS\bar{\xi}_2)^2 \quad (4.52h)$$

where

$$M_{31}=\frac{1}{2(Q^2-1)}=M_{33},\quad M_{34}=\frac{1}{2(R^2-1)^2} \quad (4.52i)$$
$$P_4(0)=M_{43}k_1\left(Q\bar{\xi}_1\right)^2,\quad M_{43}=\frac{1}{2}\left\{\frac{1}{Q^2(Q^2-1)}\right\}^2 \quad (4.52j)$$
$$P_5(0)=0, \quad (4.52k)$$
$$P_6(0)=M_{61}\bar{\xi}_1+M_{63}k_1\left(Q\bar{\xi}_1\right)^2 \quad (4.52l)$$

where

$$M_{61}=\frac{1}{2Q^2(Q^2-1)},\qquad M_{63}=\frac{-1}{Q^2(Q^2-1)^2} \quad (4.52m)$$
$$P_7(0)=0,\quad P_8(0)=M_{81}\bar{\xi}_1+M_{83}k_1\left(Q\bar{\xi}_1\right)^2 \quad (4.52n)$$

Where

$$M_{81}=-\frac{1}{2Q^2(Q^2-1)}\ ,\qquad M_{83}=-\frac{1}{Q^2(Q^2-1)^2} \quad (4.52o)$$
$$P_9(0)=0,\quad P_{10}(0)=M_{104}k_2(Q\bar{\xi}_2S)^2 \quad (4.52p)$$

Where

$$M_{104}=\frac{-2\beta}{SR^2(R^2-1)} \quad (4.52r)$$
$$P_{11}(0)=0,\ P_{12}(0)=0 \quad (4.52s)$$
$$P_{13}(0)=M_{134}k_2(QS\bar{\xi}_2)^2,\quad M_{134}=\frac{-Q^2}{SR^2(R^2-1)} \quad (4.52t)$$

$P_{14}(0) = M_{144}k_2(QS\bar{\xi}_2)^2, M_{144} = \frac{1}{R^2(R^2-1)}$ (4.52u)
$P_{15}(0) = 0,$ (4.52v)
$P_{16}(0) = M_{164}k_2(QS\bar{\xi}_2)^2, M_{164} = \frac{1}{R^2(R^2-1),}$ (4.52w)
$P_{17}(0) = 0$ (4.52x)
The solution of (4.50a,b), yields

$$A^{(2)}(t,\tau) = \alpha_2(\tau)cosQt + \gamma_2(\tau)sinQt + \frac{P_1}{Q^2} + \frac{P_2cost}{Q^2-1} + \frac{P_3cos2t}{Q^2-4}$$
$$-\frac{P_4cos2Qt}{3Q^2} - \frac{P_5sin2Qt}{3Q^2} - \frac{P_6cos(Q+1)t}{2Q+1} - \frac{P_7sin(Q+1)t}{2Q+1} + \frac{P_8cos(Q-1)t}{2Q-1}$$
$$+\frac{P_9sin(Q-1)t}{2Q-1} + \frac{P_{10}cos2Rt}{Q^2-R^2} + \frac{P_{11}sinRt}{Q^2-R^2} - \frac{P_{12}sin2Rt}{Q^2-4R^2} + \frac{P_{13}cos2Rt}{Q^2-4R^2}$$
$$+\frac{P_{14}cos(1-R)t}{Q^2-(1-R^2)^2} + \frac{P_{15}sin(1-R)t}{Q^2-(1-R^2)^2} + \frac{P_{16}cos(1+R)t}{Q^2-(1+R^2)^2} + \frac{P_{17}sin(1+R)t}{Q^2-(1+R)^2}$$ (4.53)where $Q \neq 1,\ Q \neq 2,\ Q \neq \frac{1}{2},\ Q \neq 2R,\ Q \neq (1-R),\ Q \neq (1+R)$and

$\alpha_2(0) = R_1\bar{\xi}_1 + R_2\bar{\xi}_2 + R_3\kappa_1(Q\bar{\xi}_1)^2 + R_4\kappa_2(Q\bar{\xi}_1)^2$ (4.54)
where
$R_1 = -\left[\frac{M_{11}}{Q^2} + \frac{M_{21}}{Q^2-1} + \frac{M_{31}}{Q^2-4} - \frac{M_{61}}{2Q+1} + \frac{M_{81}}{2Q-1}\right]$ (4.55a)
$R_2 = -\left(\frac{M_{12}}{Q^2} + \frac{M_{22}}{Q^2-1}\right)$(4.55b)
$R_3 = -\left[\frac{M_{13}}{Q^2} + \frac{M_{23}}{Q^2-1} + \frac{M_{33}}{Q^2-4} - \frac{M_{43}}{3Q^2} - \frac{M_{63}}{2Q^2+1} + \frac{M_{83}}{2Q-1}\right]$ (4.55c)

$$R_4 = \left[-\frac{M_{14}}{Q^2} + \frac{M_{24}}{Q^2-1} + \frac{M_{34}}{Q^2-4} - \frac{M_{43}}{3Q^2} - \frac{M_{104}}{Q^{2-}R^2} + \frac{M_{134}}{Q^{2-}4R^2}\right.$$
$$\left.+\frac{M_{144}}{Q^{2-}(1-R)^2} + \frac{M_{164}}{Q^{2-}(1+R)^2}\right]$$ (4.55d)

Also, we get using (4.50b)
$\gamma_2(0) = 0$ (4.56)
In the next substitution into (4.29), we shall need the following simplifications:

$$(1-cost)L^{(1)} = [\theta_1 cosRt + \Omega_1 sinRt + \left\{\frac{S\bar{\xi}_2}{R^2} + \frac{S\bar{\xi}_2}{2(1-R)^2}\right\} - S\bar{\xi}_2\left\{\frac{1}{R^2-1} + \frac{1}{R^2}\right\}cost$$
$$+\frac{S\bar{\xi}_2 cso2t}{2(R^2-1)} - \frac{\theta_1}{2}\ cos(1+R)t - \frac{\Omega_1}{2}\ sin(1+R)t$$
$$-\frac{\theta_1}{2}\ cos(1-R)t + \frac{\Omega_1}{2}\ sin(1-R)t\ ]$$ (4.57)

