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On the Effect of Lumped Mass Matrices on the Dynamic Response of Non-Uniform Beams Under Moving Loads

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Abstract

The effect of the Lumped Mass Matrices (LMM) on the dynamic response of non-uniform beams (NUB) under uniformly distributed moving loads is studied. The analysis was done by using the finite element numerical method. The material properties, such as the flexural rigidity and the cross-sectional area of the beams were assumed to be functions of the spatial variable x. Also, the elements stiffness, mass and centripetal acceleration matrices as well as the load vectors were derived using the Galerkin's Weighted Residual Method (GWRM). The Lumped Mass Matrices (LMM) for the problem was obtained using Archer's principle. The dynamic responses of non-uniform beams under uniformly distributed moving loads were obtained using the Newmark's numerical method. Furthermore, the dynamic responses of non-uniform beams (NUB) under moving loads using Consistent Mass Matrices (CMM) and Lumped Mass Matrices (LMM) were also compared. It was discovered that, the developed Lumped Mass matrix for the problem considered in this work, is unique and satisfies the general requirements of matrix lumping which does not require the inversion of the assembled mass matrix.

Keywords: Lumped Mass Matrices, dynamic response, non-uniform beams, Archer's principle

1.0 Introduction

Since the beginning of the last decade, there has been increasing in research in geotechnical activities across the globe in order to cater for ever increasing world population which requires continuous investments in the various transportation systems. In other word, several researchers including engineers, mathematicians and other scientists have employed various methods, in which finite element method is not an exception, to analyse the behaviour of structures under traveling loads. Foundation for research in this area was laid by Stokes [1], Wills [2], Timoshenko [3], Inglis [4] and other eminent scholars.
Analytical and numerical methods, including finite element method, have been employed by numerous scientists in analyzing problems of the dynamic responses of structures under the influence of different kinds of moving loads using consistent mass matrix which was credited to Archer [5]. The lumped- mass idealization provides a simple means of limiting the number of the degree of freedom to be considered in conducting a dynamic analysis of a structural system. The lumping procedure is mostly effective in treating systems in which a large proportion of the total mass is concentrated at discrete points [6].
Later, Wu et al [7] studied the dynamic responses of multi-span non-uniform beams under moving load using the transfer matrix method analysis to solve the moving load problem employed consistent mass matrix. Thambiratnam et al [8] also used the finite element method to investigate the free vibration of variable thickness thin beams supported on elastic foundations. Dugush et al [9] investigated the dynamic behaviour of multi-span non-uniform beams traversed by a moving load at constant and variable velocities in which both modal analysis and direct integration methods were used in their analyses.
The cost optimization of singly and doubly reinforced concrete beams was treated by Barros et al[10]. They developed a model for the optimal design of rectangular reinforced concrete with emphasis on economic bending moment, optimal area of steel, and optimal steel ratio between upper and lower steel. The modeling of uncertainty in the dynamic response of marine

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riser using probabilistic finite element technique was carried out by Olga [11] where the probabilistic solutions are compared with deterministic solutions for the same riser systems as published by the American Institute. Abiala [12] carried out the evaluation of the dynamic responses of beams under uniformly distributed moving loads in which the material properties, throughout the length of the beam under consideration are assumed to be prismatic. Recently, Abiala et al[13] investigated the effect of Lumped Mass Matrices LMM on the dynamic responses of beams under moving loads. The beams considered have problem geometry similar to the one studied in [12]

Finally, in this paper, the work in [13] is extended to include non-uniform beams. In this present work, the material properties, such as the flexural rigidity and the cross-sectional area of the beams were assumed to be functions of the spatial variable x. The approach used is similar to the one in [13], while the finite element method was employed to obtain the dynamic response of beams under uniformly distributed moving loads, and the responses were obtained using Newmark's integration method [14]. The graphical findings and comparisons between uniform beams (UB) and non-uniform beams (NUB), consistent mass matrices (CMM) and Lumped Mass Matrices (LMM) were presented.

2.0 Mathematical Problem Statement

An equation of a non-uniform Euler-Bernoulli beam carrying a load moving at a specified speed can be modeled as[7, 9, 13]:

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}]+\rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}=q(x,t) \quad (2.1)$$

Where y, x, $EI(x)$, $A(x)$, $q(x,t)$, t, and ρ are the transverse displacement, the spatial coordinate, the flexural rigidity, the cross-sectional area of the beam, the externally applied pressure loading, time, and the mass density per unit-length area of the beam respectively.

The following are the associated boundary conditions:

$$y(0,t)=y(l,t)=0$$

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=l}=0 \quad (2.2)$$

For moving load, $q(x,t)$, which in this work is assumed to be uniformly distributed , we have:

$$q(x,t)=\frac{1}{\varepsilon}[-pg-p(\frac{\partial^2 y}{\partial t^2}+2v\frac{\partial^2 y}{\partial x\partial t}+v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)] \quad (2.3)$$

Where ε is the load's length, ξ is the distance covered by the moving load, V is the moving speed of the load, P is the load (Figures 1a and1b), and $H(x)$ is the heavy-side function.

