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Certain Orthogonal Polynomials in Orthogonal Collocation Methods of Solving Fredholm Integro-Differential Equations (FIDEs)

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Abstract

This work is concerned with numerical solution of linear and nonlinear Fredholm integro-differential equations (FIDEs)using orthogonal collocation method.Here we constructed the orthogonal polynomials involved in the interval [0,1] with respect to the weight function w(x)=1. The orthogonal polynomials we constructed gave a good rate of convergence. The method is implemented on some selected problems for experimentation and the results obtained show that it is effective with excellent rate of convergence when compared with the exact solutions available in literature.

Keywords: Orthogonality, Orthogonal Collocation, FIDEs, integral equations

1.0 Introduction

This paper concerns the development of orthogonal collocation approach with orthogonal polynomial basis function for the numerical solution of linear and nonlinear FIDEs. The orthogonal collocation method has found extensive application in recent years presented in series of papers, for examples, in [1] for the case of line collocation for the solution of partial differential equations in two variables with collocation being applied in one variable, and [2] for the case of the numerical solution of ODEs . Collocation at the zeros of orthogonal polynomials was first investigated by Fox and Pascal [3] for the case of Chebychev orthogonal collocation method for the numerical solution of ordinary and partial differential equations together with integral equations.Whereas Adeniyi [4] engaged the well behaved Chebychev polynomial as basis function in a perturbed collocation approach. The orthogonal collocation method was developed by Villadsen et al [5]. It was then subsequently studied and applied to many chemical engineering problems by several authors [6 – 8]. However, the effectiveness of these integral solvers depends on the types of basis function used in developing the schemes. Various basis functions such as the Chebyshev polynomials Tn(x), the Legendre polynomial $P_n(x)$, the monomials x_r and the canonical polynomial ($Q_r(x)$, $r \geq 0$) of the Lanczos Tau method in a perturbed collocation approach have been employed for this purpose[9 - 12]. Much interest in the solution of integro-differential equations has been vented by many authors due to its applicability in many areas of science and engineering, especially in areas like electric circuit, inhibitory and exhibitory interactions, damping laws, diffusion processes. Several different approaches have been proposed in the literature to handle integro-differential equations such as the Variational iteration method (VIM), Adomian decomposition method (ADM), Homotopy perturbation method (HPM) and Differential transform method (DTM) etc. [13], which do not offer compact solution form.

The method of orthogonal collocation has played a significant role in the solution of integro-differential equations available in literature where Chebychev polynomials, Hermite polynomials, Lagrange polynomials, etc, are adopted as trial functions. In this paper, we construct orthogonal basis functions in the interval [0,1] with respect to the weight function w(x) = 1such that it is continuous, positive and converges rapidly. We employ the constructed polynomials as basis function in the approximation of the analytic solution. The proposed method is employed to solve linear and nonlinear FIDEs, in details, in section 3. Section 4 presents numerical experiments of linear and nonlinear FIDEs to verify the proposed method. The results of each numerical example indicate convergence and error analysis are discussed. Finally, the conclusion is presented in section 5.

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2.0 Derivations of Orthogonal Polynomials

Let $P_n(x)$ denotes the class of polynomials. Then, $P_n(x)$ is orthogonal if $P_n(x)$ as defined over a range [a,b] obey an orthogonality relation

$$\int_a^b \mathrm{w(x)}\varphi_m(x)\varphi_n(x)\mathrm{dx} = \mathrm{h_n}\delta_{\mathrm{mn}} \tag{1}$$

with the Kronecker delta δ_{mn} defined as

$$\delta_{\mathrm{mn}} = \begin{cases} 0, & \mathrm{m} \neq \mathrm{n} \\ 1, & \mathrm{m} = \mathrm{n} \end{cases}$$

where the weight function w(x) is continuous and positive on [a, b] such that the moments

$$\int_a^b \mathrm{w(x)}x^n\mathrm{dx}, \qquad \mathrm{n} = 0,1,2,3,\ldots \tag{2}$$

exist.

Then the integral,

$$\langle\varphi_m,\varphi_n\rangle = \int_a^b \mathrm{w(x)}\varphi_m(x)\varphi_n(x)\mathrm{dx} \tag{3}$$

denotes an inner product of polynomials $\varphi_m(x) and \varphi_n(x)$.