$\beta\eta^{(1)}L^{(1)} = \beta\left(\frac{\bar{\xi}_1}{Q^2}\right)\left[\theta_1 cosRt + \Omega_1 sinRt + S\bar{\xi}_2\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)\right]$ (4.58)

$$A^{(1)}L^{(1)} = \frac{\alpha_1\theta_1}{2}\{cos(Q+R)t + cos(Q-R)t\} + \frac{\alpha_1\Omega_1}{2}\{sin(R+Q)t$$
$$+sin(R-Q)t\} + S\bar{\xi}_2\alpha_1\left[\frac{cosQt}{R^2} - \frac{1}{2(R^2-1)}\{\cos(Q+R)\,t\right.$$
$$\left.+\cos(Q-R)\,t\}\right] + \frac{\gamma_1\theta_1}{2}\{\sin(R+Q)\,t + \sin(Q-R)\,t\}$$
$$+\frac{\gamma_1\Omega_1}{2}\{cos(R-Q)t - cos(R+Q)t\} + \gamma_1 S\bar{\xi}_2\left[\frac{sinQt}{R^2}\right.$$
$$\left.-\frac{1}{2(R^2-1)}\{sin(Q+R)t + sin(Q-R)t\}\right] + \frac{\bar{\xi}_{1\theta_1}}{Q^2}cosRt$$
$$+\frac{\bar{\xi}_{1\Omega_1}}{Q^2}sinRt + \frac{\bar{\xi}_1 s\bar{\xi}_2}{Q^2}\left\{\frac{1}{R^2} - \frac{cosRt}{(R^2}\right\} - \frac{\bar{\xi}_{1\theta_1}}{2(Q^2-)}\{cos(R+t)$$
$$+cos(R-1)t\} - \frac{\bar{\xi}_1\Omega_1}{2(Q^2-1)}\{\sin(R+1) + \sin(R-1)\,t\} - \frac{\bar{\xi}_1 S\bar{\xi}_2 cost}{R^2(Q^2-1)}$$
$$+\frac{\bar{\xi}_1\bar{\xi}_2 S}{2(Q^2-1)2(R^2-1)}\{\cos(R+1)t + \cos\,(1-R)t\}$$ (4.59)

We can further simplify (4.59) as

$$A^{(1)}L^{(1)} = \frac{\bar{\xi}_1\bar{\xi}_2 S}{(QR)^2} - \frac{\bar{\xi}_1\bar{\xi}_2 \varsigma cost}{R^2(Q^2-1)} + \left\{\frac{\bar{\xi}_{1\theta_1}}{Q^2} - \frac{\bar{\xi}_1\bar{\xi}_2 S}{Q^2(Q^2-1)}\right\} cosRt + \frac{\bar{\xi}_1\Omega_1 \sin(1+R)\,t}{Q^2}$$
$$+ \left\{\frac{\bar{\xi}_1\bar{\xi}_2}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right\} cos(1+R)t - \frac{\bar{\xi}_1\Omega_1 sin(1+R)t}{2(Q^2-1)}$$
$$+ \left\{\frac{\bar{\xi}_1\bar{\xi}_2\varsigma}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right\} cos(R-t)t + \frac{\bar{\xi}_1\Omega_1 sin(1-R)t}{2(Q^2-1)}$$
$$+ \left\{\frac{\alpha_1\theta_1}{2(R^2-1)} - \frac{S\bar{\xi}_2\alpha_1}{2(R^2-1)} - \frac{\gamma_1\Omega_1}{2}\right\} \cos(Q+R)\,t + \left\{\frac{\alpha_1\Omega_1}{2} + \frac{\gamma_1\theta_1}{2}\right.$$
$$\left. - \frac{\gamma_1 S\bar{\xi}_2}{2(R^2-1)}\right\} \sin(Q+R)\,t + \left\{\frac{\alpha_1\theta_1}{2} + \frac{\gamma_1\Omega_1}{2} - \frac{S\alpha_1\bar{\xi}_2}{2(R^2-1)}\right\} \cos(Q-R)\,t$$
$$+ \left\{\frac{\gamma_1\theta_1}{2} - \frac{\alpha_1\Omega_1}{2} - \frac{\gamma_1 S\bar{\xi}_2}{2(R^2-1)}\right\} \sin(Q-R)\,t \qquad (4.60)$$

Similarly, we get

$$A^{(1)}\zeta^{(1)} = \frac{\bar{\xi}_1}{Q^2}\left[\alpha_1 cosQt + \gamma_1 sinQt + \bar{\xi}_2\left(\frac{1}{Q^2} - \frac{cost}{Q^2-1}\right)\right] \qquad (4.61)$$