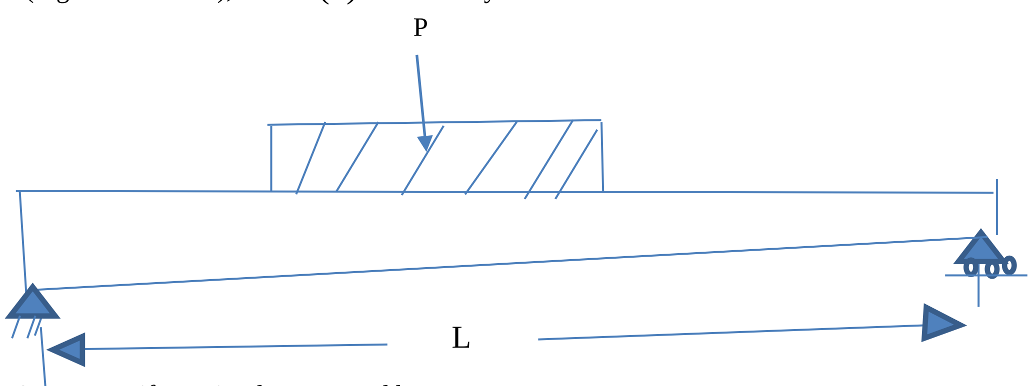

Figure 1a: Non-uniform simply supported beam

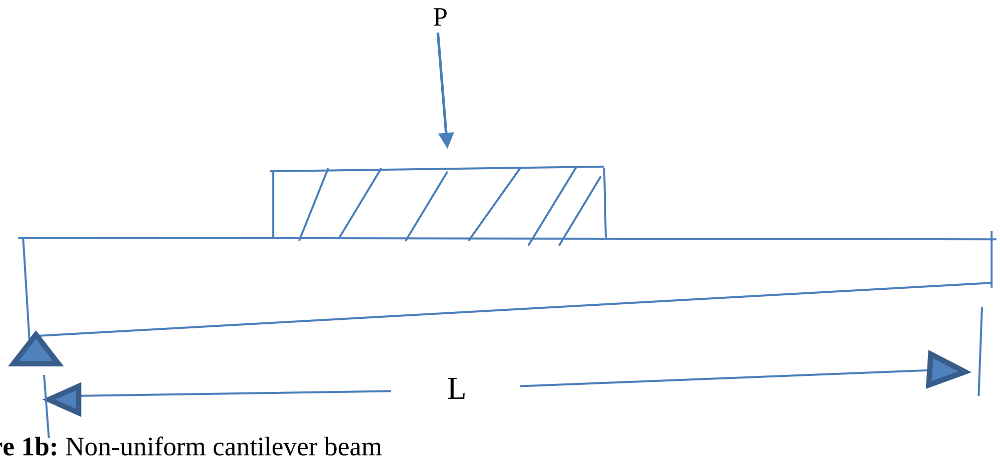

Figure 1b: Non-uniform cantilever beam

Using (2.3) in (2.1), we have:

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}]+\rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}=$$

$$1/\varepsilon[-pg-p(\frac{\partial^2 y}{\partial t^2}+2v\frac{\partial^2 y}{\partial x\partial t}+v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)] \quad (2.4)$$

we define the flexural rigidity $EI(x)$ and the mass density area $A(x)$ respectively as follows:

$$EI(x)=\sum_{r=1}^{nspan} EI_r(x-\sum_{i=1}^{r-1} L_i)[H(x-\sum_{i=1}^{r-1} L_i)-H(x-\sum_{i=1}^{r} L_i)] \quad (2.5)$$

$$A(x)=\sum_{r=1}^{nspan} A_r(x-\sum_{i=1}^{r-1} L_i)[H(x-\sum_{i=1}^{r-1} L_i)-H(x-\sum_{i=1}^{r} L_i)] \quad (2.6)$$

We remark at this juncture, that $EI(x)$ and $A(x)$ as defined above are similar to those in [9, 13].

The associated initial conditions are :

$$y(x,0)=\frac{\partial y(x,0)}{\partial t}=0 \quad (2.7)$$

Thus the initial-boundary value problem describing the behaviour of a non-uniform beam traversed by uniformly distributed moving load is governed by equations (2.2), (2.4), (2.5), (2.6), and (2.7) respectively.

The closed-form solution of the above initial-boundary value problem is either impossible or very difficult to obtain using analytical approach, hence, we employ finite element method.

3.0 The Finite Element Formulation of the Problem

The formulation of non-uniform beam element equation is similar to that of the element with uniform materials [7, 9]. Hence, we applied GWRM to equation (2.4), Rearranging and integrating twice the first term on the left-hand side, applying standard descretization approach [15] and using the approach in [16], we obtained the finite element in matrix form as

[K]{y}+[C]{ý}+[M]{ÿ}={F} (3.1)

Where,

$$[K]=\sum_{i=1}^{n}\{\int_\Omega EI(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 R}{\partial x^2}dx+pv^2/\varepsilon\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x^2}Rdx\} \quad (3.2)$$

$$[M]=\sum_{i=1}^{n}\left\{\int_\Omega \rho A(x)\frac{\partial^2 y}{\partial t^2}Rdx+p/\varepsilon\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial t^2}Rdx\right\} \quad (3.3)$$

$$[C]=\sum_{i=1}^{n}\{2pv/\varepsilon\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x\partial t}Rdx\} \quad (3.4)$$

$$\{F\}=\sum_{i=1}^{n}\{ \ -pg\big/_{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} Rdx+Q^{e} \ \} \tag{3.5}$$

4.0 Specification or Introduction of Shape Functions

By using Hermittian interpolation functions [9, 12, 15,17, 18] to interpolate the transverse displacement, Residual function and their derivatives in the above equations, therefore, from equation (3.2), we have.

$$[K_{ij}^{e}]=\begin{bmatrix} K_{11}^{e} & K_{12}^{e} & K_{13}^{e} & K_{14}^{e} \\ K_{21}^{e} & K_{22}^{e} & K_{23}^{e} & K_{24}^{e} \\ K_{31}^{e} & K_{32}^{e} & K_{33}^{e} & K_{34}^{e} \\ K_{41}^{e} & K_{42}^{e} & K_{43}^{e} & K_{44}^{e} \end{bmatrix} \tag{4.1}$$

