Transactions of the Nigerian Association of Mathematical Physics
***Volume 2, (November, 2016*), pp 49 – 58**
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Point Vortices on Spherical Surface: The Green's Function approach

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Abstract

Pointvortices on the surface of a sphere is studied by solving the Laplace-Beltrami equation using a Green's function approach. The solutions are reconstructed in spherical polar coordinates. The fundamental Green's function $G=-\frac{1}{4\pi\rho}$ is obtained where ρ depend on the relative position of the point vortices, $P_1(\xi,\eta,\varsigma)$ and $P_2(x,y,z)$, such that $\rho=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\varsigma)^2}$. The expression ρ is reconstructed using spherical polar coordinates (R,ϕ,θ) where ϕ is the azimuthal angle, θ the polar angle and R the spherical radius. The stream function $G\equiv\psi$ obtained using this method is plotted considering different values of the azimuthal angle ϕ and the polar angle θ on the surface while keeping R constant. It is found that there exist two equal point vortices at the North and South spherical poles respectively, in all the cases considered, thus flows on the surface of a sphere consists of uniform distribution of vorticity.

Keywords: Point vortices, Vorticity, Azimuthal angle, Vortex Motion, Closed surface, Green's Function

1.0 Introduction

The problem of description of point vortex motion has a long history dating as far back as the 19th century with Helmholtz initiating the two dimensional point vortex models. In the last two decades the problem has acquired renewed interest as there has arisen a need to incorporate curvature effects in both the structure and stability of vortex configurations in a view to understanding many atmospheric phenomena [1]. A number of theoretical and experimental studies have been devoted quite recently to the understanding of atmospheric vortices which are often observed in nature leading to a generalization of Euler's two dimensional vortex solution to surfaces with constant Gaussian curvature.

1.1 Vortex Flow

Vortex dynamics is one of the essential physical mechanisms in fluid mechanics, and significant work has been done on the two-dimensional vortex models, initiated by Helmholtz in 1858 as cited in [2]. The point vortex model, particularly, has played a central role in all aspects of two-dimensional vortex problems. It was formulated as a Hamiltonian dynamical system by Kirchhoff [1]. Laplace-Beltrami operator on a toroidal surface was considered in [3]. They construct the green's function for the Laplace-Beltrami operator on the surface of a three-dimensional ring torus and used a stereographic projection of the torus surface onto a planar annulus. They represent the Green's function in terms of the Schottky-Klein prime function associated with the annulus and the dilogarithm function. An application of their result to vortex dynamics on the surface of a torus was also considered.

The exact steady two-dimensional solutions of the Euler equation due to Stuart (1967) are generalized[4] to the surface of a sphere. It was shown, that mathematically in Stuart's solution there is an accumulation point of vortices at infinity but this does not cause any problem in terms of the physical interpretation of the solution because the plane is not a closed compact

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surface and the point at infinity is not a part of the surface. The sphere however is a closed compact surface; any physical meaningful generalization of Stuart's solution might therefore be expected to have a finite collection of vorticity extrema distributed over the spherical surface [5].

Vortex motion on a hyperbolic surface is studied in [14]. It is shown that Abelian Higgs vortices on a hyperbolic surface M can be constructed geometrically from holomorphic maps $f: M \longrightarrow N$ where N is also a hyperbolic surface. The fields depend on f and on the metrics of $M$and$N$. The vortex centres are the ramification points, where the derivative of f vanishes. The magnitude of the Higgs field measures the extent to which f is locally an isometry. Witten's construction of vortices on the hyperbolic plane is rederived and new examples of vortices on compact surfaces and on hyperbolic surfaces of revolution is obtained in their work.

Alobaidi and Mallier [9] considered flow on a spherical surface and used a stream function formulation to derive a nonlinear equation governing steady-state flow in that geometry. Some new vortex solutions were presented. Edwin [8] developed a mathematical framework for the dynamics of a set of point vortices on a class of differentiable surfaces conformal to the unit sphere. Other researchers [9] have considered streets of vortices on surfaces of revolution such as a sphere. With regards to distributed, have studied vortex patches on a sphere using numerical methods.

