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Exact Contradicting Solution for a η-Weak-Pseudo-Hermiticity Generators

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Abstract

An exactsolution for a non-Hermitian η weak-pseudo-Hermitian Hamiltoniansis analyzed for a class of potentials $V_{eff}(x) = V(x) + i\,S(x)$. The imaginary part $S(x)$ of the effective potential serves as the generating function for the $\hat{\eta}$ weak-pseudo-Hermitian Hamiltonians. We obtained an exact soluble solution for the non-Hermitian Hamiltonian using a simple generating function. The derived eigen functions are expressed by confluent hyper-geometric functions and a complex energy eigenvalues are obtained.

Keywords: Hermitian, Hamiltonian, Potential, Weak-Pseudo-Hermitian, Confluent Hyper-geometric functions

1.0 Introduction

Non-Hermitian PT -symmetric quantum mechanics has been an active field of study in quantum mechanics. Mostafazadeh [1-3]has analyzed series class of non-Hermitian Hamiltonians that attributes to real spectrum eigenvalues. However, a pseudo-Hermitian Hamiltonian can also be formulated to satisfy same condition without violating the PT -symmetry condition [4, 5].In particular,a class of spherically symmetric non-Hermitian Hamiltonians and their $\hat{\eta}$ weak-pseudo Hermiticity generators, where generalization of beyon the nodeless in one-dimension state was first introduced by Fityo [6]. Position-dependent massesfor $\hat{\eta}$ weak-pseudo-Hermitian d-dimensional Hamiltonians quantum particles have also been exploited [7].Edmonds and Jameshave introduced examples on complex energies in Relativistic Quantum theory [8].

In this paper we provide an exact contradicting solvable example to the $\hat{\eta}$ weak-Pseudo-Hermicity generators which equally works well for systems of non-Hermitian Hamiltonian by product of our $\hat{\eta}$ weak-pseudo-Hermiticity generators discussed in [9].

We first considernon-Hermitian η weak-pseudo-HermitianHamiltonians for a class of effective potentials $V_{eff}(x) = V(x) + i\,S(x)$, where $V(x)$ and $S(x)$ are real valued functions. The imaginary part $S(x)$ of the effective potential serves as thegenerating function that gives $V(x)$ real part of the effective potential for the $\hat{\eta}$ Weak-Pseudo- .

This paper is organized as follow. Section 2 is devoted to the model formulation of the $\hat{\eta}$ -weak-pseudo-Hermiticity generators. A contradicting example and concluding remarks are given in section 3 and 4, respectively.

2.0 Model Formulation of $\hat{\eta}$ -Weak-Pseudo-Hermiticity Generators

Let consider an invertible linear operator $\hat{\eta}$ which is Hermitian and it obey the canonical equation governed by

$$\hat{\eta} = \hat{O}^{\dagger}\,\hat{O}, \tag{1}$$

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where $\hat{O}$ and $\hat{O}^\dagger$ are linear operator known as intertwining operators given by

$$\hat{O} = \frac{\partial}{\partial x} + M(x) + iN(x)\,, \quad \hat{O}^\dagger = -\frac{\partial}{\partial x} + M(x) - iN(x)\,, \tag{2}$$

in which *M(x)* and *N(x)* are real valued functions.
For the purpose of our study we will consider non-Hermitian Schrödinger Hamiltonian operator in 1-dimension given by