For orthogonality,

$$\langle\varphi_m,\varphi_n\rangle = \int_a^b \mathrm{w(x)}\varphi_m(x)\varphi_n(x)\mathrm{dx} = 0, \quad m \neq n, [0,1] \tag{4}$$

Mamadu and Njoseh [14] constructed their orthogonal polynomials in the interval [-1,1] with respect to the weight function $w(x) = 1 + x^2$. In this work, we adopt the weight function $w(x) = 1$ such that it is continuous, positive and converges rapidly on [0,1].The construction of φ_r, r = 1, 2, 3,… of the approximant:

$$\tilde{y}(x) = \sum_{r=0}^{n} a_r\varphi_r(x) \cong y(x) \tag{5}$$

now follows:

2.1 Construction of Orthogonal Basis Function

To construct the basis function, we use the additional property that

$\varphi_n(1) = 1$

where

$$\varphi_n(x) = \sum_{r=0}^{n} C_r^{(n)}x^r \tag{6}$$

satisfies the orthogonality property (4).

Thus, the first eight constructed orthogonal polynomials $\varphi_r(x)$; $\mathrm{r} = 0(1)8$ valid in [0,1] are implemented with the aid of MAPLE 18 software and are given below.

$\varphi_0(x) = 1$

$\varphi_1(x) = -1 + 2x$

$\varphi_2(x) = 1 - 6x + 6x^2$

$\varphi_3(x) = -1 + 12x - 30x^2 + 20x^3$

$\varphi_4(x) = 1 - 20x + 90x^2 - 140x^3 + 70x^4$

$\varphi_5 = -1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5$

$\varphi_6(x) = 1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6$

$\varphi_7(x) = -1 + 56x - 756x^2 + 4200x^3 - 11550x^4 + 16632x^5 - 12012x^6 + 3432x^7$

$\varphi_8(x) = 1 - 72x + 1260x^2 - 9240x^3 + 34650x^4 - 72072x^5 + 84084x^6 - 51480x^7 + 12870x^8$

3.0 Mathematical Formulation of the Proposed Method

In this section, we first consider the mathematical formulation of the proposed method.

Case 1: The Linear aspect Of FIDEs.

Given the linear FIDEs as

$$y^{v}(x) = f(x) + \lambda\int_a^b k(x,t)y(t)dt, \ x \in [a,b] \tag{7}$$

where v is the order of the derivative, $k(x,t)$ and $f(x)$ are continuous functions, λ is a positive constant.

Substituting equation (5) into equation (12) yields a residual of the form [16]:

$$\tilde{y}^{v}(x) - \lambda\int_a^b k(x,t)\tilde{y}(t)dt \neq f(x), x \in [a,b] \tag{8}$$

We then collocate on the residual equation (8) at the zeros of the orthogonal polynomials $\varphi_r(x)$ and obtain a set of (n+1) equations in (n+1) unknowna_r. Note that r is chosen along the boundary conditions imposed so as to have equal number of $a_r$in the trial solution so as cases of over-determined or under-determined equation will be overcome [15].Thus, a Matrix solver, which in this case,is the Gaussian elimination method, is employed to solve the resulting collocation linearalgebraic equations in conjunction with equations from the boundary conditions for a unique determination of the unknown coefficientsin the approximate.

Case 2: The Nonlinear aspect Of FIDEs.

We consider the nonlinear differential equation of the form:

$$L[y(x,t)]+N[y(x,t)]=r(x) \tag{9}$$

with prescribed auxiliary conditions, where y(x,t) is unknown function, L is a linear operator and occurs to the highest order derivative, N is the non linear term, and r(x) is an non-homogeneous term.

Let solution be given as

$$\tilde{y}(x)=\sum_{r=0}^{\infty}a_r\varphi_r(x) \tag{10}$$

The nonlinear operator is decomposed as [16]

$$y_N(x,t)=\sum_{n=0}^{\infty}A_n(x,t), \tag{11}$$

where A_n is the Adomian decomposition polynomial (ADP) for all $n\in N$ given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0} \tag{12}$$

Now, let the nonlinear FIDEs be given as

$$y^v(x)=f(x)+\lambda\int_a^b k(x,t)y_N(t)dt\,,x\in[a,b]. \tag{13}$$

where $y_N(t)$ is the nonlinear term. Substituting equation (10) and (11) we get

$$\tilde{y}(x)=f(x)+\lambda\int_a^b k(x,t)\sum_{n=0}^{\infty}A_n(t)dt\,,\ x\in[a,b]$$

which implies that

$$\tilde{y}(x)=f(x)+\lambda\int_a^b k(x,t)A_n dt\,,\ x\in[a,b], \tag{14}$$

where $A_n(t)=\sum_{r=0}^{n}y_r y_{n-r}$. Thus, for every $n\geq 1,\ A_n=0.$

We also collocate on the residual equation (14) at the zeros of the orthogonal polynomials $\varphi_r(x)$ and obtain a set of (n+1) equations in (n+1) unknown a_r. As stated earlier, r is chosen along the boundary conditions imposed so as to have equal number of $a_r$in the trial solution[15]. We then employ the Gaussian elimination method to solve the resulting collocation linear algebraic equations to determine the unknown coefficients.