Kirchhoff in 1876, first derived the equations governed by point vortices in the two-dimensional plane. Such vortices have finite constant circulation but singular vorticity restricted to a point. Each vortex includes a singular velocity field entirely in the azimuthal component (with no radial component owing to zero horizontal divergence). There is no self-induced motion instead vortices move in the flow field generated by all others [1]. Sun-Chul [16] Studied latitudinal point vortex ring on a Spheroid. The point vortices on the same latitude(Called the point vortex ring) was investigate as an extension of the sphere case and the influence of a pole vortex was also discussed. Point Vortex Motion on the Surface of a sphere with impenetrable boundaries has been studied in [7] where a new example of the motion of a vortex around a straight barrier along a great circle on the spherical surfaces is presented, he studied in detail and finally a theoretical connection with a boundary value problem for a generalized Liouville-type quasilinear partial differential equation was made.

In the present work we present new method of representing the fundamental Green's function for a sphere by reconstructing ρ from Cartesian coordinate to spherical polar coordinate, we then for the first time plot the Green's function obtained using this method by varying the azimuthal angle and the polar angle on the surface of a sphere which lead to interesting point vortices in both the north and south spherical polesas shown in the plots.

2.0 Vortex Motion on a Spherical Surface

Consider vortex motion on the surface S of a sphere. The sphere is non-rotating, without loss of generality. We assume it has unit radius, in terms of standard spherical polar coordinates (r, ϕ, θ) with the latitude angle θ measured from the axis through the North Pole and ϕ is the azimuthal angle, analogous to longitude in terms of earth measuring coordinates the velocity vector has the form.

$$u = (u_r, u_\theta, u_\phi) \tag{2.1}$$

Where u_θ and u_ϕ are the zonal and the meridional components of the velocity field respectively. The incompressible nature of the flow allows the introduction of a scalar stream function $\psi(\theta, \phi)$ via

$$u = \nabla \psi \wedge e_r \tag{2.2}$$

Where e_r is the radial unit vector, it is then possible to define a scalar vorticity field $w(\theta, \phi)$ such that;

$$we_r = \nabla \wedge u \tag{2.3}$$

Where $w = -\nabla_\Sigma^2 \psi$ (2.4)

∇_Σ^2 Is the spherical Laplace – Beltrami operator defined by

$$\nabla_\Sigma^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{2.5}$$

In terms of stream function ψ , u_θ and v_ϕ are given by

$$u_\theta = -\frac{\partial\psi}{\partial\theta} \qquad v_\phi = \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{2.6}$$

In the steady case, the material conservation of vorticity is expressed by

$$\left(\frac{u_\theta}{\sin\theta}\frac{\partial\psi}{\partial\phi}+v_\phi\frac{\partial}{\partial\theta}\right)w=0 \quad (2.7)$$

With the use of (2.6) above we can write (2.7) as

$$\frac{1}{\sin\theta}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial\theta}+\frac{\partial\psi}{\partial\phi}\frac{\partial\omega}{\partial\theta}\right)=0 \quad (2.8)$$

3.0 Green's Function for a Sphere

Consider the Laplace's equation on spherical surface

$$\nabla_\Sigma^2 G=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial G}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial^2 G}{\partial\phi^2} \quad (3.1)$$

Where G, is the Green's function

The Green's function G depends only on the relative position of P_1 and P_2 and $G=G(P_2-P_1)=G(\rho)$ Then the problem for G becomes

$$\nabla_\Sigma^2 G=\frac{1}{4\pi\rho^2}\delta(\rho) \quad (3.2)$$

Because the problem is invariant under rotations we expect G to be radially symmetric, since ∇_Σ^2 and $\delta(\rho)$ are, so we look for a function $G(\rho)$

In which 3.1 becomes;

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial G}{\partial\rho})=\frac{1}{4\pi\rho^2}\delta(\rho) \quad \text{For } \rho>0, \delta(\rho)=0, \quad (3.3)$$

Carsten G [4]