Such that

$$K_{11}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{36}{l^4}-\frac{144x}{l^5}+\frac{144x^2}{l^6}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{6\eta}{l^2}+\frac{6\eta^3}{l^4}-\frac{12\eta^4}{l^5}+\frac{6\eta^2}{l^3}+\frac{24\eta^5}{5l^6})-(-\frac{6\mu}{l^2}+\frac{6\mu^3}{l^4}-\frac{12\mu^4}{l^5}+\frac{6\mu^2}{l^3}+\frac{24\mu^5}{5l^6})\ ]$$

$$K_{12}^{e}=K_{21}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{24}{l^3}-\frac{84x}{l^4}+\frac{72x^2}{l^5}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{3\eta^2}{l^2}+\frac{8\eta^3}{l^3}-\frac{15\eta^4}{l^4}+\frac{12\eta^5}{5l^5})-(-\frac{3\mu^2}{l^2}+\frac{8\mu^3}{l^3}-\frac{15\mu^4}{l^4}+\frac{12\mu^5}{5l^5})\ ]$$

$$K_{13}^{e}=K_{31}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ -\frac{36}{l^4}+\frac{144x}{l^5}-\frac{144x^2}{l^6}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{6\eta^3}{l^4}+\frac{12\eta^4}{l^5}-\frac{24\eta^5}{5l^6})-(-\frac{6\mu^3}{l^4}+\frac{12\mu^4}{l^5}-\frac{24\mu^5}{5l^6})\ ]$$

$$K_{14}^{e}=K_{41}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{12}{l^3}-\frac{6x}{l^4}+\frac{72x^2}{l^5}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (\frac{2\eta^3}{l^3}-\frac{9\eta^4}{2l^4}+\frac{12\eta^5}{5l^5})-(\frac{2\mu^3}{l^3}-\frac{9\mu^4}{2l^4}+\frac{12\mu^5}{5l^5})\ ]$$

$$K_{22}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{16}{l^2}-\frac{48x}{l^3}+\frac{36x^2}{l^4}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{2\eta^2}{l}+\frac{14\eta^3}{3l^2}-\frac{4\eta^4}{l^3}+\frac{6\eta^5}{5l^4})-(-\frac{2\mu^2}{l}+\frac{14\mu^3}{3l^2}-\frac{4\mu^4}{l^3}+\frac{6\mu^5}{5l^4})\ ]$$

$$K_{23}^{e}=K_{32}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}EI_r(x-\sum_{e=1}^{r-1}L_e)\}[\ -\frac{24}{l^3}+\frac{84x}{l^4}-\frac{72x^2}{l^5}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{4\eta^3}{l^3}+\frac{13\eta^4}{2l^4}-\frac{12\eta^5}{5l^5})-(-\frac{4\mu^3}{l^3}+\frac{13\mu^4}{2l^4}-\frac{12\mu^5}{5l^5})\ ]$$

$$K_{24}^e = K_{42}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} EI_r (x - \sum_{e=1}^{r-1} L_e)\}[\ \frac{8}{l^2} - \frac{36x}{l^3} + \frac{36x^2}{l^4}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (\frac{4\eta^3}{3l^2} - \frac{5\eta^4}{2l^3} + \frac{6\eta^5}{5l^4}) - (\frac{4\mu^3}{3l^2} - \frac{5\mu^4}{2l^3} + \frac{6\mu^5}{5l^4})\ ]$$

$$K_{33}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} EI_r (x - \sum_{e=1}^{r-1} L_e)\}[\ \frac{36}{l^4} - \frac{144x}{l^5} + \frac{144x^2}{l^6}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (\frac{6\eta^3}{l^4} - \frac{12\eta^4}{l^5} + \frac{24\eta^5}{5l^6}) - (\frac{6\mu^3}{l^4} - \frac{12\mu^4}{l^5} + \frac{24\mu^5}{5l^6})\ ]$$

$$K_{34}^e = K_{43}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} EI_r (x - \sum_{e=1}^{r-1} L_e)\}[\ -\frac{12}{l^3} + \frac{60x}{l^4} - \frac{72x^2}{l^5}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (-\frac{2\eta^3}{l^3} + \frac{9\eta^4}{2l^4} - \frac{12\eta^5}{5l^5}) - (-\frac{2\mu^3}{l^3} + \frac{9\mu^4}{2l^4} - \frac{12\mu^5}{5l^5})\ ]$$

$$K_{44}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} EI_r (x - \sum_{e=1}^{r-1} L_e)\}[\ \frac{4}{l^2} - \frac{24x}{l^3} + \frac{36x^2}{l^4}\ ]dx$$

$$+\frac{pv^2}{\varepsilon}[\ (\frac{2\eta^3}{3l^2} - \frac{2\eta^4}{l^3} + \frac{6\eta^5}{5l^4}) - (\frac{2\mu^3}{3l^2} - \frac{2\mu^4}{l^3} + \frac{6\mu^5}{5l^4})\ ]$$

$$[M_{ij}^e] = \begin{bmatrix} M_{11}^e & M_{12}^e & M_{13}^e & M_{14}^e \\ M_{21}^e & M_{22}^e & M_{23}^e & M_{24}^e \\ M_{31}^e & M_{32}^e & M_{33}^e & M_{34}^e \\ M_{41}^e & M_{42}^e & M_{43}^e & M_{44}^e \end{bmatrix} \tag{4.2}$$

$$M_{11}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} \rho A_r (x - \sum_{e=1}^{r-1} L_e)\}[\ 1 - \frac{6x^2}{l^2} + \frac{4x^3}{l^3} + \frac{9x^4}{l^4} - \frac{12x^5}{l^5} + \frac{4x^6}{l^6}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\eta - \frac{2\eta^3}{l^2} + \frac{\eta^4}{l^3} + \frac{9\eta^5}{5l^4} - \frac{2\eta^6}{l^5} + \frac{4\eta^7}{7l^6}) - (\mu - \frac{2\mu^3}{l^2} + \frac{\mu^4}{l^3} + \frac{9\mu^5}{5l^4} - \frac{2\mu^6}{l^5} + \frac{4\mu^7}{7l^6})\ ]$$

$$M_{12}^e = M_{21}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} \rho A_r (x - \sum_{e=1}^{r-1} L_e)\}[\ x - \frac{2x^2}{l} - \frac{2x^3}{l^2} + \frac{3x^4}{l^2} - \frac{7x^5}{l^4} + \frac{2x^6}{l^5}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{\eta^2}{2} - \frac{2\eta^3}{3l} - \frac{\eta^4}{2l^2} + \frac{8\eta^5}{5l^3} + \frac{\eta^6}{6l^4} + \frac{2\eta^7}{7l^5}) - (\frac{\mu^2}{2} - \frac{2\mu^3}{3l} - \frac{\mu^4}{2l^2} + \frac{8\mu^5}{5l^3} + \frac{\mu^6}{6l^4} + \frac{2\mu^7}{7l^5})\ ]$$

