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# A Conformal Field Theory Approach to Physical Properties of 1D Correlated Electron Systems

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## *Abstract*

***The Hubbard model and t-J model are two essential models for one-dimensional strongly correlated electrons. Intensive studies have explained many important aspects of these 1D systems, whose low-energy properties can be described as Tomonaga-Luttinger liquid. This article reviews current status of our understanding, which has been achieved with conformal field theory technique and numerical approaches.***

## 1.0 Introduction

The behaviour of interacting electrons in one dimension has been intensively studied. This was stimulated by the discovery of one-dimensional (1D) organic conductor (TTF-TCNQ), organic superconductor (Bechgaard salts) and carbon nanotubes. It is well known that Fermi liquid (FL) theory describes interacting electrons in three dimensions [1], and many properties of the FL theory breaks down in one dimension; hence the need for 1D physical theory to describe the physics of interacting electrons in 1D organic materials. This led to the discovery of Tomonaga-Luttinger liquid theory [2, 3]. Since then much progress has been achieved in the understanding of 1D correlated electrons.

Formerly, there were two independent approaches: the exact Bethe-Ansatz solutions for 1D models including the Hubbard model [4] and bosonization theory to describe low-energy physics of 1D weakly correlated electrons [2, 5, 6]. Later advances in conformal field theory (CFT) [7], made it possible to connect these previous two view points and understand 1D electrons in the weakly and strongly correlated regimes.

The purpose of this article is to review achievements in this field by considering the Hubbard model and *t-J* model as fundamental systems. The two 1D models are basically the same in an appropriate limit but have significant differences in general. Therefore, detailed studies on these models will bring useful insights on various aspects of 1D strongly correlated electrons. We try to cover as many researches on this field as possible.

Usually, 1D quantum systems with short-range interaction undergo phase transition at zero temperature ($T = 0$ K), which is a critical point [8]. Therefore, 1D correlated electrons can be best described with the physics of various correlation functions at zero temperature, where the exponents show power-law decay [9]. It is important to note that power-law decay of correlation function is what clearly distinguishes the Tomonaga-Luttinger (TL) liquid theory from FL theory [10]. Here the interest is in the following correlation functions: (i) density correlation function (ii) electron field correlation function, (iii) spin correlation function and (iv) singlet and triplet superconducting correlation functions. These correlation functions and their critical exponents have been studied with different techniques.

As a result the concept of TL theory, which was proposed about 35years ago [11-13] has been recognized to be applicable to 1D correlated electrons. Various techniques, which have contributed much in this field, include (i) calculation of correlation functions based on quantum Monte Carlo (QMC) simulations [14-17], (ii) calculation of correlation functions based on bosonization theory [15, 18], (iii) calculation of correlation functions based on path-integral formulation [6]and calculation of correlationfunctions and their critical exponents based on conformal field theory (CFT) [7, 19-21].

In 1996 Shaojin and Lu [22] applied the Density Matrix Renormalization Group (DMRG) method to calculate correlation functions and their critical exponents and obtained 0.125 at $k_F$, 0.75 at $3k_F$ and 1 at $5k_F$. Only theirresults at $k_F$ agree with results obtained from other techniques, but result at $3k_F$, disagrees with both numerical and analytical results from other methods [23].This discrepancy on critical exponent at the Fermi point $3k_F$ has prompted series of investigation in this field.

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Also, it has stimulated our quest in this field to investigate correlation functions and their critical exponents near odd Fermi points with the CFT technique developed by Frahm and Korepin [7]. This has provided insights and proper understanding of TL liquid properties for 1D correlated electron systems [24].
Having these in mind, we organize this article as follows. In section 2 we review the properties of the 1D Hubbard model. The 1D *t-J* model is discussed in section 3. This is immediately followed by conclusion.