4.0 Numerical Examples

To illustrate the effectiveness of this method, we use three numerical examples.We solved for $n=5$,for linear and nonlinear FIDEs. For all examples considered, the solutions are compared with the exact solutions available in the literature. The rate of convergence of each of the linear Fredholm integral equation is

$$E_r=\left|\tilde{y}_{i+1}(x)-\tilde{y}_i(x)\right|<\delta$$

where $\tilde{y}_i(x)$ is the approximate solution by the proposed method using the nth degree polynomial approximation and δ varies from 10^{-5} for $n=5$ [14].

Example 4.1

Consider the second-order linear FIDEs [16] of the form

$$u''(x)+xu'(x)-xu(x)=e^{x}-2\sin x+\int_{-1}^{1}\sin xe^{-t}u(t)dt,\ u(0)=1,\ u'(0)=1. \qquad (15)$$

The exact solution is $u(x)=e^{x}$.

Example 4.2

Consider the third-order linear FIDEs[16] of the form

$$u'''(x)=\sin x-x-\int_{0}^{\frac{\pi}{2}}xtu'(t)dt,\ \ u(0)=1,\ u'(0)=0,\ u''(0)=-1. \qquad (16)$$

The exact solution is $u(x)=\cos x.$

Example 4.3

Consider the second-order nonlinear FIDEs of the form [16]

$$u''(x)=\sinh\ x+x-\int_{0}^{1}x(\cosh^{2}(t)-u^{2}(t))dt,\ u(0)=0,\ u'(0)=1. \qquad (17)$$

The exact solution is $u(x)=\sinh\ x.$

Example 4.4

Consider the second-order nonlinear FIDEs of the form [17]

$$y''(x)+\int_{0}^{\frac{\pi}{2}}xty(t)=x-\sin x,\ y(0)=1,\ y'(0)=1. \qquad (18)$$

The exact solution is $y(x)=\sin x.$

Results are presented in **Tables 1 - 4**.

Table 1: Computed Absolute Error of examples 4.1 for n=5

X	Exact Solutions	Approximate Solutions	Absolute Error
0.00	1.0000000	1.0000000	0.0000e+00
0.10	1.1051709	1.1053100	1.3913e-04
0.20	1.2214028	1.2225823	1.1796e-03
0.30	1.3498588	1.3538517	3.9929e-03
0.40	1.4918247	1.5011905	9.3658e-03
0.50	1.6487213	1.6667198	1.7999e-02
0.60	1.8221188	1.8526201	3.0501e-02
0.70	2.0137527	2.0611421	4.7389e-02
0.80	2.2255409	2.2946179	6.7077e-02
0.90	2.4596031	2.5554720	9.5869e-02
1.00	2.7182818	2.8462319	1.2795e-01

Table 2: Computed Absolute Error of examples 4.2 for n=5

X	Exact Solutions	Approximate Solutions	Absolute Error
0.00	1.0000000	1.0000000	0.0000e+00
0.10	0.9950042	0.9950032	9.3240e-07
0.20	0.9800666	0.9800613	5.2755e-06
0.30	0.9553365	0.9553244	1.2105e-05
0.40	0.9210610	0.9210424	1.8643e-05
0.50	0.8775826	0.8775600	2.2520e-05
0.60	0.8253356	0.8253125	2.3088e-05
0.70	0.7648422	0.7648204	2.1817e-05
0.80	0.6967067	0.6966849	2.1811e-05
0.90	0.6216100	0.6215835	2.6505e-05
1.00	0.5403023	0.5402647	3.7596e-05

Table 3: Computed Absolute Error of examples 4.3 for n=5

X	Exact Solutions	Approximate Solutions	Absolute Error
0.00	0.0000000	-0.0000000	1.8100e-12
0.10	0.1001668	0.1000984	6.8372e-05
0.20	0.2013360	0.2007929	5.4307e-04
0.30	0.3045203	0.3026901	1.8302e-03
0.40	0.4107523	0.4064192	4.3371e-03
0.50	0.5210953	0.5126237	8.4716e-03
0.60	0.6366536	0.6220128	1.4641e-02
0.70	0.7585837	0.7353328	2.3251e-02
0.80	0.8881060	0.8533984	3.4708e-02
0.90	1.0265167	0.9771000	4.9417e-02
1.00	1.1752012	1.1074153	6.7786e-02