$$M_{13}^e = M_{31}^e = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} \rho A_r (x - \sum_{e=1}^{r-1} L_e)\}[\ \frac{3x^2}{l^2} - \frac{2x^3}{l^3} - \frac{9x^4}{l^4} + \frac{12x^5}{l^5} - \frac{4x^6}{l^6}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{\eta^3}{l^2} - \frac{\eta^4}{2l^3} - \frac{9\eta^5}{5l^4} + \frac{2\eta^6}{l^5} - \frac{4\eta^7}{7l^6}) - (\frac{\mu^3}{l^2} - \frac{\mu^4}{2l^3} - \frac{9\mu^5}{5l^4} + \frac{2\mu^6}{l^5} - \frac{4\mu^7}{7l^6})\ ]$$

$$M_{14}^{e}=M_{41}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ -\frac{x^3}{l}+\frac{x^3}{l^2}+\frac{3x^4}{l^3}-\frac{5x^5}{l^5}+\frac{2x^6}{l^3}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (-\frac{\eta^2}{3l}-\frac{\eta^4}{4l^2}+\frac{3\eta^5}{5l^5}-\frac{5\eta^6}{6l^4}+\frac{2\eta^7}{7l^5})-(-\frac{\mu^2}{3l}-\frac{\mu^4}{4l^2}+\frac{3\mu^5}{5l^5}-\frac{5\mu^6}{6l^4}+\frac{2\mu^7}{7l^5})\ ]$$

$$M_{22}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ x^2-\frac{4x^3}{l}+\frac{6x^4}{l^2}-\frac{4x^5}{l^3}+\frac{x^6}{l^4}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{\eta^3}{3}-\frac{\eta^4}{l}+\frac{6\eta^5}{5l^2}-\frac{2\eta^6}{3l^3}+\frac{\eta^7}{7l^4})-(\frac{\mu^3}{3}-\frac{\mu^4}{l}+\frac{6\mu^5}{5l^2}-\frac{2\mu^6}{3l^3}+\frac{\mu^7}{7l^4})\ ]$$

$$M_{23}^{e}=M_{32}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{3x^3}{l^2}-\frac{8x^4}{l^3}+\frac{7x^5}{l^4}-\frac{2x^6}{l^5}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{3\eta^4}{4l^2}-\frac{8\eta^5}{5l^3}+\frac{7\eta^6}{6l^4}-\frac{2\eta^7}{7l^5})-(\frac{3\mu^4}{4l^2}-\frac{8\mu^5}{5l^3}+\frac{7\mu^6}{6l^4}-\frac{2\mu^7}{7l^5})\ ]$$

$$M_{24}^{e}=M_{42}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ -\frac{x^3}{l}+\frac{3x^4}{l^2}-\frac{3x^5}{l^3}+\frac{x^6}{l^4}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (-\frac{\eta^4}{4l}+\frac{3\eta^5}{5l^2}-\frac{\eta^6}{2l^3}+\frac{\eta^7}{7l^4})-(-\frac{\mu^4}{4l}+\frac{3\mu^5}{5l^2}-\frac{\mu^6}{2l^3}+\frac{\mu^7}{7l^4})\ ]$$

$$M_{33}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{9x^4}{l^4}-\frac{12x^5}{l^5}+\frac{4x^4}{l^5}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{9\eta^5}{5l^4}-\frac{2\eta^6}{l^5}+\frac{4\eta^7}{7l^6})-(\frac{9\mu^5}{5l^4}-\frac{2\mu^6}{l^5}+\frac{4\mu^7}{7l^6})\ ]$$

$$M_{34}^{e}=M_{43}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ -\frac{3x^4}{l^3}+\frac{5x^5}{l^4}-\frac{2x^6}{l^5}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (-\frac{3\eta^5}{5l^3}+\frac{5\eta^6}{6l^4}-\frac{2\eta^7}{7l^5})-(-\frac{3\mu^5}{5l^3}+\frac{7\mu^6}{6l^4}-\frac{2\mu^7}{7l^5})\ ]$$

$$M_{44}^{e}=\int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e}\{\sum_{r=1}^{nspan}\rho A_r(x-\sum_{e=1}^{r-1}L_e)\}[\ \frac{x^4}{l^2}-\frac{2x^5}{l^3}+\frac{x^6}{l^4}\ ]dx$$

$$+\frac{p}{\varepsilon}[\ (\frac{\eta^5}{5l^2}-\frac{\eta^6}{3l^3}+\frac{\eta^7}{7l^4})-(\frac{\mu^5}{5l^2}-\frac{\mu^6}{3l^3}+\frac{\mu^7}{7l^4})\ ]$$

The mass matrix in equation (4.2) is called the consistent mass matrix [5]. It is obvious that the above matrix obtained for this problem satisfies the mass matrices properties for verification and debugging, that is matrix symmetry, physical symmetries, conservation of linear momentum and positive definiteness. This matrix has been used extensively, in different forms, by various researchers. In this paper, the consistent mass matrix is diagonalized in order to obtain the lumped mass matrix used for the analysis carried out.