So we find that

$$G(\rho)=-\frac{A}{\rho}+B \quad (\text{For } \rho>0) \quad (3.4)$$

Where A and B are constants, since $G\to 0$ as $\rho\to\infty$ $B=0$

Now consider the figure 1

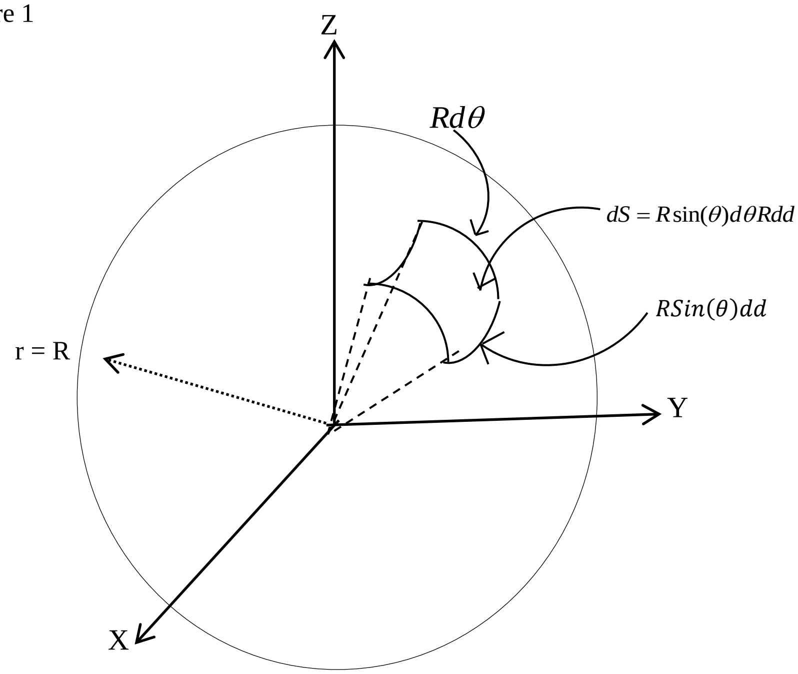

Figure 1: The Unit Area on the Surface of a Sphere

The outward normal to the surface is just unit vector pointing from the origin to the point on the surface and so

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial \rho} = \frac{A}{R^2} \tag{3.6}$$

on the surface $\rho = R$ on the sphere.

For any volume *V* that includes the origin by the divergence theorem we have.

$$\int_S \frac{\partial G}{\partial n} dS = \int_V \nabla_\Sigma^2 G dV = 1 \tag{3.7}$$

Where *S* is the surface of *V*

The unit area dS is given by $R^2 \sin\theta\, d\theta\, d\phi$ so that

$$\int_0^{2\pi}\int_0^{\pi} \frac{\partial G}{\partial \rho} R^2 \sin\theta\, d\theta\, d\phi = 1 \tag{3.8}$$

Thus

$$A \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 1 \tag{3.9}$$

$$\therefore A = \frac{1}{4\pi} \tag{3.10}$$

Now put *A* in (3.5) to get

$$G = -\frac{1}{4\pi\rho} \tag{3.11}$$

Consider two point vortices P_1 and P_2 where P_1 is the sink and p_2 as the source point, as shown in the figure 2