We now substitute the relevant terms into (4.30), using (4.57) – (4.61), to get

$$L^{(2)}_{,tt} + R^2 L^{(2)} = 2R[\theta_1' sinRt + \Omega_1' R cosRt]$$
$$+ \left(\frac{R}{Q}\right)^2 \left[\beta\left\{\theta_1 cosRt + \Omega_1 sinRt + S\bar{\xi}_2\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)\right\}\right.$$
$$+\{\theta_1 cosRt - \Omega_1 sinRt + \left(\frac{S\bar{\xi}_2}{R^2} + \frac{S\bar{\xi}_2}{2(R^2-1)}\right) - S\bar{\xi}_2\left(\frac{1}{R^2-1} + \frac{1}{R^2}\right) cost$$
$$+ \frac{S\bar{\xi}_2 cos2t}{2(R^2-)} - \frac{\theta_1\cos(1+R)\,t}{2}\Big\} - \frac{\Omega_1}{2}\sin(1+R)\,t - \frac{\theta_1}{2}\cos(1-R)\,t$$
$$\left. + \frac{\Omega_1}{2}\sin\,(1-R)t\Big\}\right] - R^2\left[\left(\frac{\beta\bar{\xi}_1}{Q^2}\right)\left\{\theta_1 cost + \Omega_1 sinRt + S\bar{\xi}_2\left(\frac{1}{R^2} - \frac{cost}{R^2-1}\right)\right\}\right.$$
$$+ \left(\frac{\bar{\xi}_2}{Q^2}\right)\left\{\alpha_1 cosQt + \gamma_1 sinQt + \bar{\xi}_1\left(\frac{1}{Q^2} - \frac{cost}{Q^2-1}\right)\right\} + \frac{\bar{\xi}_1\bar{\xi}_2 S}{(QR)^2} - \frac{\bar{\xi}_1\bar{\xi}_2 S cost}{R^2(Q-1)^2}$$
$$+ \left(\frac{\bar{\xi}_1\theta_1}{Q^2} - \frac{\bar{\xi}_1\bar{\xi}_2 S}{Q^2(Q^2-1)}\right) cost + \frac{\bar{\xi}_1\Omega_1 sinRt}{Q^2}$$
$$+ \left(\frac{\bar{\xi}_1\bar{\xi}_2}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right)\cos(1+R)\,t - \frac{\bar{\xi}_1\Omega_1 \sin\,(1+R)t}{2(Q^2-1)}$$
$$+ \left(\frac{\bar{\xi}_1\bar{\xi}_2 S}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right)\cos(1-R)\,t + \frac{\bar{\xi}_1\Omega_1 \sin\,(1-R)t}{2(Q^2-1)}$$
$$+ \left(\frac{\alpha_1\theta_1}{2(R^2-1)} - \frac{\alpha_1 S\bar{\xi}_2}{2(R^2-1)} - \frac{\gamma_1\Omega_1}{2}\right)\cos(Q+R)\,t + \left(\frac{\alpha_1\Omega_1}{2} + \frac{\gamma_1\theta_1}{2}\right.$$
$$\left. - \frac{\gamma_1 S\bar{\xi}_2}{2(R^2-1)}\right)\sin(Q+R)\,t + \left(\frac{\alpha_1\theta_1}{2} + \frac{\gamma_1\Omega_1}{2} - \frac{S\bar{\xi}_2\alpha_1}{2(R^2-1)}\right)\cos\,(Q-R)t$$
$$\left. + \left(\frac{\gamma_1\theta_1}{2} - \frac{\alpha_1\Omega_1}{2} - \frac{\gamma_1 S\bar{\xi}_2}{2(R^2-1)}\right)\sin(Q-R)\,t\right] \qquad (4.62)$$

The solution of (4.62) is subject to

$$L^{(2)}(0,) = 0,\ L^{(2)}_{,t}(0,0) + L^{(2)}_{,t}(0,0) = 0 \qquad (4.63)$$

To ensure a uniformly valid asymptotic solution in t, we equate to zero in (4.62), the coefficients of $cosRt$ and $sinRt$ and get respectively

$$\theta_1{}' + \psi\Omega_1 = 0 \qquad (4.64a)$$
$$\Omega_1' - \psi\theta_1 = \frac{R\bar{\xi}_1\bar{\xi}_2 S}{2Q^2(Q^2-1)} \qquad (4.64b)$$
$$\psi = \frac{R(\beta+1)(1-\bar{\xi}_1)}{2Q^2} \qquad (4.65)$$

From (4.64a) we get, after differentiating with respect to τ

$$\Omega_1{}' = \frac{\theta_1{}''}{\psi}$$

On substituting this in (4.64b), we get

$$\theta_1'' + \psi^2\theta_1 = -\frac{R\bar{\xi}_1\bar{\xi}_2 SQ}{2Q^2(Q^2-1)} \quad (4.66a)$$

We solve (4.66a) subject to

$$\theta_1(0) = \frac{S\bar{\xi}_2}{R^2(R^2-1)}, \qquad \theta_1'(0) = 0 \quad (4.66b)$$

The solution of (4.66a,b) is

$$\theta_1(\tau) = A\cos\psi\tau - \frac{R\bar{\xi}_1\bar{\xi}_2 S\psi}{2(\psi^2-1)Q^2(Q^2-1)} \quad (4.67a)$$

$$A = \frac{R\bar{\xi}_1\bar{\xi}_2 S\psi}{2(\psi^2-1)Q^2(Q^2-1)} + \frac{S\bar{\xi}_2}{R^2(R^2-1)} \quad (4.67b)$$

Similarly, from (4.64b), we get

$$\theta_1' = \frac{\Omega_1''}{\psi}$$

On substituting this in (4.64a), we get

$$\Omega_1'' + \psi^2\Omega_1 = 0 \quad (4.68a)$$

$$\Omega_1(0) = 0, \qquad \Omega_1'(0) = \frac{\psi\varsigma\bar{\xi}_2}{R^2(R^2-1)} + \frac{R\bar{\xi}_1\bar{\xi}_2 S}{2Q^2(Q^2-1)} \quad (4.68b)$$

On solving (4.68a,b), we get

$$\Omega_1(\tau) = C\sin\psi\tau, \quad C = \frac{S\bar{\xi}_2}{R^2(R^2-1)} + \frac{R\bar{\xi}_1\bar{\xi}_2 S}{2\psi Q^2(Q^2-1)} \quad (4.69)$$

We now re-arrange the remaining equation in (4.62) to get

$$L^{(2)}_{,tt} + R^2L^{(2)} = P_{18} + P_{19}cost + P_{20}cos2t + P_{21}\cos(1+R)\,t$$
$$+P_{22}\sin(1+R)\,t + P_{23}\cos(1-R)\,t + P_{24}\sin(1-R)\,t$$
$$+P_{25}cosQt + P_{26}\text{sinQt} + P_{27}\cos(Q+R)\,t + P_{28}\sin(Q+R)t$$
$$+P_{29}\cos\,(\text{Q}-\text{R})\text{t} + P_{30}\sin(Q-R)\,t \quad (4.70)$$

where

$$P_{18} = \left[\left(\frac{R}{Q}\right)^2\left(\frac{S\bar{\xi}_2\beta}{R^2}\right) + \left(\frac{R}{Q}\right)^2\left(\frac{S\bar{\xi}_2}{R^2} + \frac{S\bar{\xi}_2}{2(R^2-1)}\right) - \frac{R^2\beta\bar{\xi}_1\bar{\xi}_2 S}{(QR)^2}\right.$$
$$\left. - \frac{R^2\bar{\xi}_1\bar{\xi}_2}{Q^4} - \frac{R^2\bar{\xi}_1\bar{\xi}_2 S}{(QR)^2}\right] \quad (4.71a)$$