5.0 Construction of the Lumped Mass Matrix

It is pertinent to mention here that the consistent matrix $[M^e]$ derived from the weighted-integral formulation of the governing differential equation is symmetric, positive-definite, and nondiagonal. The solution of the global equations associated with the consistent mass matrices requires inversion of the assembled mass matrices. If the mass matrix is diagonal, then, the assembled equations can be solved explicitly, thus saving the computational time. The explicit nature of the lumped mass matrix motivates the analysts to find rational ways of diagonalizing the mass matrix. In this paper, we employed the procedures used in [5] to construct the required Lumped Mass Matrices. Such that

$$\beta = \sum_{i=1}^{n} M_{ii}^{e}, \quad M_{ii}^{e} = \begin{cases} \neq 0 & if \ i \ is \ odd \\ =0, & if \ i \ is \ even \end{cases} \tag{5.1}$$

$$M_{ii}^{e^*} = \rho A \frac{M_{ii}^{e}}{\beta} \tag{5.2}$$

$$M_{ij}^{e} = \begin{cases} \neq 0 & if \ i=j \\ =0 & if \ i \neq j \end{cases} \tag{5.3}$$

Which yields

$$[M^{e^*}] = \begin{bmatrix} M_{11}^{e^*} & 0 & 0 & 0 \\ 0 & M_{22}^{e^*} & 0 & 0 \\ 0 & 0 & M_{33}^{e^*} & 0 \\ 0 & 0 & 0 & M_{44}^{e^*} \end{bmatrix} = \frac{\rho A}{\beta} \begin{bmatrix} M_{11}^{e} & 0 & 0 & 0 \\ 0 & M_{22}^{e} & 0 & 0 \\ 0 & 0 & M_{33}^{e} & 0 \\ 0 & 0 & 0 & M_{44}^{e} \end{bmatrix} \tag{5.4}$$

where

$$\beta = M_{11}^{e} + M_{33}^{e} = \int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} \rho A_r (x - \sum_{e=1}^{r-1} L_e)\}[\ 1 - \frac{6x^2}{l^2} + \frac{4x^3}{l^3} + \frac{9x^4}{l^4} - \frac{12x^5}{l^5} + \frac{4x^6}{l^6}\ ]dx$$

$$+ \frac{p}{\varepsilon}[\ (\eta - \frac{2\eta^3}{l^2} + \frac{\eta^4}{l^3} + \frac{9\eta^5}{5l^4} - \frac{2\eta^6}{l^5} + \frac{4\eta^7}{7l^6}) - (\mu - \frac{2\mu^3}{l^2} + \frac{\mu^4}{l^3} + \frac{9\mu^5}{5l^4} - \frac{2\mu^6}{l^5} + \frac{4\mu^7}{7l^6})\ ] +$$

$$\int_{\sum_{e=1}^{r-1} L_e}^{\sum_{e=1}^{r} L_e} \{\sum_{r=1}^{nspan} \rho A_r (x - \sum_{e=1}^{r-1} L_e)\}[\ \frac{9x^4}{l^4} - \frac{12x^5}{l^5} + \frac{4x^4}{l^5}\ ]dx$$

$$+ \frac{p}{\varepsilon}[\ (\frac{9\eta^5}{5l^4} - \frac{2\eta^6}{l^5} + \frac{4\eta^7}{7l^6}) - (\frac{9\mu^5}{5l^4} - \frac{2\mu^6}{l^5} + \frac{4\mu^7}{7l^6})\ ]$$

Equation (5.4) is the developed Lumped Mass matrix for the problem considered in this work, which is unique and satisfies the general requirements of matrix lumping. Using this in place of the consistent matrix does not require the inversion of the assembled mass matrix.

6.0 Numerical Examples

A simply supported structural beam element was modeled(Fig.2a in order to illustrate the procedure employed in this paper. The total length of the beam, L=10m, the mass density per beam length $\rho = 7.04gm^3$,the beam's element area $A = 20m^2$, and the load's length $\varepsilon = 0.5m$.

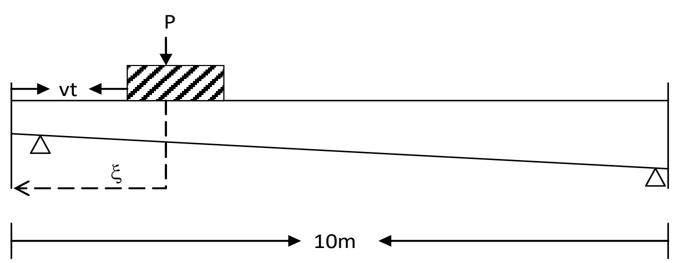

Fig. 2a: A non-uniform beam under moving load.

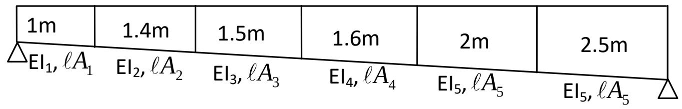

Fig. 2b: A discretized non- uniform beam under moving load.

The beam element is discretized into 6 non-uniform element model (Fig.2b) with the length of each element given as $L_1 = 1m$, $L_2 = 1.4m$, $L_3 = 1.5m$, $L_4 = 1.6m$, $L_5 = 2m$, $L_6 = 2.5m$, and the flexural rigidities $EI_1 = 2.7728\times10^5 Nm$, $EI_2 = 3.9947\times10^5 Nm$, $EI_3 = 8.2858\times10^5 Nm$, $EI_4 = 2.6179\times10^6 Nm$, $EI_5 = 6.3936\times10^6 Nm$, $EI_6 = 9.3936\times10^6 Nm$, while $A_{11} = 2m^2$, $A_{12} = 2.8m^2$, $A_{13} = 3m^2$, $A_{14} = 3.2m^2$, $A_{15} = 4m^2$, $A_{61} = 5m^2$. To obtain the effect of the velocity on the dynamic response of non-uniform beam elements to moving loads using the derived Lumped Mass Matrices (LMM), the velocity is varied from 3m/s to 9m/s. Similarly, the responses at different sizes of the load's length , and span- length of the beam element were also presented, while a comparison between the responses of non-uniform simply supported using the Consistent Mass Matrices(CMM) and Lumped Mass Matrices(LMM) is also discussed. The solutions of problems of non-uniform beams under moving loads formed the basis for this comparison, which led to the following additional conclusions:

***(a) Effects of velocity on the dynamic response of the non-uniform beam*:**

The effect of increasing in velocity on the dynamic response of non-uniform simply supported beam under distributed moving load using the Consistent Mass Matrices(CMM) is shown in figure3. It shows that for the initial velocity V_0 smaller than a certain value**,** denoted by $V_0^/$, the value of the deflections(y) increases with increasing in velocity. However, for $V_0 > V_0^/$, the foregoing trend just reverses, the critical value of the initial velocity for this problem is $V_0^/ = 5m/s$, while the reverse case is shown in figure4, the implication is that after exceeding the critical value of the velocity, the deflections decreases as the velocity increases.