## 2.0 One-Dimensional Hubbard Model

## 2.1 Correlation Functions and their Critical Exponents

The Hubbard Hamiltonian [7]

$$H=-\sum_{j=1}^{N}\sum_{\sigma}\left(\psi^{\dagger}_{j+1,\sigma}\psi_{j,\sigma}+\psi^{\dagger}_{j,\sigma}\psi_{j+1,\sigma}\right)+4u\sum_{j=1}^{N}n_{j\uparrow}n_{j\downarrow}+\mu\sum_{j=1}^{N}\left(n_{j\uparrow}+n_{j\downarrow}\right)+\frac{\mathrm{H}}{2}\sum_{j=1}^{N}\left(n_{j\uparrow}-n_{j\downarrow}\right), \tag{2.1}$$

is the simplest model of correlated electrons. Here electrons are only allowed to hop to nearest neighbor sites. $n_{j,\sigma}=\psi^{\dagger}_{j,\sigma}\psi_{j,\sigma}$ is the number of spin $\sigma$ electron at site $j$ , $\psi^{\dagger}_{j,\sigma}(\psi_{j,\sigma})$ is creation(annihilation) field operator, $4u>0$ is the on-site Coulomb repulsion, $\mu$ is the chemical potential, and H is an external magnetic field.
In the language of CFT, correlation functions have been calculated for various quantum numbers and the critical exponents can be extracted from these correlation functions. The general expression for correlation function [7] at zero temperature is

$$G(x,t)=\sum A_k\frac{\exp(-2iD_c k_{F,\uparrow}x)\exp[-2i(D_c+D_s)k_{F,\downarrow}x]}{(x-iv_c t)^{2h_c^+}(x+iv_c t)^{2h_c^-}(x-iv_s t)^{2h_s^+}(x+iv_s t)^{2h_s^-}} \tag{2.2}$$

Where the conformal dimensions $h_c^{\pm}\left(h_s^{\pm}\right)$ for holon (spinon) excitations are given by

$$2h_c^{\pm}(\Delta N,D)=\left(Z_{cc}D_c+Z_{sc}D_s\pm\frac{Z_{ss}\Delta N_c-Z_{cs}\Delta N_s}{2\det Z}\right)^2+2N_c^{\pm} \tag{2.3}$$

$$2h_s^{\pm}(\Delta N,D)=\left(Z_{cs}D_c+Z_{ss}D_s\pm\frac{Z_{cc}\Delta N_s-Z_{sc}\Delta N_c}{2\det Z}\right)^2+2N_s^{\pm} \tag{2.4}$$

The $v_c\left(v_s\right)$ is velocity for holon (spinon) excitations, *t* is time, $k_{F,\uparrow}\left(k_{F,\downarrow}\right)$ is Fermi points with spin up (down) respectively. The positive integers $N_{c,s}^{\pm}$ , for holon and spinon describes particle-hole excitations, with $N_{c,s}^{+}(N_{c,s}^{-})$ being the number of occupancies a particle at the right (left) Fermi level jumps to, $\Delta N_c(\Delta N_s)$ represents the change in number of electrons (down-spin) with respect to the ground state, $D_c\left(D_s\right)$ is the quantum number of particles which transfer from one Fermi level of the holon (spinon) to the other, and both $D_c$ and $D_s$ are either integer or half-odd integer values. Finally, Z is the well-known dressed charge $2\times 2$ matrix describing anomalous behaviour of critical exponents [7, 24].
At zero magnetic field, the density correlation function is calculated for the set of quantum numbers $(D_c,\ D_s,\ \Delta N_c,\ \Delta N_s,\ N_{c,s}^{\pm})=$ (1, -1, 0, 0, 0), (1, 0, 0, 0, 0). Therefore, the equal time (*t*→0, *x*→0) density correlation function is obtained as

$$G_{nn}(r,0)-n_c^2\approx A_1\frac{\cos(2k_F r)}{r^{\theta_1}}+A_2\frac{\cos(4k_F r)}{r^{\theta}} \tag{2.5}$$

The exponents

$$\theta_1=1+\frac{\theta}{4}\ ,\qquad \theta=2\xi_c^2 \tag{2.6}$$

and $\xi_c$ is the solution of the dressed charged matrix. As a result of the range of variation of $\theta$ , the dominant contribution to the correlation function Eqn. (2.5) is the first term on the right hand side, for all finite values of interaction *u*. The oscillations with Fermi points $2k_F$ appears in free electron gas. While the oscillations with $4k_F$ are a product ofinteraction, i.e., as $u\to\infty$ the amplitude $A_1$ vanishes and the leading oscillating contribution to the correlation function is the second term on the right hand side of Eqn. (2.5). For any density below half-filling ($0<n_c<1$), the value of $\theta$ increases from 2 to 4 as the Coulomb repulsion *u* decreases from $\infty$ to 0. This is shown in Figure 2.1.