$$P_{19} = \left[-\left(\frac{R}{Q}\right)^2\frac{\beta\bar{\xi}_2 S}{R^2-1} - S\bar{\xi}_2\left(\frac{1}{R^2-1} + \frac{1}{R^2}\right) + \frac{R^2\beta\bar{\xi}_1\bar{\xi}_2 S}{Q^2(Q^2-1)}\right.$$
$$\left. - \frac{R^2\bar{\xi}_1\bar{\xi}_2}{Q^2(Q^2-1)} + \frac{\bar{\xi}_1\bar{\xi}_2 S}{R^2(Q^2-1)}\right] \quad (4.71b)$$

$$P_{20} = \left[\frac{R^2\bar{\xi}_2 S}{2Q^2(R^2-1)}\right] \quad (4.71c)$$

$$P_{21} = \left[-\frac{R^2\theta_1}{2Q^2} - R^2\left(\frac{\bar{\xi}_1\bar{\xi}_2}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right)\right] \quad (4.71d)$$

$$P_{22} = \left[\frac{\Omega_1 R^2}{2Q^2} + \frac{R^2\bar{\xi}_1\Omega_1}{2(Q^2-1)}\right] \quad (4.71e)$$

$$P_{23} = \left[-\left(\frac{R}{Q}\right)^2\left(\frac{\theta_1}{2}\right) - R^2\left(\frac{\bar{\xi}_1\bar{\xi}_2 S}{2(Q^2-1)(R^2-1)} - \frac{\bar{\xi}_1\theta_1}{2(Q^2-1)}\right)\right] \quad (4.71f)$$

$$P_{24} = \left[\frac{R^2\Omega_1}{2Q^2} + \frac{R^2\bar{\xi}_1\Omega_1}{2(Q^2-1)}\right] \quad (4.71g)$$

$$P_{25} = \left[\frac{-R^2\bar{\xi}_2\alpha_1}{Q^2}\right] \quad (4.71h)$$

$$P_{26} = \left[\frac{-R^2\bar{\xi}_2\gamma_1}{Q^2}\right] \quad (4.71i)$$

$$P_{27} = \left[\frac{-R^2}{2}\left(\frac{\alpha_1\theta_1}{(R^2-1)} + \frac{S\bar{\xi}_2\alpha_1}{R^2-1} + \gamma_1\Omega_1\right)\right] \quad (4.71j)$$

$$P_{28} = \left[\frac{-R^2}{2}\left(\alpha_1\Omega_1 + \gamma_1\theta_1 - \frac{\gamma_1 S\bar{\xi}_2}{R^2-1}\right)\right] \quad (4.71k)$$

$$P_{29} = \left[\frac{-R^2}{2}\left(\alpha_1\theta_1 + \gamma_1\Omega_1 - \frac{\alpha_1 S\bar{\xi}_2}{R^2-1}\right)\right] \quad (4.71l)$$

$$P_{30} = \left[\frac{-R^2}{2}\left(\gamma_1\theta_1 - \alpha_1\Omega_1 - \frac{\gamma_1 S\bar{\xi}_2}{R^2-1}\right)\right] \quad (4.71m)$$

where

$$P_{18}(0) = M_{182}S\bar{\xi}_2 + M_{185}\bar{\xi}_1\bar{\xi}_2 \quad (4.72a)$$

$M_{182} = \frac{(\beta+1)}{Q^2} + \frac{R^2}{2Q^2(R^2-1)}$ (4.72b)

$M_{185} = -\left[\frac{S(\beta+1)}{Q^2} + \left(\frac{R}{Q}\right)^2\right]$ (4.72c)

$P_{19}(0) = M_{192}S\bar{\xi}_2 + M_{195}\bar{\xi}_1\bar{\xi}_2$ (4.73a)

$M_{192} = \frac{-R^2\beta}{Q^2(R^2-1)} - \left(\frac{1}{R^2-1} + \frac{1}{R^2}\right)$ (4.73b)

$M_{195} = \frac{R^2\beta S}{Q^2(R^2-1)} - \frac{-R^2}{Q^2(Q^2-1)} + \frac{S}{R^2(Q^2-1)}$ (4.73c)

$P_{20}(0) = M_{202}S\bar{\xi}_2$, $M_{202} = \frac{R^2}{2Q^2(R^2-1)}$ (4.74a)

$P_{21}(0) = M_{212}S\bar{\xi}_2 + M_{215}\bar{\xi}_1\bar{\xi}_2$ (4.74b)

$M_{212} = \frac{-1}{2Q^2(R^2-1)}$

$M_{215} = -R^2\left[\frac{1}{2(Q^2-1)(R^2-1)} - \frac{S}{2R^2(Q^2-1)(R^2-1)}\right]$ (4.74c)

$P_{22}(0) = 0,$ $P_{23}(0) = M_{232}S\bar{\xi}_2 + M_{235}\bar{\xi}_1\bar{\xi}_2$ (4.75a)

$M_{232} = \frac{-1}{2Q^2(R^2-1)}$, $M_{235} = \frac{-R^2S}{2}\left[\frac{1}{(Q^2-1)(R^2-1)} - \frac{1}{Q^2-1}\right]$ (4.75b)

$P_{24}(0) = 0$, $P_{25}(0) = M_{255}\bar{\xi}_1\bar{\xi}_2$, $M_{255} = \frac{-R^2}{Q^4(Q^2-1)}$ (4.76)

$P_{26}(0) = 0$, $P_{27}(0) = M_{275}\bar{\xi}_1\bar{\xi}_2$ (4.77a)

$M_{275} = \frac{-R^2S}{2}\left[\frac{1}{(QR^2)(Q^2-1)(R^2-1)^2} + \frac{1}{Q^2(Q^2-1)(R^2-1)}\right]$ (4.77b)

$P_{28}(0) = 0$, $P_{29}(0) = M_{295}\bar{\xi}_1\bar{\xi}_2$

$M_{295} = \frac{SR^2}{2Q^2(Q^2-1)(R^2-1)}\left[\frac{1}{R^2} - 1\right]$ (4.74b)

