Transactions of the Nigerian Association of Mathematical Physics
Volume 2, (November, 2016), **pp 65 – 70**
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On the Convergence of Variational Iteration Method for the Numerical Solution of Nonlinear Integro-Differential Equations

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Abstract

The methodology of Variational iteration method has gone viral in the solution of integro-differential equations in recent years as available in literature. The quest of convergence of the method has not been explicit. This paper majors in deriving sufficient conditions of convergence of the method. The derived sufficient conditions of convergence of VIM was experimented on a selected problem of first order nonlinear integro-differential equations for experimentation and the result obtained shows that the condition obtained are indeed sufficient.

Keywords: Mean-value theorem, Schwartz inequality, Hilbert space, Lagrange multiplier, linearization, perturbation

1.0 Introduction

Much interest in the solution of integro-differential equations has been vented by many authors due to its applicability in many areas of science and engineering, especially in areas like electric circuit, inhibitoryand exhibitory interactions, damping laws, diffusion processes, etc, [1]. Most conventional integro-differential equations solvers do not have analytic solutions.Thus, numerical techniques and approximations become imperative. Solvers such as the Adomian Decomposition Method (ADM) [2], Laplace Adomian Decomposition Method (LADM) [3], Differential Transform Method (DTM) [4], Homotopy Perturbation Method (HPM) [5], The Tau method [6], Variational Iteration Method (VIM), [7],etc, has been standard methods in physics and mathematics for the numerical solution of integro-differential equations (of order either integer or fractional) via linearization or perturbation methods.

The VIM was first proposed in 1998 by He [8]. The method was proved effective in solving linear and nonlinear integro-differential equations by several authors. He [8] applied VIM for autonomous ordinary differential system. Also, He [9] applied the VIM to nonlinear problems and its applications. Similarly, Saberi-Nadjafi et al [10] employed the VIM for the numerical solution of systems of integro-differential equations. Also, Matinfar et al [11] worked on the solution of systems of integro-differential equations by VIM.

In this paper, the convergence analysis of the VIM is investigated. This investigation was instigated by the work of Batiha et al. [12] who stated an open question on the convergence analysis of the VIM. Thus, we proved the convergence analysis of VIM with some derived sufficient conditions. Consequently, section 2 presents the basic methodology of VIM and the construction of the correction functional for the stated integro-differential equations. Section 3 offers numerical stimulation on two selected nonlinear integro-differential equations of first order. Also, approximate solutions obtained here are compared with existing results in the literature. Section 4 presents a critical thought on the convergence of VIM on the given integro-differential equations. Consequently, the convergence analysis is implemented on example 1 as an illustration. Finally, the conclusion and open questions for further improvement of this work are presented.

2.0 Variational Iteration Method

To illustrate thismethod, we consider the nonlinear differential equation of the form:

$$Ly + Ny = r(x) \tag{1}$$

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with prescribed auxiliary conditions, where y is unknown function, L is a linear operator and occurs to the highest order derivative, N is the non linear term, and r(x) is a non-homogeneous term.

VIM is a method based on the general Lagrange multiplier scheme [12]. The main innovation is by a linearization process used as a trial function in the solution of the stated problem. Thus, more precise approximations are obtained at some discreet point.

Thus we can construct correction functional for equation (1) as:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda\left[Ly_k(\xi) + N\tilde{y}_k(\xi) - r(\xi)\right]d\xi,\ k \geq 0 \tag{2}$$

where λ is a general Lagrange multiplier, $\tilde{y}_k = 0$, i.e, $\tilde{y}_k$ is a restricted variable.

Let an integro-differential equation be given as:

$$u'(x) = \omega(x) + F(t) \tag{3}$$

where

$$F(t) = \int_0^x f(t, y(t), y'(t)dt\,.$$

We make successive approximation for the approximate Lagrange multiplier for equation (3) as follows:

Let the correction functional be given as

$$\delta y_{k+1}(x) = \delta[y_k(x) + \int_0^x \lambda(s)\left[(y_k)_s(s) - \omega - F_s(t)\right]ds] \tag{4}$$

where

$$F_s(t) = \int_0^x f(t, \tilde{y}(t), y'(t)dt$$

with $\delta\tilde{y}_k = 0$.