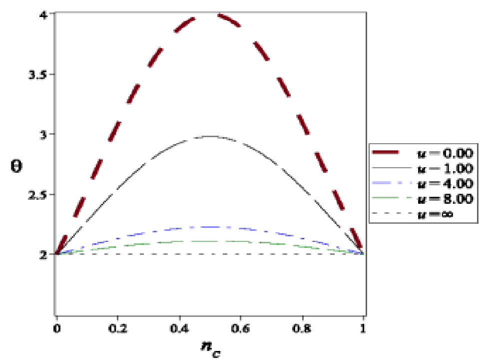


**Fig. 2.1**. The exponent $\theta$ ofdensity correlation function in 1D Hubbard model below half-fillingas a function of $n_c$ for various values of *u*.

With the excitations (0, 1, 0, 0, 0), (1, 0, 0, 0, 0), the spin correlation function has the same form as the density correlation function except that the constant term $n_c$ is replaced by the magnetization *m*. Using these excitations the spin correlation function for zero magnetic field is obtained as

$$G^{z}_{\sigma\sigma}(r,0) \approx m^2 + A_1 \frac{\cos(2k_F r)}{r^{\theta_1}} + A_2 \frac{\cos(4k_F r)}{r^{\theta}} \qquad (2.7)$$

## 2.2 Presence of Magnetic Field Effects

In 1D correlated electrons, magnetic field has significant effects on the spin degrees of freedom[7, 25].First is the mixing of phase operators of spin and charge densities, which results from a difference in spin-up and spin-down Fermi velocities, $v_{F\uparrow} \neq v_{F\downarrow}$, caused by magnetic field. Secondly, is the suppression of scattering between right-going and left-going states (i.e., due to $k_{F\uparrow} \neq k_{F\downarrow}$). These effects are seen in bosonization theory [1]. Also in CFT, magnetic field effect appears in various Fermi points. That is, the $3k_F$ manifests as $k_{F\uparrow} + 2k_{F\downarrow}$, $5k_F$ manifests as $2k_{F\uparrow} + 3k_{F\downarrow}$, and so on. For small magnetic field, the conformal dimensions take the form [26, 27]

$$2h_c^{\pm} = \left[\left(D_c + \tfrac{1}{2}D_s\right) \pm \tfrac{1}{2}\Delta N_c\right]^2 - \frac{4B}{\pi^2 B_c}\left[\left(D_c + \tfrac{1}{2}D_s\right) \pm \tfrac{1}{2}\Delta N_c\right] D_s \qquad (2.8)$$

$$2h_s^{\pm} = \tfrac{1}{2}\left[D_s \pm \left(\Delta N_s - \tfrac{1}{2}\Delta N_c\right)\right]^2 + \frac{1}{4\ln\left(\frac{B_0}{B}\right)}\left[D_s^2 - \left(\Delta N_s - \tfrac{1}{2}\Delta N_c\right)^2\right] + \frac{1}{2}\left(\frac{1}{4\ln\left(\frac{B_0}{B}\right)}\right)^2 [D_s^2 + 3\left(\Delta N_s - \tfrac{1}{2}\Delta N_c\right)^2] \qquad (2.9)$$

The electron field correlation function is calculated for the excitations (-1/2, 1/2,1, 0, 0), (-1/2, -1/2,1, 0, 0), (3/2, -1/2,1, 0, 0), (-3/2, -1/2,1, 0, 0), (5/2, -1/2,1, 0, 0), (-5/2, -1/2,1, 0, 0) and (7/2, -1/2,1, 0, 0), and we obtain