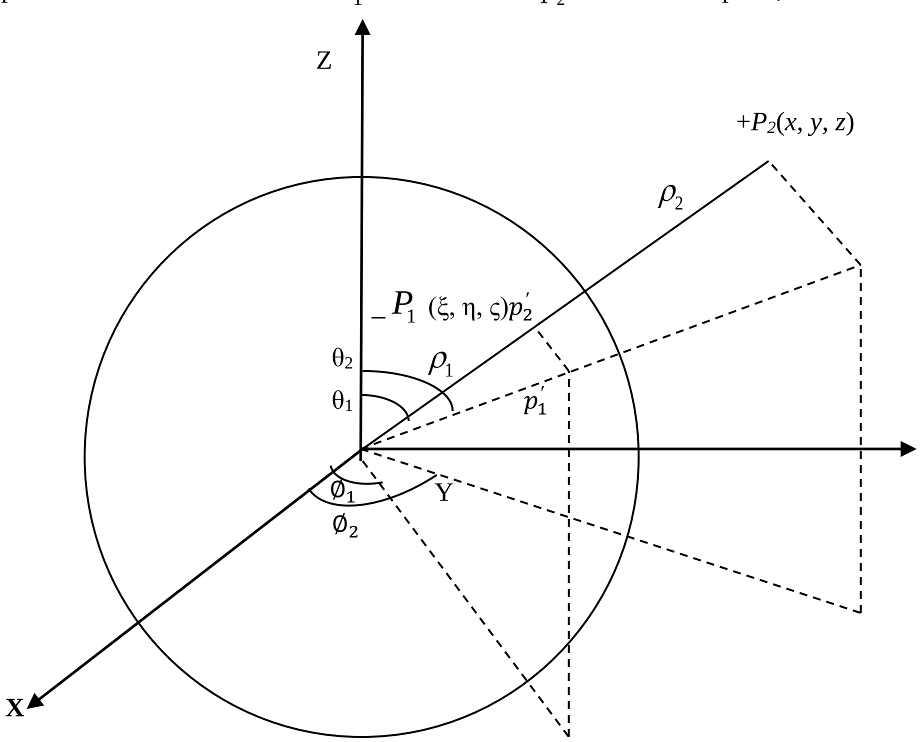

Figure 2: The Motion of Point Vortices on a Sphere

From the figure 2 $OP_1 = \rho_1$ and

$$OP_2 = \rho_2 \text{, with } \rho_1\rho_2 = a^2 \text{ i.e } (OP_1 \bullet OP_2 = a^2) \quad (3.12)$$

$$\rho = OP_2 - OP_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2} \quad (3.13)$$

We can now write

$$G = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2}}\right] \quad (3.14)$$

Where $x = \rho_1 \sin\theta_1 \cos\phi_1$ $y = \rho_1 \sin\theta_1 \sin\phi_1$ $z = \rho_1 \cos\theta_1$ (3.15)

And $\xi = \frac{a^2}{\rho_1}\sin\theta_2 \cos\phi_2$ $\eta = \frac{a^2}{\rho_1}\sin\theta_2 \sin\phi_2$ $\varsigma = \frac{a^2}{\rho_1}\cos\theta_2$ (3.16)

$$\Rightarrow (x-\xi)^2 + (y-\eta)^2 + (z-\varsigma)^2$$

$$= \frac{a^4}{\rho_1^2}\sin^2\theta_2 \sin^2\phi_2 - 2a^2 \sin\theta_1 \sin\theta_2 \cos\phi_1 \cos\phi_2 + \rho_1^2 \sin^2\theta_1 \cos^2\phi_1$$

$$+\frac{a^4}{\rho_1^2}\sin^2\theta_2 \cos^2\phi_2 - 2a^2 \sin\theta_1 \sin\theta_2 \sin\phi_1 \sin\phi_2 + \rho_1^2 \sin^2\theta_1 \sin^2\phi_1$$

$$+\frac{a^4}{\rho_1^2}\cos^2\theta_2 - 2a^2 \cos\theta_1 \cos\theta_2 + \rho_1^2 \cos^2\theta_1$$

$$= \frac{a^4}{\rho_1^2}\left[\sin^2\theta_2(\cos^2\phi_2 + \sin^2\phi_2) + \cos^2\theta_2\right]$$

$$-2a^2\left[\sin\theta_1 \sin\theta_2(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) + \cos\theta_1 \cos\theta_2\right]$$

$$+\rho_1^2\left[\sin^2\theta_1(\cos^2\phi_1 + \sin^2\phi_1) + \cos^2\theta_1\right]$$

$$= \frac{a^4}{\rho_1^2} - 2a^2\left[\sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) + \cos\theta_1 \cos\theta_2\right] + \rho_1^2$$

Now to fix ψ at the surface we take $\rho_1 = a = R$ where R the radius of the sphere.