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + V_{eff}(x) \tag{3}$$

where $\hbar = 2, m = 1,$ and $V_{eff}(x)$ known as the effective potential written as

$$V_{eff}(x) = V(x) + iS(x) \tag{4}$$

Accordingly, the Hamiltonian operator in eq. (3) is said to be pseudo-Hermitian if it satisfies the relation [10],

$$\hat{H}^\dagger = \hat{\eta}\hat{H}\hat{\eta}^{-1} \tag{5}$$

and hence one may obtain a real energy spectrum. Moreover, the two intertwining operators as well as the invertible, Hermitian operator η satisfies an intertwining relation

$$\hat{\eta}\hat{H} = \hat{H}^\dagger\hat{\eta}. \tag{6}$$

Substituting eq. (2) into eq. (1) imply

$$\hat{\eta} = \left(\frac{\partial}{\partial x} + M(x) + iN(x)\right)\left(-\frac{\partial}{\partial x} + M(x) - iN(x)\right) \tag{7}$$

expanding eq. (7) and simplifying we get

$$\hat{\eta} = -\frac{\partial^2}{\partial x^2} - 2iN(x)\frac{\partial}{\partial x} + M^2(x) + N^2(x) - M'(x) - iN'(x). \tag{8}$$

Evaluating the intertwining relation eq. (6) by substituting eq. (3) and eq. (8) and after an explicit calculation one finds

$$S(x) = -2N'(x), \tag{9}$$

and subsequently

$$M^2(x) - M'(x) = \frac{2N(x)N''(x) - N'^2(x) + \alpha}{4N^2(x)}, \tag{10}$$

which implies

$$V(x) = \frac{2N(x)N''(x) - N'^2(x) + \alpha}{4N^2(x)} - N^2(x) + \beta. \tag{11}$$

in which α and β are real constants.
From eq. (9), we obtain

$$N(x) = -\frac{1}{2}\int S(x)\,dx. \tag{12}$$

Subsequently from eq. (9), we get

$$N'(x) = -\frac{1}{2}S(x), \qquad N''(x) = -\frac{1}{2}S'(x). \tag{13}$$

Substituting eq. (13) into eq. (10), we have

$$M^2(x) - M'(x) = \frac{1}{2}\frac{N''(x)}{N(x)} - \frac{N'^2(x)}{4N^2(x)} + \frac{\alpha}{4N^2(x)}, \tag{14}$$

and in terms of *S(x)* potential function gives

$$M^2(x)-M'(x)=\frac{1}{2}\left(-\frac{1}{2}\int S(x)dx\right)^{-1}\left(-\frac{1}{2}S'\right)-\frac{1}{4}S^2\left[4\left(\frac{1}{2}\int S(x)dx\right)^2\right]^{-1}$$

$$+\frac{\alpha}{4}\left[\frac{1}{2}\int S(x)dx\right]^{-2}, \qquad (15)$$

after simplification eq. (15) becomes

$$M^2(x)-M'(x)=\frac{S'(x)}{2}\left[\int S(x)dx\right]^{-1}-\frac{S^2(x)}{4}\left[\int S(x)dx\right]^{-2}+\alpha\left[\int S(x)dx\right]^{-2}. \qquad (16)$$

Also substituting eq. (13) into eq. (11) and solving for $V(x)$ we obtain

$$V(x)=\frac{S'(x)}{2}\left[\int S(x)dx\right]^{-1}-\frac{S^2(x)}{4}\left[\int S(x)dx\right]^{-2}+\alpha\left[\int S(x)dx\right]^{-2}$$

$$-\frac{1}{4}\left[\int S(x)dx\right]^2+\beta. \qquad (17)$$

Finally, we will compute the $V_{eff}(x)$ real part potential *V(x)* of the Hamiltonian equation provided by the equation (4). However, eq. (17) will be used to determine the real potential function *V(x)*, using the imaginary part of the effective potential, *S(x)* as a generating function and with some adjustable values of integration constant α and β that would yield an exact solution to the Hamiltonian operator equation (3).