$$\begin{aligned} G^{\uparrow}_{\psi\psi^{\dagger}}(x,t) \approx\ & A_1 \frac{\cos(k_{F,\uparrow}x)}{|x+iv_c t|^{\alpha_{c1}}\,|x+iv_s t|^{\alpha_{s1}}} + A_2 \frac{\cos[(k_{F,\uparrow}+2k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c2}}\,|x+iv_s t|^{\alpha_{s2}}} + A_3 \frac{\cos[(3k_{F,\uparrow}+2k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c3}}\,|x+iv_s t|^{\alpha_{s3}}} \\ & + A_4 \frac{\cos[(3k_{F,\uparrow}+4k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c4}}\,|x+iv_s t|^{\alpha_{s4}}} + A_5 \frac{\cos[(5k_{F,\uparrow}+4k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c5}}\,|x+iv_s t|^{\alpha_{s5}}} + A_6 \frac{\cos[(5k_{F,\uparrow}+6k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c6}}\,|x+iv_s t|^{\alpha_{s6}}} \\ & + A_7 \frac{\cos[(7k_{F,\uparrow}+6k_{F,\downarrow})x]}{|x+iv_c t|^{\alpha_{c7}}\,|x+iv_s t|^{\alpha_{s7}}} \end{aligned} \qquad (2.10)$$

For the result (2.10), both holon and spinon excitations are responsible for the oscillations in the electron field correlation function. We note that $3k_F$ manifests as $k_{F\uparrow} + 2k_{F\downarrow}$, $5k_F$ appears as $3k_{F\uparrow} + 2k_{F\downarrow}$, $7k_F$ as $3k_{F\uparrow} + 4k_{F\downarrow}$, $9k_F$ as $5k_{F\uparrow} + 4k_{F\downarrow}$, $11k_F$ as $5k_{F\uparrow} + 6k_{F\downarrow}$ and $13k_F$ as $5k_{F\uparrow} + 6k_{F\downarrow}$, respectively. From the magnetic field dependence of the singularities at these Fermi points, we determine the critical exponents $\alpha_1$, $\alpha_3$, $\alpha_5$, $\alpha_7$, $\alpha_9$, $\alpha_{11}$ and $\alpha_{13}$ defined in

$$
\begin{aligned}
n(k) &\sim \left|k-k_{F\uparrow}\right|^{\alpha_1}\operatorname{sgn}(k-k_{F\uparrow}) && \text{for } k \sim k_{F\uparrow} \\
&\sim \left|k-k_{F\uparrow}-2k_{F\downarrow}\right|^{\alpha_3}\operatorname{sgn}(k-k_{F\uparrow}-2k_{F\downarrow}) && \text{for } k \sim k_{F\uparrow}+2k_{F\downarrow} \\
&\sim \left|k-3k_{F\uparrow}-2k_{F\downarrow}\right|^{\alpha_5}\operatorname{sgn}(k-3k_{F\uparrow}-2k_{F\downarrow}) && \text{for } k \sim 3k_{F\uparrow}+2k_{F\downarrow} \\
&\sim \left|k-3k_{F\uparrow}-4k_{F\downarrow}\right|^{\alpha_7}\operatorname{sgn}(k-3k_{F\uparrow}-4k_{F\downarrow}) && \text{for } k \sim 3k_{F\uparrow}+4k_{F\downarrow} \quad (2.11) \\
&\sim \left|k-5k_{F\uparrow}-4k_{F\downarrow}\right|^{\alpha_9}\operatorname{sgn}(k-5k_{F\uparrow}-4k_{F\downarrow}) && \text{for } k \sim 5k_{F\uparrow}+4k_{F\downarrow} \\
&\sim \left|k-5k_{F\uparrow}-6k_{F\downarrow}\right|^{\alpha_{11}}\operatorname{sgn}(k-5k_{F\uparrow}-6k_{F\downarrow}) && \text{for } k \sim 5k_{F\uparrow}+6k_{F\downarrow} \\
&\sim \left|k-7k_{F\uparrow}-6k_{F\downarrow}\right|^{\alpha_{13}}\operatorname{sgn}(k-7k_{F\uparrow}-6k_{F\downarrow}) && \text{for } k \sim 7k_{F\uparrow}+6k_{F\downarrow}
\end{aligned}
$$

Unlike the momentum distribution for Fermi liquid that exhibits a jump at the Fermi surface, the momentum distribution for TL liquid theory shows power-law singularity, and this characterizes TL liquid theory in contrast to FL theory. This is as a result of large quantum fluctuations present in one dimension, and therefore the low-energy properties of TL liquid are described by collective motion of Fermions instead of elementary quasi-particles.