$P_{30}(0) = 0$

On solving (4.70) with (4.63), we get

$L^{(2)}(t,\tau) = \theta_2(\tau)cosRt + \Omega_2(\tau)sinRt + \frac{P_{18}}{R^2} + \frac{P_{19}cost}{R^2-1} + \frac{P_{20}cos2t}{R^2-4}$
$-\frac{P_{21}\cos(1+R)\,t}{2R+1} - \frac{P_{22}\sin(1+R)\,t}{2R+1} + \frac{P_{23}\cos(1-R)\,t}{2R-1} + \frac{P_{24}\sin(1-R)\,t}{2R-1}$
$+\frac{P_{25}\cos Qt}{R^2-Q^2} + \frac{P_{26}sinQt}{R^2-Q^2} + \frac{P_{27}\cos(Q-R)t}{Q(2R+Q)} - \frac{P_{28}\sin(Q-R)t}{Q(2R+Q)}$
$+\frac{P_{29}\cos(Q-R)t}{Q(2R-Q)} + \frac{P_{30}\sin(Q-R)t}{Q(2R-Q)}$ (4.79)

where

$\theta_2(0) = M_{312}S\bar{\xi}_2 + M_{315}\bar{\xi}_1\bar{\xi}_2$(4.80a)

$M_{312} = -\frac{M_{182}}{R^2} - \frac{M_{192}}{R^2-1} - \frac{M_{202}}{R^2-4} + \frac{M_{212}}{2R+1} - -\frac{M_{232}}{2R-1}$ (4.80b)

$M_{315} = -\frac{M_{185}}{R^2} - \frac{M_{195}}{R^2-1} - \frac{M_{215}}{2R+1} + \frac{M_{235}}{2R-1} — \frac{M_{255}}{R^2-Q^2}$
$+\frac{M_{275}}{Q(2R+Q)} - \frac{M_{295}}{Q(2R-Q)}$ (4.80c)

$\Omega_2(0) = 0$(4.80d)

where$R \neq 1,\ R \neq 2,\ R \neq \frac{1}{2}\ ,\ R \neq Q,\ R \neq \frac{Q}{2}$.

So far, we write

$\xi_1^{(l)} = A^{(1)}\epsilon + A^{(2)}\epsilon^2 + \ldots$, $\xi_2^{(l)} = L^{(1)}\epsilon + L^{(2)}\epsilon^2 + \ldots$ (4.81)

4.2 Maximum Displacement

In order to determine the dynamic buckling loadλ_D, we shall first determine the maximum displacements of the buckling modes $\xi_1^{(l)}$ and $\xi_2^{(l)}$ and in either case, the condition for maximum displacement is

$\xi_{\alpha,t}^{(l)} + \epsilon\xi_{\alpha,\tau}^{(l)} = 0$, $\alpha = 1,2.$ (4.82)

We let $t_a^{(1)}$ and $\tau_a^{(1)}$ be the values of t and τ respectively at maximum displacement of $\xi_1^{(l)}$, while $t_a^{(2)}$ and $\tau_a^{(2)}$ are similar values for $\xi_2^{(l)}$.

We expand $t_a^{(1)}$ and $\tau_a^{(1)}$ asymptotically as

$t_a^{(1)} = t_0^{(1)} + \epsilon t_1^{(1)} + \epsilon^2 t_2^{(1)} + \cdots$ (4.83a)

$\tau_a^{(1)} = \epsilon t_a^{(1)} = \epsilon[t_0^{(1)} + \epsilon t_1^{(1)} + \epsilon^2 t_2^{(1)} + \cdots]$ (4.83b)

Now, for α=1 in (4.82), we get

$\xi_{1,t}^{(l)} + \epsilon\xi_{1,\tau}^{(l)} = 0$ (4.84)

which is evaluated at $(t_a^{(1)}, \tau_a^{(1)})$.

If we substitute for $\xi_1^{(l)}$ from (4.81) in (4.84) and equate the coefficient of powers of ϵ, we get the following, evaluated at $(t_0^{(1)}, 0)$

O(ϵ): $A_{,t}^{(1)}(t_0^{(1)}, 0) = 0$ (4.85a)

O(ϵ^2): $t_1^{(1)} A_{,tt}^{(1)} + t_0^{(1)} A_{,t\tau}^{(1)} + A_{,t}^{(2)} = 0$ (4.85b)

etc.

From (4.85a), we get, after simplification

$Qsint_0^{(1)} - \sin Q t_0^{(1)} = 0$ (4.86a)

For Q small, equation (4.86) easily yields

$t_0^{(1)} \approx Q$ (4.86b)

From (4.85b), we get

$t_1^{(1)} = \frac{-t_0^{(1)} A_{,t\tau}^{(1)} + A_{,t}^{(2)}}{A_{,tt}^{(1)}}$ (4.87)

where, we have substituted for $t_0^{(1)}$ in (4.87).

The maximum value of $\xi_1^{(l)}$, namely $\xi_{1a}^{(l)}$ is obtained by evaluating (4.81) at $(t_a^{(1)}, \tau_a^{(1)})$ which is achieved by using (4.83a,b). i.e.

$\xi_{1a}^{(l)} = \xi_1^{(l)}(t_a^{(1)}, \tau_a^{(1)}) = \epsilon A^{(1)} + \epsilon^2\left(t_1^{(1)} A_{,t}^{(1)} + t_0^{(1)} A_{,\tau}^{(1)} + A^{(2)}\right) + \cdots$ (4.88)

which is evaluated at $(t_0^{(1)}, 0)$.