(b) Effects of load's length:

In order to investigate the influence of the load's length on the dynamic response of non-uniform beam having the same properties as those of the one in Figure 3, but with $\varepsilon = 0.5$, $\varepsilon = 0.7$, $\varepsilon = 0.9$ respectively were studied. This shows that the deflections(y) increases with increasing in load's length as described in Figure5.

***(c) Effects of the span-length of the beam element* :**

Furthermore, the span- lengths of $L = 10m$, $16m$ and $22m$ of the beam elements were used to study the influence of the span-length on the dynamic response of non-uniform beam having the same physical properties as those in figure3. It was observed that the deflections increases with increasing in the span- length of the beam, this is shown in figure6.

(d) Effects of changing in boundary conditions:

However, if the boundary condition is changed from simply supported type to a cantilever one, the behavioural pattern of the responses is in other way round (Figure7). That is, the deflections(y) decreases with increasing in velocity after exceeding the critical value of the velocity $V_0^/ = 5m/s$ (Figure 8). In Figure 9, it shows that for cantilever beam, the amplitude increases with increases in the load's length which is in conformity with that of simply supported beam. Furthermore, if the span length of the beam is increased, the amplitude also increases just like in the simply supported case, this is shown in figure 10.

7.0 Conclusion

In this research work, the effect of the Lumped Mass Matrix (LMM) on the dynamic response of non-uniform structural beams under uniformly distributed moving loads is studied. The dynamic response of beams subjected to uniformly distributed moving loads using finite element method, employing the Newmark's β numerical technique for the evaluation of the resulted equations in order to obtain the effects of velocity of the moving load and load's length on the response of beams. The velocities of the moving loads and load's length have significant effects on the dynamic response of non-uniform beams under uniformly distributed moving loads. In addition to the fact that the results obtained, for the effects of velocity of the moving loads, is in agreement with those in the existing literatures, most especially, the work in [13].

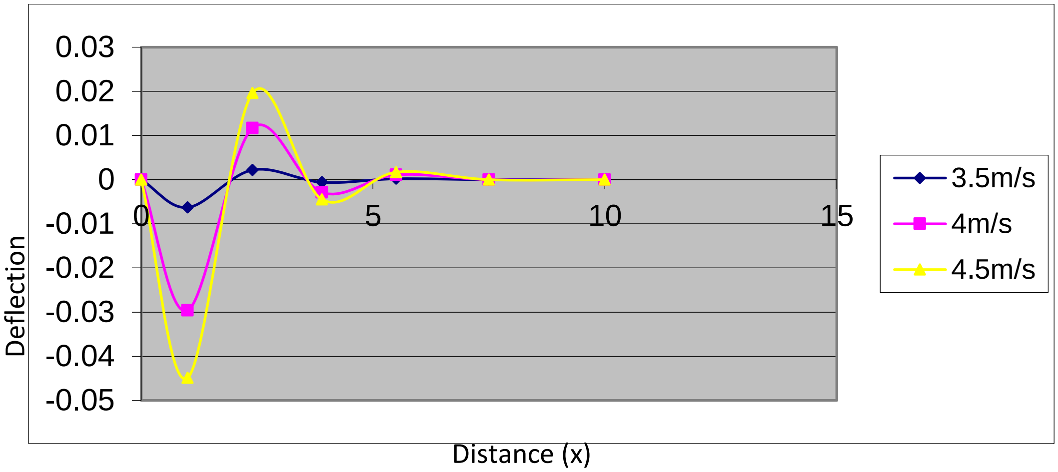

Figure 3: Effect of increasing in velocity on the dynamic response of a non-uniform simply supported beam using CMM.

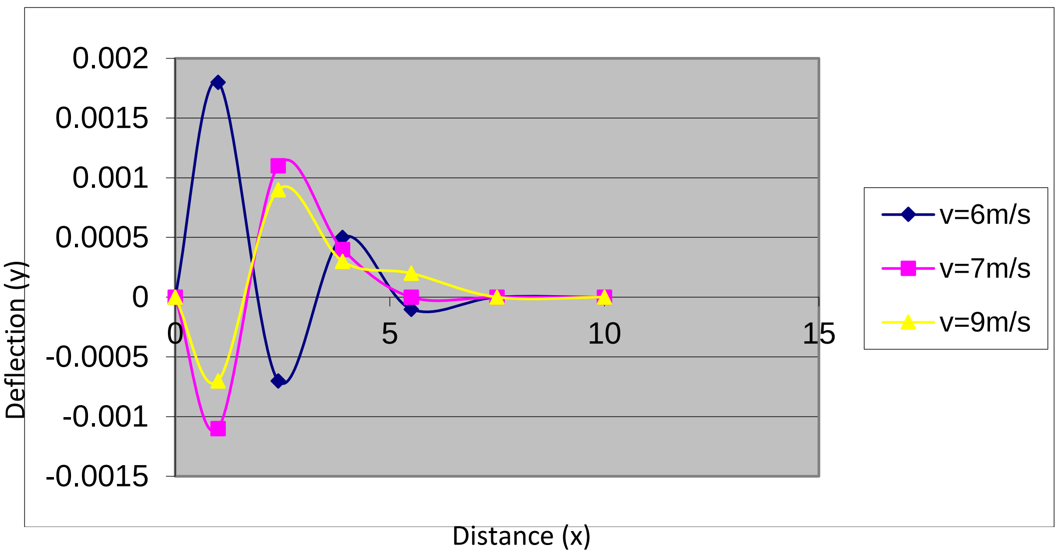

Figure 4: Effect of exceeding the critical value of the velocity on the dynamic response of non-uniform simply supported beam under moving load using CMM