In the presence of magnetic field effects, the momentum distribution functions Eqn. (2.11) around various Fermi wave numbers for the electron field correlator exhibits typical power-law singularities of Tomonaga-Luttinger liquid. As the magnetic field goes to zero,these critical exponents areobtained as tabulated in Table 2.1. The critical exponents around $k_F$ and $3k_F$ have been obtained earlier [23].

**Table 2.1:** Critical exponents for Hubbard model.

| Fermi wave numbers | Critical Exponents $\alpha$ |
|---|---|
| $k_F$ | 0.125 |
| $3k_F$ | 1.125 |
| $5k_F$ | 3.125 |
| $7k_F$ | 6.125 |
| $9k_F$ | 10.125 |
| $11k_F$ | 15.125 |
| $13k_F$ | 21.125 |

The result at $k_F$ and $3k_F$ agrees with results from QMC. However, results at $3k_F$ and $5k_F$ disagrees with Shaojin and Lu's DMRG results [22]. The results at $7k_F$, $9k_F$, $11k_F$, and $13k_F$ first appeared in [24, 27].

## 3.0 One-Dimensional *t-J* Model

One-dimensional *t-J* model [21] is known to have spin-1/2 electronsaround nearest neighbour lattice sites with hopping matrix element –t < 0. Double occupation of every lattice site is not allowed, i.e., each lattice is constrained to have either one electron (with spin-up or spin-down) or none. This accounts for strong correlation in this model.The *t-J* model has some similarities to the repulsive Hubbard model, but they are also different in many aspects. First the *t-J* model can have a region with Luttinger parameter $K_\rho > 1$ where superconductivity correlations are the most dominant. Secondly, this model also show phase separation [28, 29]. Emery et al. [30]discussed the phase separation in the 2D *t-J* model, while Ogata et al. and Schulz [31, 32] studied the phase separation in the 1D *t-J* model.The phase separation in 1D systems is not trivial because of large quantum fluctuations [33]. These have stimulated both theoretical and experimental interests in the study of the properties of 1D *t-J* model.

## 3.1 Correlation Functions of *t-J* Model

As mentioned earlier, there are different approaches to the study of correlation function. Here we shall review only the CFT approach where critical exponents can be extracted easily from correlation functions. The 1D version of the *t-J* Hamiltonian [21] is given by

$$
H = -t\sum_{i,\sigma}\left(c^\dagger_{i,\sigma}c_{i+1,\sigma} + c^\dagger_{i+1,\sigma}c_{i,\sigma}\right) + 2J\sum_i\left(S_iS_{i+1} - \tfrac{1}{4}n_in_{i+1}\right) - \mu\sum_i n_i - \tfrac{1}{2}\mathrm{H}\sum_i\left(n_{i\uparrow} - n_{i\downarrow}\right) \qquad (3.1)
$$

Where $c^\dagger_{i,\sigma}(c_{i,\sigma})$ is the spin-σ electron creation (annihilation) operators at lattice site $i$, $S_i = c^\dagger_{i,\sigma}S_{\sigma\sigma}c_{i+1,\sigma}$ with the spin-1/2 matrix $S$, the number operator $n_{i,\sigma} = c^\dagger_{i,\sigma}c_{i+1,\sigma}$, $n_i = n_{i,\uparrow} + n_{i,\downarrow}$ and $\mu$ and H are the chemical potential and external magnetic field, respectively. Here the correlation function at zero temperature is estimated using the relation.

$$G(x,t) \approx \frac{\exp\left(i\left(2\pi - 2k_{F,\uparrow} - 2k_{F,\downarrow}\right)D_c x\right)\exp\left(i\left(2\pi - 2k_{F,\uparrow}\right)D_s x\right)}{\left(x - iv_c t\right)^{2\Delta_c^+}\left(x + iv_c t\right)^{2\Delta_c^-}\left(x - iv_s t\right)^{2\Delta_s^+}\left(x + iv_s t\right)^{2\Delta_s^-}} \quad (3.2)$$

Where $D_c,\ D_s,\ k_{F\sigma},\ x,\ v_x,\ t$, have the same meaning as in Hubbard model. The conformal dimensions for both holon and spinon excitations at zero magnetic field are given by

$$\Delta_c^{\pm}(I,D) = \frac{1}{2}\left(\frac{I_c}{2\xi_c} \pm \xi_c\left(D_c + \frac{D_s}{2}\right)\right)^2 + N_c^{\pm}$$
$$\Delta_s^{\pm}(I,D) = \frac{1}{4}\left(I_s - \frac{I_c}{2} \mp D_s\right)^2 + N_s^{\pm} \quad (3.3)$$