Here we get

$A^{(1)}(t_0^{(1)}, 0) = \frac{\bar{\xi}_1}{Q^2(Q^2-1)}[cosQ^2 + (Q^2 - 1) - Q^2 cosQ]$ (4.89)

$A_{,t}^{(1)}(t_0^{(1)}, 0) = 0$ (4.90)

$A_{,\tau}^{(1)}(t_0^{(1)}, 0) = \frac{\bar{\xi}_1[1+\bar{\xi}_1 k_1(1+\beta)]sinQ^2}{2Q^3(Q^2-1)}$ (4.91)

$$A^{(2)}(t_0^{(1)}, 0) = [R_1\bar{\xi}_1 + R_2\bar{\xi}_2 + R_3 k_1(Q\bar{\xi}_1)^2 + R_4 k_2(QS\bar{\xi}_2)^2]cosQ^2$$
$$+\frac{1}{Q^2}\{M_{11}\bar{\xi}_1 + M_{12}\bar{\xi}_2 + M_{13}k_1(Q\bar{\xi}_1)^2 - M_{14}k_2(QS\bar{\xi}_2)^2\}$$
$$+\{M_{21}\bar{\xi}_1 + M_{22}\bar{\xi}_2 + M_{23}k_1(Q\bar{\xi}_1)^2 + M_{24}k_2(QS\bar{\xi}_2)^2\}\frac{cosQ}{Q^2-1}$$
$$+\{M_{31}\bar{\xi}_1 + M_{33}k_1(Q\bar{\xi}_1)^2 + M_{34}k_2(QS\bar{\xi}_2)^2\}\frac{\cos(2Q)}{Q^2-4}$$
$$-\{M_{43}k_1(Q\bar{\xi}_1)^2\}\frac{\cos(2Q^2)}{3Q^3} - \{M_{61}\bar{\xi}_1 + M_{63}k_1(Q\bar{\xi}_1)^2\}\frac{\cos[Q(Q+1)]}{2Q+1}$$
$$+\{M_{81}\bar{\xi}_1 + M_{83}k_1(Q\bar{\xi}_1)^2\}\frac{\cos[Q(Q-1)]}{2Q-1} + \{M_{104}k_2(QS\bar{\xi}_2)^2\}\frac{\cos(2RQ)}{Q^2-R^2}$$
$$+\{M_{134}k_2(QS\bar{\xi}_2)^2\}\frac{\cos(2RQ)}{Q^2-4R^2} + \{M_{144}k_2(QS\bar{\xi}_2)^2\}\frac{\cos[Q(1-R)]}{Q^2-(1-R)^2}$$
$$+\{M_{164}k_2(QS\bar{\xi}_2)^2\}\frac{\cos[Q(1+R)]}{Q^2-(1+R)^2} \quad (4.92)$$

After a re-arrangement of terms in (4.88), using (4.89) – (4.92), we get

$\xi_{1a}^{(l)} = \epsilon\bar{\xi}_1\xi_{1a}^{(1)} + \epsilon^2\left[\bar{\xi}_1\xi_{1a}^{(2)} + \bar{\xi}_2\xi_{1a}^{(3)} + k_1\bar{\xi}_1^{\,2}\xi_{1a}^{(4)} + k_2\bar{\xi}_2^{\,2}\xi_{1a}^{(5)}\right] + \cdots$ (4.93a)

where

$\xi_{1a}^{(1)} = \frac{1}{Q^2(Q^2-1)}[cosQ^2 + (Q^2 - 1) - Q^2 cosQ]$ (4.93b)

$$\xi_{1a}^{(2)} = \frac{\sin Q^2}{2Q^2(Q^2-1)} + R_1\cos Q^2 + \frac{M_{11}}{Q^2} + \frac{M_{21}cosQ}{Q^2-1} + \frac{M_{31}\cos 2Q}{Q^2-4}$$
$$-\frac{M_{61}\cos[Q(Q+1)]}{2Q+1} + \frac{M_{81}\cos[Q(Q-1)]}{2Q-1} \quad (4.93c)$$

$$\xi_{1a}^{(3)} = R_2 + \frac{M_{12}}{Q^2} + \frac{M_{22}\cos Q}{Q^2-1} \quad (4.93d)$$

$$\xi_{1a}^{(4)} = \frac{(1+\beta)\sin Q^2}{2Q^2(Q^2-1)} + R_3 Q^2 \cos Q^2 + M_{13}Q^2 + \frac{M_{23}Q^2\cos Q}{Q^2-1} + \frac{M_{33}Q^2\cos 2Q}{Q^2-4}$$
$$- \frac{M_{43}\cos 2Q^2}{3} - \frac{M_{63}Q^2\cos[Q(Q+1)]}{2Q+1} - \frac{M_{83}Q^2\cos[Q(Q-1)]}{2Q-1} \quad (4.93e)$$

$$\xi_{1a}^{(5)} = R_4(QS)^2\cos Q^2 - M_{14}S^2 + \frac{M_{24}(QS)^2 cosQ}{Q^2-1} + \frac{M_{34}(QS)^2\cos(2Q)}{Q^2-4}$$
$$+ \frac{M_{104}(QS)^2\cos(QR)}{Q^2-R^2} + \frac{M_{134}(QS)^2\cos(2RQ)}{Q^2-4R^2} + \frac{M_{144}(QS)^2\cos[Q(1-R)]}{Q^2-(1-R)^2}$$
$$+ \frac{M_{164}(QS)^2\cos[Q(1+R)]}{Q^2-(1+R)^2} \quad (4.93f)$$

To determine the maximum displacement, $\xi_{2a}^{(l)}$ of $\xi_2^{(l)}$, we let

$$t_a^{(2)} = t_0^{(2)} + \epsilon t_1^{(2)} + \epsilon^2 t_2^{(3)} + \cdots \quad (4.94a)$$
$$\tau_a^{(2)} = \epsilon t_a^{(2)} = \epsilon[t_0^{(2)} + \epsilon t_1^{(2)} + \epsilon^2 t_2^{(3)} + \cdots] \quad (4.94b)$$

From (4.82) and for α=2, the condition for maximum displacement is

$$\xi_{2,t}^{(l)} + \xi_{2,\tau}^{(l)} = 0 \quad (4.95)$$

On substituting (4.94a,b) into (4.95) and equating coefficients of powers of ϵ, we get

$O(\epsilon)$: $L_{,t}^{(1)}(t_0^{(2)},0) = 0$

(4.96a) $O(\epsilon^2)$: $\quad t_1^{(2)}L_{,tt}^{(1)}(t_0^{(2)},0) + t_0^{(2)}L_{,t\tau}^{(1)}(t_0^{(2)},0) + L_{,t}^{(2)}(t_0^{(2)},0) = 0 \quad (4.96b)$

From (4.96a), we get

$$Rsint_0^{(2)} - sinRt_0^{(2)} = 0 \quad (4.97a)$$

This yields

$$t_0^{(2)} \approx R. \quad (4.97b)$$

From (4.96b), we get

$$t_1^{(2)} = \frac{1}{L_{,tt}^{(1)}}(t_0^{(2)}L_{,t\tau}^{(1)} + L_{,t}^{(2)}) \quad (4.97c)$$

The maximum displacement $\xi_{2a}^{(l)}$ is now obtained by evaluating the second equation in (4.81) at $(t_a^{(2)},\tau_a^{(2)})$ and using (4.94b). This gives

$$\xi_{2a}^{(l)} = \xi_2^{(l)}(t_a^{(2)},\tau_a^{(2)}) = \epsilon L^{(1)} + \epsilon^2(t_1^{(2)}L_{,t}^{(1)} + t_0^{(2)}L_{,\tau}^{(1)} + L^{(2)}) + \cdots \quad (4.98)$$

which is evaluated at $(t_0^{(2)},0)$.