3.0 A Contradicting Example

First we consider a simple generating function given by

$$S(x)=-\frac{1}{2}\frac{k}{x^2}. \qquad (18)$$

Substituting eq. (18) into eq. (17) one finds

$$V(x)=\frac{3}{4x^2}+\frac{4\alpha x^2}{k^2}-\frac{1}{16}\frac{k^2}{x^2}+\beta, \qquad (19)$$

Thus, we choose the arbitrary constant $\alpha=\frac{1}{4}$, β=0, we get

$$V(x)=\frac{3}{4x^2}+\frac{x^2}{k^2}-\frac{k^2}{16x^2}. \qquad (20)$$

Hence, Hamiltonian operator

$$\hat{H}=-\frac{\partial^2}{\partial x^2}+V(x)+iS(x),$$

implies

$$\hat{H}=-\frac{\partial^2}{\partial x^2}+\left(\frac{3}{4x^2}+\frac{x^2}{k^2}-\frac{k^2}{16x^2}-\frac{ik}{2x^2}\right), \qquad (21)$$

Which can be simplified to

$$\hat{H}=-\frac{\partial^2}{\partial x^2}+\left(\frac{3}{4}-\frac{k^2}{16}-\frac{ik}{2}\right)\frac{1}{x^2}+\frac{x^2}{k^2}. \qquad (22)$$

For simplicity we will consider a new substitution, *l* so that

$$l(l+1)=\frac{3}{4}-\frac{k^2}{16}-\frac{ik}{2}, \quad \omega^2=\frac{1}{\mathrm{k}^2}. \qquad (23)$$

Then, substituting eq. (23) into eq. (22), we obtain

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + \frac{l(l+1)}{x^2} + \omega^2 x^2. \tag{24}$$

Accordingly, Schrodinger equation $\hat{H}\psi(x) = E\psi(x)$, imply

$$-\frac{\partial^2\psi(x)}{\partial x^2} + \left(\frac{l(l+1)}{x^2} + \omega^2 x^2\right)\psi(x) = E\psi(x). \tag{25}$$

Now, if we choose *E=2E*, eq. (25) yields

$$\frac{1}{2}\frac{\partial^2\psi(x)}{\partial x^2} - \left(\frac{l(l+1)}{2x^2} + \frac{\omega^2 x^2}{2}\right)\psi(x) = -E\psi(x). \tag{26}$$

We can observe that eq. (26) is one dimensions analogy of the 3-D harmonic oscillator which can be solved in spherical coordinates [11, 12]. Since the potential is only radial dependent, the angular part of the solution is a spherical harmonic. However, Let us define a variable

$$z = \gamma x. \tag{27}$$

So that

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \gamma\frac{\partial}{\partial z}, \tag{28}$$

and

$$\frac{\partial^2}{\partial r^2} = \left(\frac{\partial z}{\partial r}\right)^2 \frac{\partial^2}{\partial z^2} + \frac{\partial^2 z}{\partial r^2}\frac{\partial}{\partial z} = \gamma^2 \frac{\partial^2}{\partial z^2}. \tag{29}$$

Then eq. (26) becomes

$$\frac{\partial^2\psi(z)}{\partial z^2} + \left(\frac{2E}{\gamma^2} - \frac{l(l+1)}{z^2} - \frac{\omega^2 z^2}{\gamma^4}\right)\psi(z) = 0. \tag{30}$$

We now introduce another new substitution

$$q = \frac{2E}{\gamma^2}\ ,\ \gamma = \sqrt{\omega}, \tag{31}$$

and subsequently

$$q = \frac{2E}{\omega}, \tag{32}$$

which upon substituting eq. (31) into eq. (30) we get

$$\frac{\partial^2\psi(z)}{\partial z^2} + \left(q - \frac{l(l+1)}{z^2} - z^2\right)\psi(z) = 0 \tag{33}$$

From eq. (33) if we consider $(z \to \infty)$ for large *z*. Then eq. (33) can be written in more convenient form as

$$\frac{\partial^2\psi(z)}{\partial z^2} - z^2\psi(z) = 0. \tag{34}$$

Eq. (34) admits a solution in the form

$$\psi(z) = Ae^{-\frac{z^2}{2}} + Be^{\frac{z^2}{2}}, \tag{35}$$

where A and B are constants. Since at infinity $B = 0,$ then

$$\psi(z) = Ae^{-\frac{z^2}{2}}. \tag{36}$$

And subsequently for $(z \to 0)$ small *z*, eq. (33), implies

$$\frac{\partial^2\psi(z)}{\partial z^2}-\frac{l(l+1)}{z^2}\psi(z)=0. \quad (37)$$