Thus;

$$= \frac{R^4}{R^2} - 2R^2\left[\sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) + \cos\theta_1 \cos\theta_2\right]$$

$$= 2R^2\left[1 - \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) - \cos\theta_1 \cos\theta_2\right] \quad (3.17)$$

Therefore, by substituting (3.17) in to (3.14) we have

$$G = -\frac{1}{4\pi R}\left(\frac{1}{\sqrt{2\left[1 - \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)\right]}}\right) \quad (3.18)$$

The Green's function G we have just obtained is the stream function describing a flow with single point vortex surrounded by a uniform distribution of vorticity.

Let's now deduce the stream function ψ for this system. Suppose the point vortex is at (θ_2, ϕ_2) on the surface of the sphere. Then ψ satisfies;

$$\nabla^2\psi = \frac{1}{4\pi\rho^2}\delta\left(\theta_1,\phi_1,\theta_2,\phi_2\right) \tag{3.19}$$

Thus identify ψ as the Green's function of the Laplace-Beltrami operator on the sphere, hence $\psi \equiv G$.

4.0 Analysis of the Solution

We consider different cases by keeping θ_1 and ϕ_1 constant, that is keeping P_1 fixed and P_2 is moving on the surface, taking different values of θ_2 and ϕ_2

We also observe that the solution has two equal points-vortex singularities at the North and South spherical poles.

To study the solution, it is natural to fix the latitude angle θ_1 at which number of vortices extrema occur. We assume $0<\theta_1 \le \pi/2$ so that the layer of vorticity extrema is situated in the northern hemisphere, this is done without loss of generality because the circulations of the two antipodal point vortices are identical. Solutions corresponding to $\pi/2<\theta_1 \le 0$ can be obtained simply by turning the sphere upside down

The circulations of the point vortices at the north and south poles are equal and are given by $\Gamma_N = 2\pi(N-1) = \Gamma_S$ where N is the number of point vortices at the latitude angle [5].

Using this Crowdy's formula we can easily determine the number of point vortices for a given circulation, conversely we can get the circulation for a given number of point vortices. Consider for example case 1, the given circulation is $\Gamma_N = 6\pi$ implies $6\pi = 2\pi(N-1)$, thus $N=4$. Clearly it can be seen from the contour plot and this is applicable to all the cases considered.

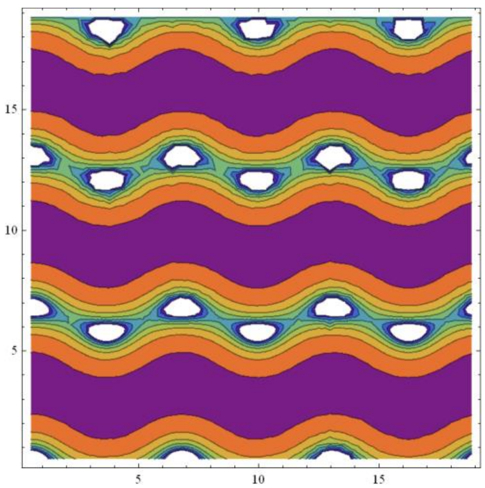

Figure 3: Showing point vortices at the northern hemisphere with polar angle $\pi/6$

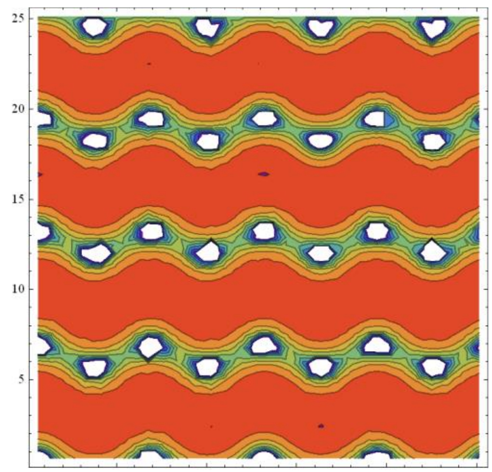

Figure 4: Showing point vortices at the northern hemisphere with polar angle $\pi/5$