Here we have

$$L^{(1)}(t_0^{(2)},0) = \frac{S\bar{\xi}_2}{R^2(R^2-1)}[cosR^2 + (R^2-1) - R^2 cosR] \quad (4.99)$$
$$L_{,t}^{(1)}(t_0^{(2)},0) = 0 \quad (4.100)$$
$$L_{,\tau}^{(1)}(t_0^{(2)},0) == \frac{(1+\beta)(1-\bar{\xi}_1)S\bar{\xi}_2}{2Q^2R(R^2-1)} + \frac{RS\bar{\xi}_1\bar{\xi}_2}{2Q^2(Q^2-1)} \quad (4.101)$$

$$L^{(1)}(t_0^{(2)},0) = (M_{312}S\bar{\xi}_2 + M_{315}\bar{\xi}_1\bar{\xi}_2)cosR^2 + \frac{1}{R^2}(M_{182}S\bar{\xi}_2 + M_{185}\bar{\xi}_1\bar{\xi}_2)$$
$$+(M_{192}S\bar{\xi}_2 + M_{195}\bar{\xi}_1\bar{\xi}_2)\frac{cosR}{R^2-1} + (M_{202}S\bar{\xi}_2)\frac{\cos(2R)}{R^2-4}$$
$$-(M_{212}S\bar{\xi}_2 + M_{215}\bar{\xi}_1\bar{\xi}_2)\frac{\cos\{(1+R)R\}}{2R+1}$$
$$+(M_{232}S\bar{\xi}_2 + M_{235}\bar{\xi}_1\bar{\xi}_2)\frac{\cos\{(1-R)R\}}{2R-1} + (M_{255}\bar{\xi}_1\bar{\xi}_2)\frac{\cos(QR)}{R^2-Q^2}$$
$$-(M_{275}\bar{\xi}_1\bar{\xi}_2)\frac{\cos\{(Q+R)R\}}{Q(2R+Q)} + (M_{295}\bar{\xi}_1\bar{\xi}_2)\frac{\cos\{(Q-R)R\}}{Q(2R-Q)} \quad (4.102)$$

We can now write $\xi_{2a}^{(l)}$ as

$$\xi_{2a}^{(l)} = \epsilon S\bar{\xi}_2\xi_{2a}^{(1)} + \epsilon^2\bar{\xi}_1\bar{\xi}_2\xi_{2a}^{(2)} + \cdots \quad (4.103)$$

Where

$$\xi_{2a}^{(1)} = \frac{1}{R^2(R^2-1)}\{\cos(R^2) + (R^2-1) - R^2 cosR\}$$
$$+ \frac{1+\beta}{2Q^2R(R^2-1)}M_{312}cosR^2 + \frac{M_{182}}{R^2} + \frac{M_{192}cosR}{R^2-1} + \frac{M_{202}\cos(2R)}{2R-4}$$

$$-\frac{M_{212}\cos\{(1+R)R\}}{2R+1}+\frac{M_{232}\cos\{(1-R)R\}}{2R-1} \quad (4.104)$$

$$\xi_{2a}^{(2)}=\frac{-(1+\beta)}{2Q^2R(R^2-1)}+\frac{R}{2Q^2(Q^2-1)}+M_{315}cosR^2+\frac{M_{185}}{R^2}+\frac{M_{195}cosR}{R^2-1}$$
$$-\frac{M_{215}\cos\{(1+R)\,R\}}{2R+1}+\frac{M_{235}\cos\{(1-R)R\}}{2R-1}+\frac{M_{255}\cos\,(QR)}{R^2-Q^2}$$
$$-\frac{M_{275}\cos\{(Q+R)R\}}{Q(2R+Q)}+\frac{M_{295}\cos\{(Q-R)R\}}{Q(2R-Q)} \quad (4.105)$$

The net maximum displacement ζ_a is

$$\zeta_a=f_1\epsilon+f_1\epsilon^2+\cdots \quad (4.106)$$

where

$$f_1=\bar{\xi}_1\xi_{1a}^{(1)}+S\bar{\xi}_2\xi_{2a}^{(1)} \quad (4.107)$$

$$f_2=\bar{\xi}_1\xi_{1a}^{(2)}+\bar{\xi}_2\xi_{1a}^{(3)}+k_1\bar{\xi}_1{}^2\xi_{1a}^{(4)}+k_2\bar{\xi}_2{}^2\xi_{1a}^{(5)}+S\bar{\xi}_1\bar{\xi}_2\xi_{2a}^{(2)}. \quad (4.108)$$

Equation (4.106) is similar to the second equation of (3.16a), where we have ignoredc_3, and so the maximization (2.7) yields an equation similar to (3.22) and reads:

$$\epsilon=\frac{1}{4}\left(\frac{f_1}{f_2}\right) \quad (4.109)$$

On substituting in (4.109) for f_1 and f_2 from (4.107) and (4.108), we get

$$\lambda_D\left(\frac{\omega_1}{\omega_0}\right)^2=\frac{\frac{1}{4}\left(\bar{\xi}_1\xi_{1a}^{(1)}+S\bar{\xi}_2\xi_{2a}^{(1)}\right)}{\bar{\xi}_1\xi_{1a}^{(2)}+\bar{\xi}_2\xi_{1a}^{(3)}+k_1\bar{\xi}_1{}^2\xi_{1a}^{(4)}+k_2\bar{\xi}_2{}^2\xi_{1a}^{(5)}+S\bar{\xi}_1\bar{\xi}_2\xi_{2a}^{(2)}} \quad (4.110)$$