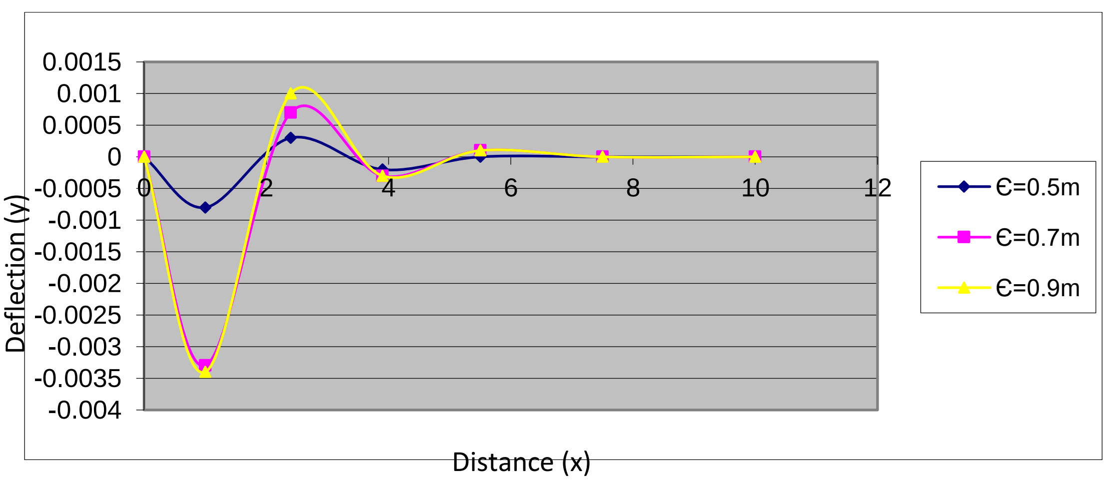

Figure 5: Effect of increasing in load's length on the dynamic response of non-uniform simply supported beam under moving load using CMM

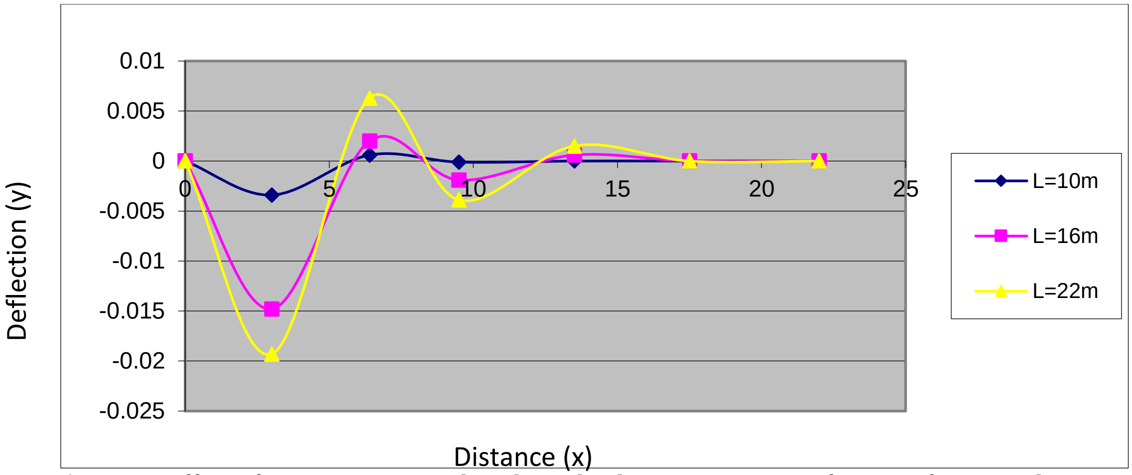

Figure 6: Effect of increases in span-length on the dynamic response of non-uniform simply supported beam under moving load using CMM

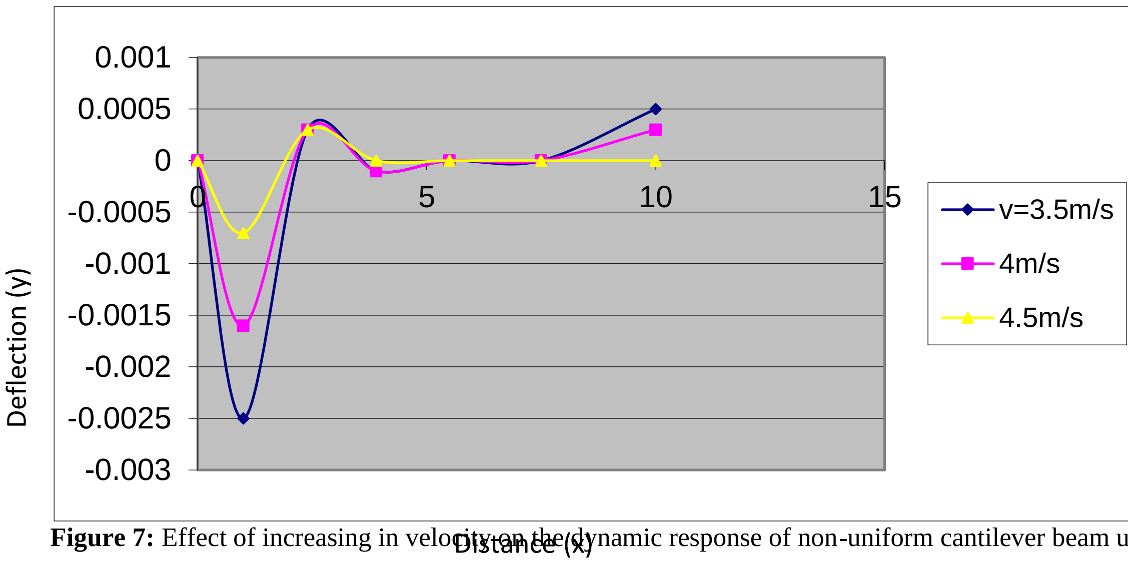

Figure 7: Effect of increasing in velocity on the dynamic response of non-uniform cantilever beam under moving load