Hence, eq. (37) also admits a solution given by

$$\psi(z)=Cz^{l+1}+Dz^{-l}, \quad (38)$$

where C and D are constants. Also $D=0.$ In order not to obtain infinite at $z=0.$ Thus, we have

$$\psi(z)=Cz^{l+1}. \quad (39)$$

Furthermore, proposing a solution to eq. (33) of the type

$$\psi(z)=z^{l+1}e^{-\frac{Z^2}{2}}F(z). \quad (40)$$

Differentiating eq. (40), we obtain

$$\psi'(z)=\left(\left[(l+1)-z^2\right]F(z)+zF'(z)\right)z^l e^{-\frac{z^2}{2}}, \quad (41)$$

and hence

$$\psi''(z)=lz^{l-1}e^{\frac{z^2}{2}}\left[\left(l+1-z^2\right)F(z)+zF'(z)\right]-z^{l+1}e^{\frac{z^2}{2}}\left(\left(l+1-z^2\right)F(z)+zF'(z)\right)$$

$$+z^l e^{\frac{z^2}{2}}\left[(-2z)F(z)+\left(l+1-z^2\right)F'(z)+zF'(x)\right]. \quad (42)$$

Substituting eq. (42) and eq. (40) into eq. (33), we have

$$zF''(z)+\left(l-z^2+l+1-z^2\right)F'(z)+\left(l(l+1)z^{-1}-zl\right)F(z)$$

$$+\left(-z\left(l+1-z^2\right)-2z+qz-l(l+1)z^{-1}-z^3\right)F(z)=0, \quad (43)$$

simplifying further gives

$$zF''(z)+2\left(l+1-z^2\right)F'(z)+(q-2l-3)zF(z)=0. \quad (44)$$

Let define a new variable

$$\xi=z^2, \quad (45)$$

and using the transformation

$$\frac{\partial}{\partial z}=\frac{\partial\xi}{\partial z}\frac{\partial}{\partial\xi}=2\xi^{\frac{1}{2}}\frac{\partial}{\partial\xi}, \quad (46)$$

in which $\frac{\partial^2}{\partial z^2}=4\xi\frac{\partial^2}{\partial\xi^2}+2\frac{\partial}{\partial\xi}.$ (47)

Thus, upon substituting eq. (45), eq. (46) and eq. (47) into the differential equation (44) it follows that

$$4\xi^{\frac{3}{2}}F''(\xi)+\left[4(l+1-\xi)+2\right]\xi^{\frac{1}{2}}F'(\xi)+(q-2l-3)\xi^{\frac{1}{2}}F(\xi)=0. \quad (48)$$

Hence, dividing through eq. (48) by $4\xi^{\frac{1}{2}}$ simplifies to

$$\xi F''(\xi)+\left(l+\frac{3}{2}-\xi\right)F'(\xi)+\frac{(q-2l-3)}{4}F(\xi)=0. \quad (49)$$

We may observe that the above differential equation (49) is well known as Confluent Hyper-geometric equation given in general form as

$$\xi F''(\xi)+(c-\xi)F'(\xi)-aF(\xi)=0. \quad (50)$$

However, we may find it exact solutions by power series method. Comparing our differential equation (50) with the Confluent Hyper-geometric equation for simplicity, we choose to define.

$$Q = l + \frac{3}{2},\ b = -\frac{\left(q - 2l - 3\right)}{4}, \tag{51}$$

then the differential equation (49) becomes

$$\xi F''\left(\xi\right) + \left(Q - \xi\right)F'\left(\xi\right) - bF\left(\xi\right) = 0. \tag{52}$$