The case when there are point vortices, each of circulation 6π at the northern hemisphere and these correspond to $N=4$, while $\pi/6 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 6\pi$

The case when there are point Vortices, each of circulation 8π at the northern hemisphere and these correspond to $N=5$, while $\pi/5 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 8\pi$

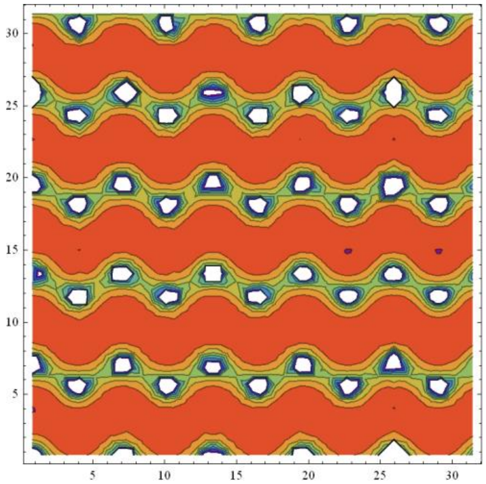

Figure 5: Showing point vortices at the northern hemisphere with polar angle $\pi/4$

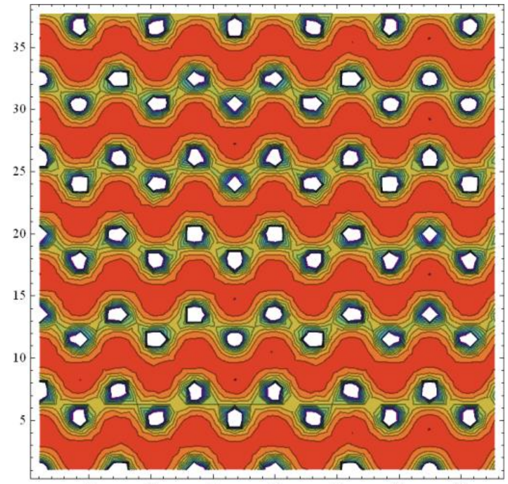

Figure 6: Showing point vortices at the northern hemisphere with polar angle $\pi/3$

The case when there are point Vortices, each of circulation 10π at the northern hemisphere and these correspond to $N=6$, while $\pi/4 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 10\pi$

The case when there are point Vortices, each of circulation 12π at the northern hemisphere and these correspond to $N=7$, while $\pi/3 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 12\pi$

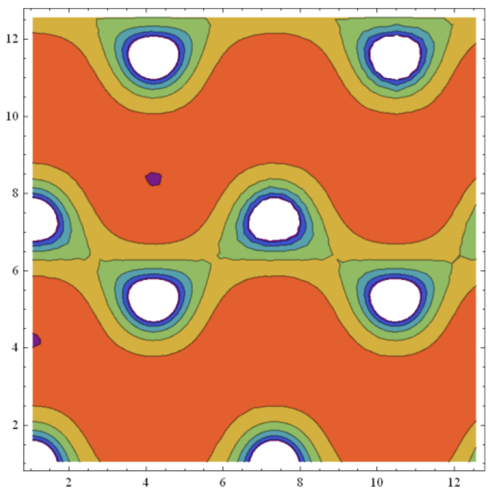

Figure 7: Showing point vortices at the northern hemisphere with polar angle $\pi/3$

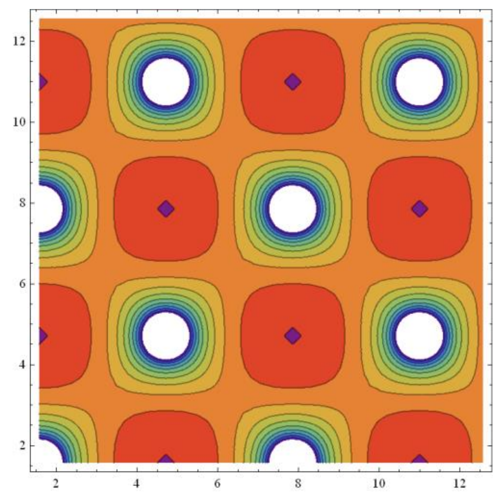

Figure 8: Showing point vortices at the equator with polar angle $\pi/2$

The case when there are point Vortices, each of circulation 4π at the northern hemisphereand these correspond to $N=3$, while $\pi/3 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 4\pi$