Here, $I_c\left(I_s\right)$ is the change in number of electrons (down-spin) with respect to the ground state.
For vanishing magnetic field, the charge density correlation function is obtained as

$$N(r,0) \approx n_c^2 + \frac{A_0}{r^2} + A_2\frac{\cos(2k_F r)}{r^{\alpha_s}} + A_4\frac{\cos(4k_F r)}{r^{\alpha_c}} \quad (3.4)$$

Where, $\alpha_s = 1 + \alpha_c/4$, $\alpha_c = 2\xi_c^2$ and $\xi_c = \sqrt{2}\left(1 - 5\sin(\frac{\pi n_c}{4})/12\right)$. Eqn. (3.4) is obtained with $(D_c,\ D_s,\ I_c,\ I_c,\ N_{c,s}^{\pm}) =$ (0, 0, 0, 0, 1), (1, 0, 0, 0, 0), (-1, 0, 0, 0, 0). Similarly, the superconducting singlet and triplet correlation functions are estimated with the excitations (1/2, 0, 2, 1, 0) for singlet and (0, 0, 2, 2, 0) for triplet, and we obtained the singlet pair correlator as

$$P_s(r,0) \approx \frac{\cos(2k_F r)}{r^{\beta_s}}, \quad (3.5)$$

and triplet pair correlator as

$$P_t(r,0) \approx \frac{1}{r^{\beta_t}} \quad (3.6)$$

The superconducting exponents are given by

$$\beta_s = \frac{4}{\alpha_c} + \frac{\alpha_c}{4}, \quad \beta_t = 1 + \frac{4}{\alpha_c} \quad (3.7)$$

If the exponents $\beta_s$ and $\beta_t$ are plotted against electron concentration or band filling ($v = n_c/2$) as shown in Fig. 3.1, one observes that the superconducting correlations get more enhanced as holes are doped into the half-filled band [20, 34]

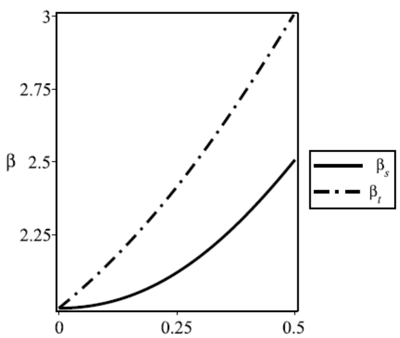


**Fig. 3.1**. Superconducting correlation exponents as a function of *v*.
$\beta_s$ and $\beta_t$ are for singlet and triplet pairs, respectively.

The momentum distribution for singlet and triple pair are obtained as

$$\tilde{P}_s(k) \approx \left|k - 2k_F\right|^{\zeta_s}\left[\mathrm{sgn}(k - 2k_F)\right]^2$$
$$\tilde{P}_t(k) \approx \left|k\right|^{\zeta_t} \quad (3.8)$$

Where the superconducting critical exponents are obtained as

$$\zeta_s = \beta_s - 1, \quad \zeta_t = \beta_t - 1 \quad (3.9)$$

Lastly, the long-distance behaviour of the electron field correlation function at zero magnetic field is calculated for the excitations (0, 1/2, 1, 1, 0), (1, -1/2, 1, 1, 0), (2, -3/2, 1, 1, 0), (3, -5/2, 1, 1, 0), (4, -7/2, 1, 1, 0), (5, -9/2, 1, 1, 0) and (6, -11/2, 1, 1, 0). We thus obtain the momentum distribution functions as