$\lambda(s)$ is required in equation (4). Thus, $\delta\omega(x)$ is linearized together with $\delta\tilde{y}_k = 0$ such that equation(4) becomes

$$\delta y_{k+1}(x) = \delta[y_k(x) + \int_0^x \lambda(s)\left[(y_k)_s(s)\right]ds] \tag{5}$$

Integrating equation (5) by part, and finding $\left.\dfrac{\partial y_{k+1}(x)}{\partial y_k(x)}\right|_{s=t} = 0$, $\dfrac{\partial y_{k+1}(x)}{\partial y'_k(s)} = 0$, we obtain

$$\lambda(s)\big|_{s=t} = 1 \quad \text{and} \quad \lambda'(s) = 0 \tag{6}$$

Substituting (6) into equation (4), we obtain

$$y_{k+1}(x) = y_k(x) - \int_0^x \left[(y_k)_s(s) - \omega - F_s(t)\right]ds \tag{7}$$

Now, the general structure of the scheme after successive approximations for $k \geq 0$ of equation (7) is as follows:

Let the initial approximation of equation (7) be given as $y_0(x) = F(u_0)$. Then subsequent approximations for $k \geq 0$ of equation (7) can as analyzedas:

Given

$$F(x_0 + h) = f(x_0) + hf'(x_0) + h\frac{f''(x_0)}{2!} + \ldots$$

It implies that

$$y_1(x) = F(u_0 + u_1)$$

Similarly,

$$y_2(x) = F(u_0 + u_1 + u_2)$$

$$y_3(x) = F(u_0 + u_1 + u_2 + u_3)$$

$$y_4(x) = F(u_0 + u_1 + u_2 + u_3 + u_4)$$
$$\vdots$$
$$y_k(x) = F(s_k),\ k \geq 1,$$

where $s_k = \sum_{i=0}^{k} u_i$ is *n*- component truncated series.

3.0 Numerical Experiments

To illustrate the effectiveness of the proposed method, we demonstrate the method with two numerical examples.

Example 1

Consider the nonlinear integro-differential equation [3]

$$u'(x) = -1 + \int_0^x u^2(t)dt \quad 0 \leq x \leq 1,\quad u(0) = 0. \tag{8}$$

Let our initial approximationbe$y_0(x) = -x$. The iteration formula for equation (8) is:

$$y_{k+1}(x) = y_k(x) - \int_0^x [(y_k)_s(s) - \omega(x) - F_s(t)]ds \tag{9}$$

where $\omega(x) = -1$ and $F_s(t) = \int_0^x u^2(t)dt$.

Thus the first four iterates are easily obtained from equation (9) via MAPLE 18 software:

$$y_1 := -x + \frac{1}{12}x^4$$

$$y_2 := -x + \frac{1}{12}x^4 + \frac{1}{12960}x^{10} + \frac{1}{252}x^7$$

$$y_3 := -x + \frac{1}{12}x^4 + \frac{1}{6048}x^{10} + \frac{1}{252}x^7 + \frac{1}{77598259200}x^{22} - \frac{1}{558472320}x^{19} + \frac{109}{914457600}x^{16} - \frac{37}{7076160}x^{13}$$

$$\begin{aligned} y_4 := & -x + \frac{1}{12}x^4 + \frac{1}{6048}x^{10} + \frac{1}{252}x^7 + \frac{34109}{170758496563200}x^{22} \\ & - \frac{4799}{677613081600}x^{19} + \frac{2663}{11887948800}x^{16} - \frac{1}{157248}x^{13} \\ & + \frac{1}{1246448394990169620480000 0}x^{46} - \frac{1}{391328412985769656320000}x^{43} \\ & + \frac{241247}{59943018919370607820800000}x^{40} - \frac{1709}{405316465646329036800000}x^{37} \\ & + \frac{253524431}{76477009517988426547200000}x^{34} - \frac{1312457}{6288693911429529600000}x^{31} \\ & + \frac{24354871}{22152431455745310720 0}x^{28} - \frac{4507}{901604861853696}x^{25} \end{aligned}$$

Table 1: Results obtained using VIM are comparedwith J. Manafianheris [3] experiment of LADM for example 1.