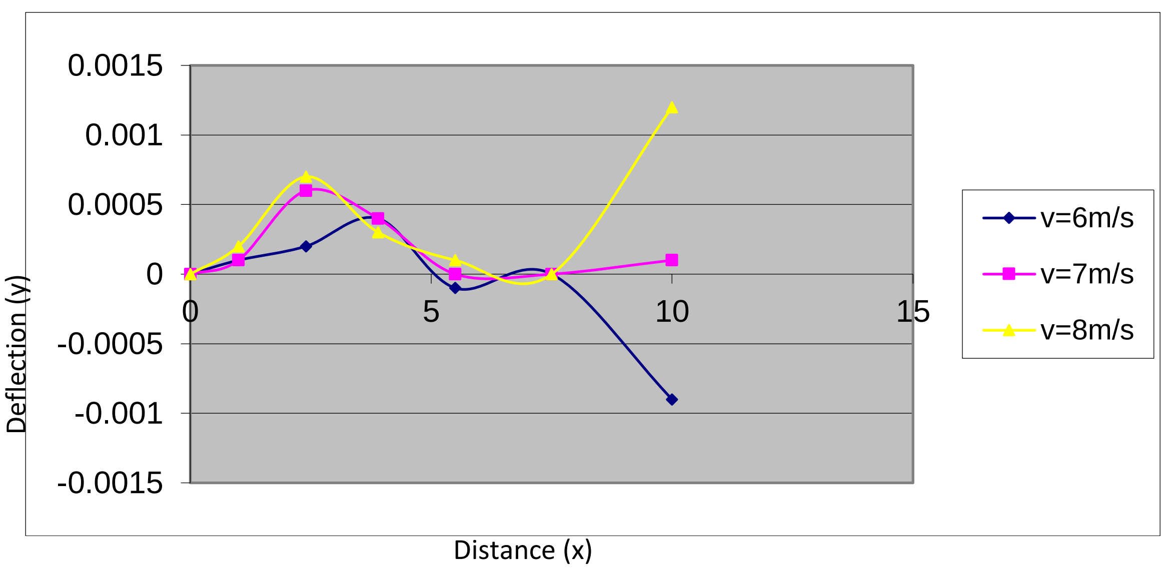

Figure 8: Effect of exceeding critical value of the velocity on the response of cantilever beam

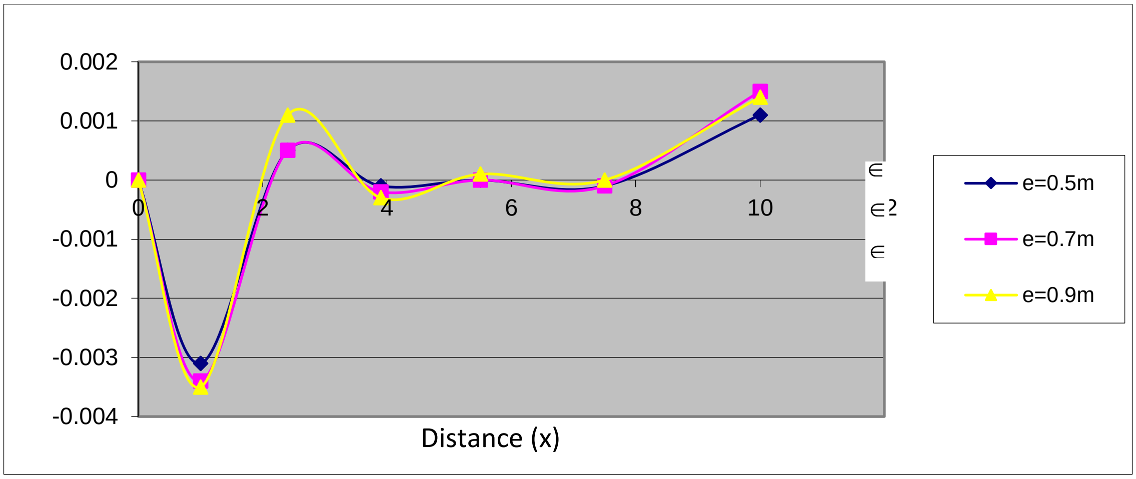

Figure 9: Effects of increases in load's length on the dynamic response of the cantilever beam

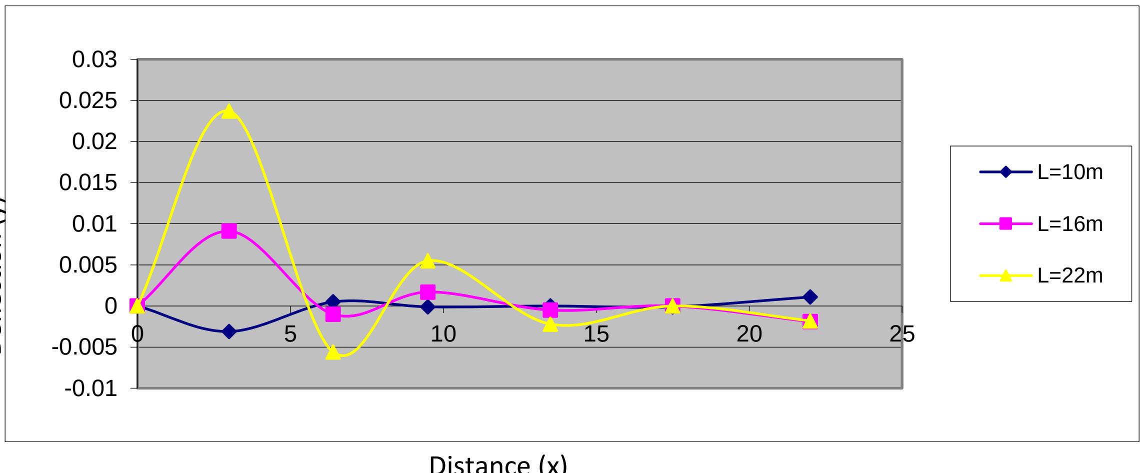

Figure 10: Effects of increases in the length of the beam on the dynamic response of cantilever beam.

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