$$
\begin{aligned}
n(k) &\sim \left|k-k_{F\uparrow}\right|^{\zeta_1}\,\mathrm{sgn}(k-k_{F\uparrow}) && \text{for } k \sim k_F \\
&\sim \left|k-3k_F\right|^{\zeta_3}\,\mathrm{sgn}(k-3k_F) && \text{for } k \sim 3k_F \\
&\sim \left|k-5k_F\right|^{\zeta_5}\,\mathrm{sgn}(k-5k_F) && \text{for } k \sim 5k_F \\
&\sim \left|k-7k_F\right|^{\zeta_7}\,\mathrm{sgn}(k-7k_F) && \text{for } k \sim 7k_F \\
&\sim \left|k-9k_F\right|^{\zeta_9}\,\mathrm{sgn}(k-9k_F) && \text{for } k \sim 9k_F \\
&\sim \left|k-11k_F\right|^{\zeta_{11}}\,\mathrm{sgn}(k-11k_F) && \text{for } k \sim 11k_F \\
&\sim \left|k-13k_F\right|^{\zeta_{13}}\,\mathrm{sgn}(k-13k_F) && \text{for } k \sim 13k_F
\end{aligned} \qquad (3.10)
$$

Conformal field theory has enabled us to extract the critical exponents from the momentum distribution functions. As $\nu$ goes to half-filling, we obtain the critical exponents around the Fermi points as shown in Table 3.1.

**Table 3.1**: Critical Exponents for t-J Model.

| Fermi wave numbers | Critical Exponents $\zeta$ |
|---|---|
| $k_F$ | 0.125 |
| $3k_F$ | 1.125 |
| $5k_F$ | 5.125 |
| $7k_F$ | 12.125 |
| $9k_F$ | 22.125 |
| $11k_F$ | 35.125 |
| $13k_F$ | 51.125 |

In the language of bosonization [5, 11, 35, 36] $K_\rho$ and $K_\sigma$ are parameters governing the decay of correlation functions. Critical exponents for systems without a spin gap are related to these Luttinger parametersas shown in Table3.2[37].

**Table 3.2:**.Relationship between CFT exponents and Luttinger parameter for systems without spin gap.

| | Exponents |
|---|---|
| $2k_F - \mathrm{CDW}$ | $\zeta = K_\rho$ |
| $4k_F - \mathrm{CDW}$ | $\zeta = 4K_\rho$ |
| singlet pair | $\beta_s = \frac{1}{K_\rho} + K_\sigma$ |
| tinglet pair | $\beta_s = \frac{1}{K_\rho} + \frac{1}{K_\sigma}$ |
| $n(k)$ at $k_F$, $3k_F$, $5k_F$,... | $\zeta = \frac{1}{4}\left(\frac{1}{K_\rho} + K_\sigma - 2\right)$ |

In the absence of magnetic field, $K_\sigma = 1$. Therefore, relating our results in Table 3.1with the Luttinger parameter $K_\rho$ in Table 3.2gives the results in Table 3.3.

**Table 3.3**. Critical exponents for *t-J* model and corresponding Luttinger parameter.

| | Exponents for t-J model | | |
|---|---|---|---|
| $k$ | $\zeta$ | $K_\rho < 1$ | $K_\rho > 1$ |
| $k_F$ | 0.125 | 0.5 | 2 |
| $2k_F$ | 0.5 | 0.5 | - |
| $3k_F$ | 1.125 | 0.2 | 6.34 |
| $4k_F$ | 1 – 3 | 0.25 - 0.75 | - |
| $5k_F$ | 5.125 | 0.05 | 22.45 |
| $7k_F$ | 12.125 | 0.02 | 50.48 |
| $9k_F$ | 22.125 | 0.01 | 90.49 |
| $11k_F$ | 35.125 | 0.005 | 142.49 |
| $13k_F$ | 51.125 | 0.005 | 206.49 |
| Singlet pairing | 1.5 | 0.67 | - |
| Triplet pairing | 2 | 0.5 | - |

One major significance of the Luttinger parameter is that exponents with $K_\rho < 1$ is repulsive Tomonaga-Luttinger liquid, $K_\rho = 1$ is Fermi liquid and $K_\rho > 1$ is attractive Tomonaga-Luttinger liquid.

## 3.2 Phase Diagram

Our results can be summarized in the phase diagram shown in Figure 3.2, with contour lines for several values of $K_\rho$.