Using (3.23), we can relate the dynamic buckling load λ_D to the static buckling load λ_S and get

$$\frac{\lambda_D}{\lambda_S}\left(\frac{\omega_1}{\omega_0}\right)^2=\frac{\left(\bar{\xi}_1\xi_{1a}^{(1)}+S\bar{\xi}_2\xi_{2a}^{(1)}\right)\left(\bar{\xi}_1+k_1\bar{\xi}_1{}^2-k_2\bar{\xi}_2{}^2+\bar{\xi}_2{}^2-\bar{\xi}_1\bar{\xi}_2\right)}{\left(1+\bar{\xi}_1+\bar{\xi}_2{}^2\right)\left(\bar{\xi}_1\xi_{1a}^{(2)}+\bar{\xi}_2\xi_{1a}^{(3)}+k_1\bar{\xi}_1{}^2\xi_{1a}^{(4)}+k_2\bar{\xi}_2{}^2\xi_{1a}^{(5)}+S\bar{\xi}_1\bar{\xi}_2\xi_{2a}^{(2)}\right)} \quad (4.111)$$

The results (4.110) and (4.111) are asymptotic and are valid for $0<\frac{\omega_1}{\omega_0}<1$.

Meanwhile, the results (4.110) and (4.111) are carefully arranged so as to highlight the contributions of each of the terms in the governing equations (2.3) – (2.5) to the buckling process. For example, the coefficients of $\bar{\xi}_1$ and $\bar{\xi}_2$ in (4.110) and (4.111) are the contributions of the coupling terms $\xi_1\xi_0$ and $\xi_2\xi_0$ in (2.4) and (2.5) respectively. Similarly the coefficients of $\bar{\xi}_1{}^2$ and $\bar{\xi}_2{}^2$ in (4.110) and (4.111) are the contributions of $k_1\bar{\xi}_1{}^2$ and $-k_2\bar{\xi}_2{}^2$ respectively, while the coefficients of $\bar{\xi}_1\bar{\xi}_2$ is the contribution of the coupling term $\xi_1\xi_2$ in (2.5).

Thus, if we set $\bar{\xi}_1=0$, the result from (4.111) becomes

$$\frac{\lambda_D}{\lambda_S}\left(\frac{\omega_1}{\omega_0}\right)^2=\frac{-S\bar{\xi}_2\xi_{2a}^{(1)}k_2\bar{\xi}_2{}^2}{\left(1+\bar{\xi}_2{}^2\right)\left(\bar{\xi}_2\xi_{1a}^{(3)}+k_2\bar{\xi}_2{}^2\xi_{1a}^{(5)}\right)}. \quad (4.112)$$

We thus observe that:

(i) There is no contribution from the coupling term $\xi_1\xi_0$.

(ii) There is no contribution from the nonlinear term $k_1\bar{\xi}_1{}^2$.

(iii) There is no contribution from the coupling between the buckling modes, namely $\xi_1\xi_2$.

(iv) The only nonlinear term initiating buckling is $k_2\bar{\xi}_2{}^2$.

(v) By neglecting $\bar{\xi}_1$, we have automatically neglected the effect of the nonlinear term $k_1\bar{\xi}_1{}^2$.

We recall that this was Danielson's problem for the case of step loading, when he neglected both $\bar{\xi}_1$ and $k_1\bar{\xi}_1{}^2$ on the assumption that it was the effects of coupling between the buckling modes, i.e. the term $\xi_1\xi_2$ that dominated the buckling process. Danielson obtained the following result:

$$\frac{\lambda_D}{\lambda_S}=\frac{\frac{1}{6}\left\{4-\left(\frac{\omega_1}{\omega_0}\right)^2\right\}}{\frac{\lambda_S}{\lambda_C}+\left(\frac{32}{27}\right)^{\frac{1}{2}}\left(\frac{10}{9}\right)\left(\frac{\omega_1}{\omega_0}\right)\left(\frac{\omega_2}{\omega_0}\right)\left(1-\frac{\lambda_S}{\lambda_C}\right)^2} \quad (4.113)$$

where $\lambda_C$is the classical buckling load. Our equivalent result can be obtained from (4.112) by setting β=0, (wherever it occurs).

Similarly, if we set $\bar{\xi}_2=0$ (assuming that axisymmetric imperfections dominate the buckling problem), we get the following result from (4.111)

$$\frac{\lambda_D}{\lambda_S}\left(\frac{\omega_1}{\omega_0}\right)^2=\frac{\left(\bar{\xi}_1\xi_{1a}^{(1)}\right)\left(\bar{\xi}_1+k_1\bar{\xi}_1{}^2\right)}{\left(1+\bar{\xi}_1\right)\left(\bar{\xi}_1\xi_{1a}^{(2)}+k_1\bar{\xi}_1{}^2\xi_{1a}^{(4)}\right)} \quad (4.114)$$

From (4.114), we observe the following:

(i) The effects of the coupling term $\xi_2\xi_0$ is absent.

(ii) The effects of the nonlinear term $k_2\bar{\xi}_2{}^2$ is also absent.

(iii) The effects of the coupling term $\xi_1\xi_2$ is absent.

(iv) The only nonlinear term initiating buckling is the term $k_1\bar{\xi}_1{}^2$.

From the foregoing, we conclude that for the effects of the coupling between the buckling modes to be felt (i.e. the effects of $\xi_1\xi_2$), the imperfections in the shapes of the modes coupling must not be neglected, that is, we must retain both $\bar{\xi}_1$ and $\bar{\xi}_2$.

We note that neglecting both $\bar{\xi}_1$ and $k_1\bar{\xi}_1{}^2$ is superfluous (Danielson, 1969), because, by simply neglecting $\bar{\xi}_1$ we automatically neglect the of $k_1\bar{\xi}_1{}^2$.

A similar observation holds for $\bar{\xi}_2$ and $k_2\bar{\xi}_2{}^2$.

5.0 Conclusion

The analysis here can easily be extended to cover other loading histories apart from step load. Though, we have limited our investigation to the case where the load amplitude during the pre-loading and the step loading application are the same, the analysis can be extended to the cases where we intermittently prescribe and change the pre-loading load amplitude.

6.0 References

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