We now proceed by proposing a solution of the form

$$F\left(\xi\right) = \sum_{r=0}^{\infty} c_r \xi^{n+r}, \tag{53}$$

accordingly, differentiating eq. (54) 1st and 2nd , we get

$$F'\left(\xi\right) = \sum_{r=0}^{\infty}\left(n+r\right)c_r\xi^{n+r-1}\ , F''\left(\xi\right) = \sum_{\mathrm{r}=0}^{\infty}\mathrm{c_r}\left(n+r\right)\left(n+r-1\right)\xi^{\mathrm{n+r-2}}. \tag{54}$$

Putting eq. (54) and eq. (53) into the differential equation (52) yields

$$\sum_{r=0}^{\infty}c_r\left(n+r\right)\left(n+r-1\right)\xi^{n+r-1} + \sum_{r=0}^{\infty}Q\left(n+r\right)c_n\xi^{n+r-1} - \sum_{r=0}^{\infty}\left(n+r\right)c_r\xi^{n+r} - \sum_{r=0}^{\infty}bc_r\xi^{n+r} = 0. \tag{55}$$

Collecting like terms of eq. (55) we have

$$\sum_{r=0}^{\infty}c_r\left(n+r\right)\left[\left(n+r-1\right)+Q\right]\xi^{n+r-1} - \sum_{r=0}^{\infty}\left[\left(n+r\right)+b\right]c_r\xi^{n+r} = 0. \tag{56}$$

Putting $r = r + 1$ in the 1st summation terms, gives

$$\sum_{r=-1}^{\infty}c_{r+1}\left(n+r+1\right)\left[n+r+Q\right]\xi^{n+r} - \sum_{r=0}^{\infty}\left[\left(n+r\right)+b\right]c_r\xi^{n+r} = 0. \tag{57}$$

Expanding the 1st summation, implies

$$c_0 n\left[n-1+Q\right]\xi^{n-1} + \sum_{r=0}^{\infty}\left\{c_{r+1}\left(n+r+1\right)\left[n+r+Q\right] - \left[\left(n+r\right)+b\right]c_r\right\}\xi^{n+r} = 0. \tag{58}$$

From eq.(58) if $c_0 \neq 0,$ then we may find an indicial equation of the form

$$n\left[n-1+Q\right] = 0, \tag{59}$$

whose roots implies

$$n = 0, \qquad n = 1 - Q. \tag{60}$$

Hence from eq (58), we deduce the recurrence relation is given by

$$c_{r+1} = \frac{\left[n+r+b\right]c_r}{\left(n+r+1\right)\left[n+r+Q\right]}, \tag{61}$$

We now proceed by substitution of the indicial solution $n = 0$ into the recurrence relation simplifies to

$$c_{r+1} = \frac{\left[r+b\right]c_r}{\left(r+1\right)\left(r+Q\right)}. \tag{62}$$

And after an explicit calculation it follows that eq.(62) satisfied a solution in the form

$$F_1\left(\xi\right) = 1 + \frac{b}{Q}z + \frac{b(b+1)}{Q(Q+1)}\frac{z^n}{2!} + \ldots = {}_1F_1(b, Q, \xi), \tag{63}$$

where $Q \neq 0,$ and subsequently $n = 1 - Q,$ we have

$$c_{r+1} = \frac{\left[r+1-Q+b\right]}{\left(r+2-Q\right)\left(r+1\right)}c_r, \tag{64}$$

this satisfy also to give the relation

$$F_2\left(\xi\right) = 1 + \frac{(b-Q+1)}{(2-Q)}z + \frac{(b-Q+1)(b-Q+2)}{(2-Q)(3-Q)}\frac{z^n}{2!} + \ldots, \tag{65}$$

yielding the second solution as

$$F_2(\xi) = {}_1F_1(b-Q+1, 2-Q, \xi). \tag{66}$$

Hence, for $c \neq 0, \pm 1, \pm 2, \pm 3,$ the complete two independent solutions to the differential equation (52) becomes

$$F(\xi) = A\,{}_1F_1(b, Q, \xi) + B z^{1-Q}\,{}_1F_1(b-Q+1, 2-Q, \xi), \tag{67}$$

where A and $B$are constants.