The case when there are point Vortices, each of circulation 14π at equator and these correspond to $N=8$, while $\pi/2 \le (\theta_1,\theta_2),(\phi_1,\phi_2) \le 4\pi$

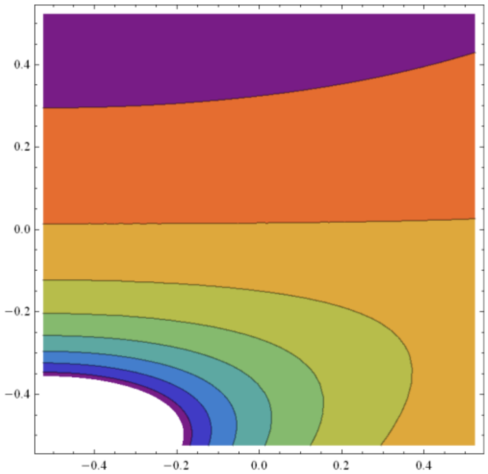

Figure 9: Showing point vortices at the southern hemisphere with polar angle $-\pi/6$

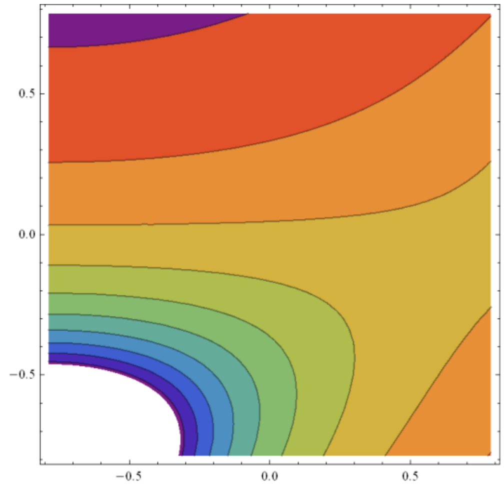

Figure 10: Showing point vortices at the southern hemisphere with polar angle $-\pi/4$

The case when there are point Vortices, each of circulation $\pi/6$ at southern hemisphere and these correspond to $N = \frac{13}{12}$, while $-\pi/6 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le \pi/6$

The case when there are point Vortices, each of circulation $\pi/4$ at southern hemisphere and these correspond to $N = \frac{9}{8}$, while $-\pi/4 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le \pi/4$

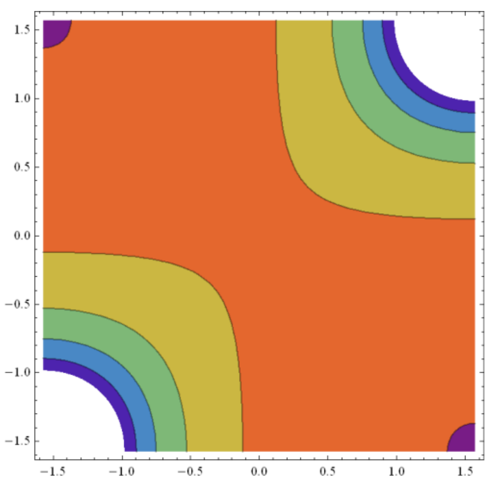

Figure 11: Showing point vortices at the southern hemisphere with polar angle $-\pi/2$

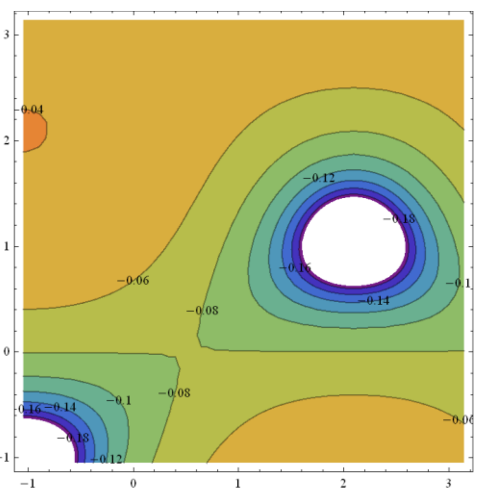

Figure 12: Showing point vortices at the southern hemisphere with polar angle $-\pi/3$