1	Second-order approximation		Third-order approximation		Fourth-order approximation	
X	VIM	LADM	VIM	LADM	VIM	LADM
0.0000	0.0000000000	0.00000000	0.0000000000	0.00000000	0.0000000000	0.000000000
0.0938	-0.0937935492	-0.09379355	-0.0937935492	-0.09379355	-0.0937935492	-0.09379355
0.2188	-0.2186091064	-0.2186091	-0.2186091064	-0.2186091	-0.2186091064	-0.2186091
0.3125	-0.3117064256	-0.3117065	-0.3117064248	-0.3117065	-0.3117064248	-0.3117065
0.4062	-0.4039385242	-0.4039385	-0.4039385134	-0.4039385	-0.4039385135	-0.4039385
0.5000	-0.4948225933	-0.4948227	-0.4948225078	-0.4948225	-0.4948225080	-0.4948225
0.6250	-0.6124314696	-0.6124321	-0.6124306791	-0.6124306	-0.6124306816	-0.6124306
0.7188	-0.6969446250	-0.6969474	-0.6969414490	-0.6969413	-0.6969414640	-0.6969414
0.8125	-0.7771007409	-0.7771104	-0.7770900316	-0.7770897	-0.7770901040	-0.7770901
0.9062	-0.8519653870	-0.8519942	-0.8519338831	-0.8519325	-0.8519341765	-0.8519343
1.0000	-0.9205577601	-0.9206350	-0.9204746881	-0.9204697	-0.9204757191	-0.9204761

Example 2:
Consider the nonlinear integro-differential equation [12]

$$u'(x) = 1 + \int_0^x u(t)\frac{du(t)}{dt}dt, \quad 0 \le x \le 1, u(0) = 0. \qquad (10)$$

Let the initial approximation be $u_0 = x$. The first three iterates are using easily obtained from equation (10) via MAPLE 18 software:

$$u_1 := x + \frac{1}{12}x^4$$

$$u_2 := x + \frac{1}{12}x^4 + \frac{1}{2016}x^8 + \frac{1}{90}x^6$$

$$u_2 = x + \frac{1}{12}x^4 + \frac{1}{1243662336}x^{18} + \frac{1}{21772800}x^{16} + \frac{187}{165110400}x^{14}$$

$$+ \frac{1}{71280}x^{12} + \frac{1}{110880}x^{11} + \frac{1}{12960}x^{10} + \frac{1}{3240}x^9 + \frac{1}{252}x^7$$

Resultsare presented in **table 2**. Results obtained with the VIM were compared with results obtained in Batiha et al. [12] VIM using 3-iterate solution for example 2.

Table 2: Results obtained are compared with Batiha et al. [12] VIM experiment of example 2.

X	Third-order VIM approximation	Batiha et al Third-order VIM approximation
0.0000	0.0000000000	0.00000000
0.0938	0.0938064513	0.0938065
0.2188	0.2189910846	0.2189913
0.3125	0.3132958930	0.3132982
0.4062	0.4084760508	0.4084907
0.5000	0.5052400214	0.5053032
0.6250	0.6378687824	0.6381768
0.7188	0.7414585409	0.7422988
0.8125	0.8498042408	0.8518520
0.9062	0.9645523437	0.9691418
1.0000	1.0877116170	1.0973681

4.0 Convergence Analysis

Here, we will study the convergence analysis of VIM applied to equation (3) of this present work. Let $H = ((a,b) \times [0,T]$, where H is Hilbert space. Also, let

$$y : H \to R$$

with

$$\int_H y^2(x,s)dsd\psi < +\infty.$$

Theorem 1 (**Sufficient Condition of Convergence**).