Accordingly, we may deduce the wave function eq. (40) in the form

$$\psi(z) = z^{l+1} e^{-\frac{z^2}{2}} F(z), \tag{68}$$

in which

$$F(z) = A\,{}_1F_1(b, Q, z^2) + B z^{1-Q}\,{}_1F_1(b-Q+1, 2-Q, z^2), \tag{69}$$

and

$$z = \gamma x, \qquad Q = l + \frac{3}{2}, \; b = -\frac{(q-2l-3)}{4}. \tag{70}$$

Finally, the general solution becomes

$$\psi(x) = (\gamma x)^{l+1} e^{-\frac{(\gamma x)^2}{2}} F(\gamma x), \tag{71}$$

where

$$F(\gamma x) = A\,{}_1F_1(b, Q, \gamma^2 x^2) + B(\gamma x)^{1-Q}\,{}_1F_1(b-Q+1, 2-Q, \gamma^2 x^2), \tag{72}$$

in which

$$Q = l + \frac{3}{2}, \; b = -\frac{(q-2l-3)}{4}. \tag{73}$$

We shall now consider $b = -n_r$, as $x \to \infty$ for the wave function to be well behaved in order to deduce the energy equation. Hence, it follows that the relation

$$b = -\frac{(q-2l-3)}{4} = -n_r. \qquad n_r = 0, 1, 2.... \tag{74}$$

Therefore, $$q = 2l + 3 + 4n_r, \tag{75}$$

hence from eq. (32) it follows that $q = \frac{2E}{\omega}$ upon substituting into eq. (74) and solving for E_n. We may obtain the energy eigenvalues given by

$$E_{n_r} = \left(2n_r + l + \frac{3}{2}\right)\omega. \tag{76}$$

And since $E = 2E$, then energy eigenvalues becomes

$$E_{n_r} = (4n_r + 2l + 3)\omega, \tag{77}$$

where $\omega^2 = \frac{1}{k^2}$ and $l = -(\frac{3}{4} - \frac{ik}{4})$ or $l = (\frac{1}{2} - \frac{ik}{4})$.

Thus, for simplicity we choose to work with $l = (\frac{1}{2} - \frac{ik}{4}),$ and finally the complex energy eigenvalues becomes

$$E_{n_r} = \left(4n_r + 4 - \frac{ik}{2}\right)\omega, \tag{78}$$

where $\omega^2 = \frac{1}{k^2}$.

However, the complete wave function would be given by

$$\psi(x) = A(\gamma x)^{l+1} e^{-\frac{(\beta x)^2}{2}} {}_1F_1(b, Q, \gamma^2 x^2), \quad (79)$$

where A is normalization constant. $Q = l + \frac{3}{2}$, and $b = -\frac{(q-2l-3)}{4}$.

To find the normalization constant using the condition $\int_o^\infty |\psi(x)|^2 dx = 1$. We choose for simplicity $b = 0$, thus

$$ {}_1F_1(0, Q, \gamma^2 x^2) = 1, \quad (80)$$

and subsequently

$$\psi(x) = A(\gamma x)^{l+1} e^{-\frac{(\gamma x)^2}{2}}, \quad (81)$$

upon substitution in the normalization condition gives

$$A^2 \int_o^\infty (\gamma x)^{2(l+1)} e^{-(\gamma x)^2} dx = 1. \quad (82)$$

Comparing eq. (82) with the standard integral given by

$$\int_o^\infty x^{2n} e^{-x^2} dx = \frac{1}{2}\Gamma(n + \frac{1}{2}), \quad (83)$$