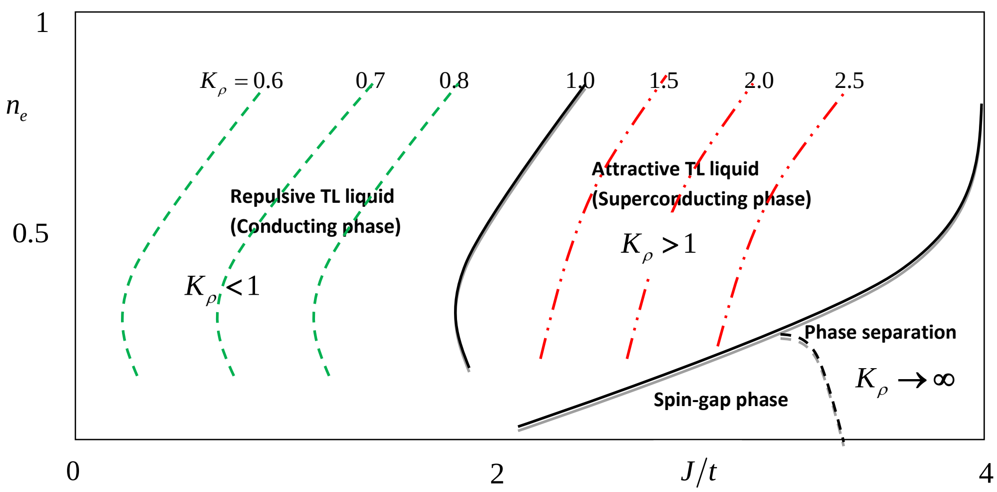


**Fig. 3.2**: Phase diagram of 1D *t-J* model from CFT for $0 < n_e \le 1$ and in the range $0 < J/t \le 4$, with curves representing the contours of constant Luttinger parameter $K_\rho$ [38-40].

We obtained four different phases:in the region of small *J/t*, the ground state of the *t-J* model is repulsive TL liquid with $K_\rho < 1$. In this region, the spin correlation dominates the long-distance behaviour. This signifies a conducting (or metallic) phase. Increasing *J/t*, these correlations are suppressed and the ground state changes to attractive TL liquid with $K_\rho > 1$. It is dominated by singlet pairing correlations. This signifies superconducting phase. When the attraction among the particles is so strong, the system separates into particle-rich and electron-rich regions. This region is called phase separation. In the limit $J \to \infty$ all the particles join up in a singlet Island, which can be described by the Heisenberg model forming an electron state phase, where the kinetic fluctuations are strongly quenched and only spin fluctuations remains [40]. The slope of the phase separation line indicates that the separation occurs between the empty phase( $n_e = 0$ ) and a finite density phase. It was observed in quantum Monte Carlo simulations [41] that for $J/t > 3.5$, the ground state becomes a fully phase separated state between $n_e = 0$ and $n_c = 1$. Lastly, the spin-gap state lies between the TL liquid and phase-separated regions at small densities.

The phase diagram determined by the present method (CFT) is consistent with results of the exact diagonalization on a 16-site ring [42] and the Variational Monte Carlo method [40].Also, critical exponents in Table 3.3 around various Fermi points fall into the conducting phase with $K_\rho < 1$ and superconducting phase with $K_\rho > 1$. This corresponds to small and large values of $J/t$ for the *t-J* model [39]. $K_\rho \approx 1$, signals FL regime (3D metallic regime). It is clear that, increasing the Fermi wave number increases the critical exponents, which corresponds to increase in $J/t$. This favours the superconducting phase and the system is said to be in the attractive TL liquid regime.

## 4.0 Conclusion

We have shown that for 1D correlated electrons, a consistent understanding is being achieved for low-energy properties from weak to strong correlation regimes. In particular, the conformal field theory has been used to describe the physics of various correlation functions at zero temperature for 1D Hubbard model. This is supported by numerical studies. Therefore, the properties of 1D Hubbard model are well understood to exhibit Tomonaga-Luttinger liquid properties. One of the characteristics of TL liquid theory is the occurrence of power-law decayincorrelation functions at low energies.

The evaluation of critical correlation exponents with finite-size scaling in conformal field theory has shown that this method correctly reproduce not only the global features of Tomonaga-Luttinger liquid theory but also the phase diagram, with conducting and superconducting regions present inthe *t-J* model.Also, it is observed that critical exponents associated to higher Fermi points correspond to Luttinger parameterwith $K_\rho > 1$, and this signifies superconductivity.

## 5.0 References


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