The case when there are point Vortices, each of circulation $\pi/2$ at southern hemisphere and these correspond to $N = \frac{5}{4}$, while $-\pi/2 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le \pi/2$

The case when there are point Vortices, each of circulation π at southern hemisphere and these correspond to $N = \frac{3}{2}$, while $-\pi/3 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le \pi$

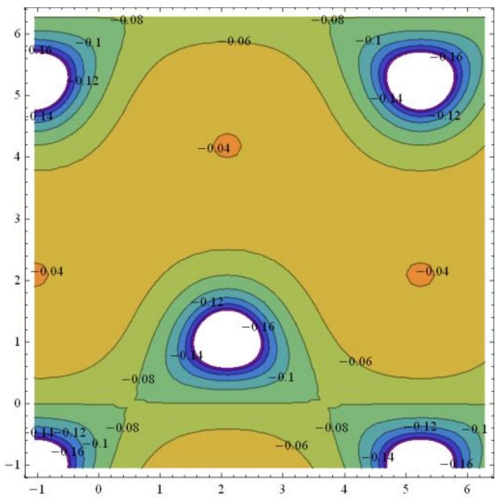

Figure 13: Showing point vortices at the southern hemisphere with polar angle $-\pi/3$

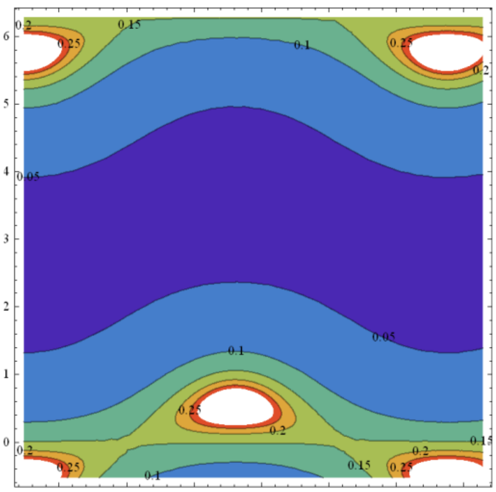

Figure 14: Showing point vortices at the southern hemisphere with polar angle $-\pi/6$

The case when there are point Vortices, each of circulation 2π at southern hemisphere and these correspond to $N = 2$, while $-\pi/3 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le 2\pi$

The case when there are point Vortices, each of circulation 2π at southern hemisphere and these correspond to $N = 2$, while $-\pi/6 \le (\theta_1, \theta_2), (\phi_1, \phi_2) \le 2\pi$

5.0 Summary and Conclusion

A point vortex is a model of a flow in which the vorticity is zero except at the point itself where the vorticity is infinite so that there is no zero circulation around the point.

The study of point vortices on two dimensional manifolds such as the cylinder, sphere and torus, has a long history, dating back to Helmholtz, initiating the point vortex model and Kirchhoff and Lin formulating it as a Hamiltonian dynamical system.

In this paper we are concerned with the point vortices on spherical surfaces, using the Green's function approach. We present new method of representing the fundamental Green's function for a sphere by reconstructing ρ from Cartesian coordinate to spherical polar coordinate. We obtained the free space green's function by solving the Laplace's equation on a sphere and reconstructed the solution in spherical coordinates which is termed the stream function, the solution was plotted for the first time by considering different cases varying the azimuthal angle ϕ and the polar angle θ while keeping R constant as the unit radius

Flows on a sphere are very important because of its applications to planetary atmosphere. Spherical surface consists of one or more periodic rows of point vortices, the simplest of which is a single infinite row of identical vortices. These have important applications in engineering and geophysics.

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