we have $A = \frac{\sqrt{2}}{\left[\Gamma(l + \frac{3}{2})\right]^{\frac{1}{2}}}$ (84)

hence the complete general wave function is given by

$$\psi_n(x) = \frac{\sqrt{2}}{\left[\Gamma(l + \frac{3}{2})\right]^{\frac{1}{2}}} \frac{(\gamma x)^{l+1}}{n!} e^{-\frac{(\gamma x)^2}{2}} {}_1F_1(b, Q, \gamma^2 x^2), \quad (85)$$

where

$$Q = l + \frac{3}{2}, \; b = -\frac{(q-2l-3)}{4} \text{ and } l = (\frac{1}{2} - \frac{ik}{4}). \quad (86)$$

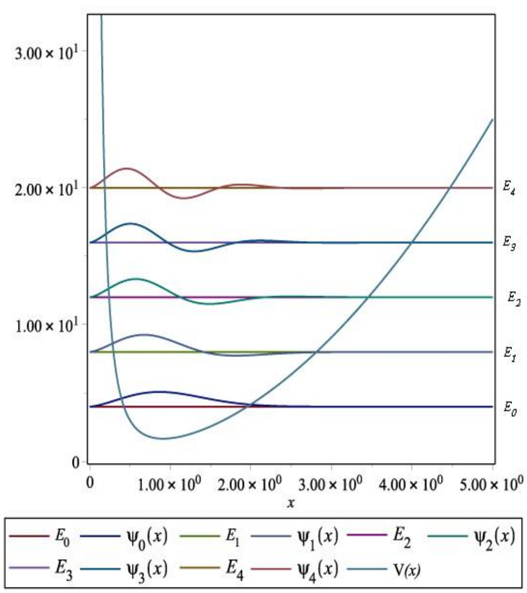

Figure 1: Plots of wave function, Energy levels and Potential, $l = 1$.

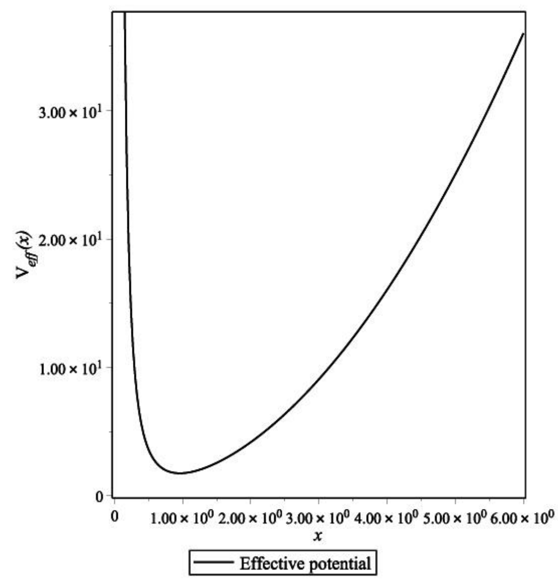

Figure 2: Effect potential graph, $l = 1$.

4.0 Conclusion

In this study, we introduced our model formulation for the η -weak-pseudo-Hermiticity generators analyzed in [6].We obtained an exact solvable contradicting solution for the η weak-pseudo-hermicity generators of non-Hermitian Hamiltonian which results to a complex energy eigenvalues with the eigenfunctions in form of confluent hyper-geometric functions, which extended the work of Mustafa and Mazharimousavi [9].However, only properlychosen of integration constants α and β withthe *S(x)*generating function of the effective potential, $V_{eff}(x) = V(x) + iS(x)$ would yield an exact solution to the Hamiltonian operator equation (3).For simplicity, we choose to work with $l = (\frac{1}{2} - \frac{ik}{4})$, to obtain our results of energy eigenvalues with the eigenfunction.

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