Table 4: Computed Absolute Error of examples 4.4 for n=5

X	Exact Solutions	Approximate Solutions	Absolute Error
0.00	0.0000000	-0.0000000	5.3310e-11
0.10	0.0998334	0.0998339	5.2613e-07
0.20	0.1986693	0.1986699	5.8900e-07
0.30	0.1986693	0.2955197	4.7850e-07
0.40	0.3894183	0.3894162	2.1319e-06
0.50	0.4794255	0.4794219	3.6026e-06
0.60	0.5646425	0.5646379	4.5892e-06
0.70	0.6442177	0.6442121	5.5608e-06
0.80	0.7173561	0.7173485	7.5833e-06
0.90	0.7833269	0.7833153	1.1580e-05
1.00	0.8414710	0.8414540	1.6953e-05

5.0 Conclusion

We have usedcollocation method based on certain orthogonal polynomial basis which we constructed to solve linear and nonlinear FIDEs.The numerical results obtained using our polynomials show an excellent rate of convergent even as n increases, which is shown in **Tables 1 - 4**. Thus, the method is accurate, efficient and effective for the solution of integro-differential equations of Fredholm type.

It is also evident that the method offers several advantages which include, among others;

(i) cost-effectiveness as no extra interpolation is required in other to achieve several output of solution;

(ii) excellent convergence rate; and

(iii) ease of implementation.

6.0 References

[1] Kantorovich, L.V. (1934). On a New Method for Solving Partial Differential Equations. Doklady Acad. Sci. USSR, 2(8): 532- 536.

[2] Fraser, R.A., Jones, W.P., and Skan, S.W. (1937) Approximation to Functions and to the Solution of Differential Equations.Gt. Brit. Aero. Res. Council Rept. and memo, 1799; Reprinted in Gt. Brit. Air Ministry Aero. Rs. Comm. Tech. Rept., 1: 517 -549.

[3] Fox, L and Parker, I.B. (1968). Chebychev polynomials in numerical analysis. Oxford University Press, Oxford, London.

[4] Adeniyi, R.B. Adeyefa, E.O and Alabi, M.O (2006).A Continuous Formulation of some Non-Perturbed Multistep Collocation Approach using ChebyshevPolynomials as Basis Functions,Journal of the Nigerian Association of Mathematical Physics, 10:261-274.

[5] Finlayson, B.A. (1980). Orthogonal Collocation on Finite Elements-Progress and Potential.Mathematics and Computer in stimulation, 22(1):11 - 17.

[6] Michelsen, M.L. and Villadsen, J.V. (1972). A Convenient Computational Procedure for Collocation Constants, Chemical Engineering Journal, 4(1):64 - (68).

[7] Biscaia, E.C. Jnr, Mansur, M.B., Salum, A. and Castro, R.M.Z (2001). A Moving Boundary Problem and Orthogonal Collocation in Solving a Dynamic Liquid Surfactant Membrane Model including Osmosis and Breakage.Brazilian Journal of Chemical Engineering, 18: 2.

[8] Soliman,M., Al-Zeghayer,Y., and Ajbar (2014). A Modified Orthogonal Collocation Method for Reaction Diffusion Problems.Brazilian Journal of Chemical Engineering, 31(4): 967- 975.

[9] Lanczos, C. (1938). Trigonometric Interpolation of Empirical Analytical Functions.J. Math. Phys., 17: 123-199.

[10] Lanczos, C. (1956).Applied Analysis. Englewood Cliffs (NJ), Prentice-Hall, New Jersey.

[11] Lambert, D.J. (1973). Computational Methods in Ordinary Differential Equations. New York: John Willey.

[12] Fatunla, S.O (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations. New York, Academic Press INC.

[13] Batiha, B, Noorani, M.S.M. and Hashim, I. (2008). Numerical Solutions of the Nonlinear Integro-Differential Equations.Int. J. Open Problems Compt. Math., 1 (1).

[14] Mamadu, J. E. and Njoseh, I. N. (2016). Numerical Solutions of Volterra Equations using Galerkin Method with Certain Orthogonal Polynomials. Journal of Applied Mathematics and Physics,JAMP, 4:367-382.

[15] Olagunju, A.S. and Joseph Folake L. (2013).Third-Kind Chebychev Polynomials $V_r(x)$ in Collocation Methods of Solving Boundary Value Problems. IOSR Journal of Mathematics, 7(2):42-47.

[16] Manafianheris, J. (2012). Solving the Integro-Differential Equations using the Modified Laplace Adomian Decomposition Method.Journal of Mathematical Extension, 6(1): 41-55.

[17] Hosseini, S.M. and Shahmorad, S. (2003).Tau Numerical Solution of Integro-Differerential Equations with Arbitrary Polynomial Bases. Journal of Applied Mathematical Modelling, 27: 145-154