Define $u'(x)$ by $D(u)$ in equation (3) such that $D(u) = \omega(x) + \int_0^x f(t, y(t), y'(t))dt.$ Then,VIM convergesif the following are satisfied:

i. $(D(u) - D(v), u - v) \ge \tau\|u - v\|^2,\ \tau > 0,\ \ u, v \in H$

ii. For $\Omega > 0$, $\exists\ \mathrm{I}(\Omega) > 0$ such that $\|u\| \le \Omega$, $\|v\| \le \Omega$, $u, v \in H$, then

$$(D(u) - D(v), u - v) \ge \mathrm{I}(\Omega)\|u - v\|\|r\|,\ r \in H.$$

Proof

For $\tau > 0$, $u, v \in H$, we have that

$$(D(u) - D(v), u - v) = \int_0^x ((f(t, y(t), y'(t) - g(t, y(t), y'(t))dt, u - v),$$

where

$$D(v) = \omega(x) + \int_0^x g(t, y(t), y'(t))dt.$$

Applying the Schwartz inequality, we get

$$\int_0^x ((f(t,y(t),y'(t)-g(t,y(t),y'(t))dt,u-v) \le \tau_1 \| f(t,y(t),y'(t))-g(t,y(t),y'(t))\| \|u-v\|$$

By the conventional use of the mean value theorem we obtain

$$\int_0^x ((f(t,y(t),y'(t)-g(t,y(t),y'(t))dt,u-v) \ge \xi \|u-v\|^2$$, where $\xi = \frac{1}{2}\tau_1 \Omega^2$.

Hence,

$$(D(u)-D(v),u-v) \ge \tau \|u-v\|^2$$

holds with $\tau = \frac{1}{2}\tau_1\Omega^2$.

Similarly, for $\Omega > 0$, $\exists\ \mathrm{I}(\Omega) > 0$ such that $\|u\| \le \Omega$, $\|v\| \le \Omega$, $u,v \in H$, then

$$(D(u)-D(v),r) = \int_0^x ((f(t,y(t),y'(t)-g(t,y(t),y'(t))dt,r) \le \Omega^2 \|u-v\|\|r\|$$

$$= \mathrm{I}(\Omega)\|u-v\|\|r\|.$$

Thus, the second condition. This completes the proof.

Illustration with Example 1.

VIM applied to the nonlinear Voltera integro-differential equation as follows

$$D(u) = -1 + \int_0^x u^2(t)dt,$$

without initial conditions converges to a particular solution [3].

Proof

By theorem 1, we get

$$(D(u)-D(v),u-v) = \int_0^x ((u^2(t)-v^2(t))dt,u-v),\ \tau > 0,\ u,v \in H$$

By Schwartz inequality, we get

$$\int_0^x ((u^2(t)-v^2(t))dt,u-v) \le \tau_1 \|u-v\|^2 \|u-v\|.$$

Using the mean value theorem, we obtain

$$\int_0^x ((u^2(t)-v^2(t))dt,u-v) \ge \xi \|u-v\|^2,$$

where $\xi = \frac{1}{3}\tau_1\Omega^3$, $(u,v) \in \xi$, $\|u\| \le \Omega$, $\|v\| \le \Omega$.

Hence,

$$(D(u)-D(v),u-v) \ge \tau \|u-v\|^2$$

holds with $\tau = \frac{1}{3}\tau_1\Omega^3$.

Also, for $\Omega > 0$, $\exists\ \mathrm{I}(\Omega) > 0$ such that $\|u\| \le \Omega$, $\|v\| \le \Omega$, $u,v \in H$, then

$$(D(u)-D(v),r) = \int_0^x (u^2(t)-v^2(t))dt,r) \le \Omega^3 \|u-v\|\|r\| = \mathrm{I}(\Omega)\|u-v\|\|r\|.$$

Thus, the second condition.

5.0 Conclusion

In this paper, we have considered the convergence of VIM for the solution of nonlinear integro-differential equations. For this course, we derived sufficient conditions of convergence of the VIM and this was implemented on a selected first order Volterra integro-differential equation which shows convergence to a particular solution rapidly.

The quest of finding a mathematical scheme for the initial approximation is highly imperative for VIM. Thus, it is worth the attention of future researchers to critically consider this aspect for the numerical solution of first order and higher order nonlinear integro-differential equations.

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