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On The Reproduction Number of Vaccination Model

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Abstract

In this work, an SEIR epidemic model was developed to investigate the efficacy of vaccination on the control of epidemic diseases. The dynamics of the compartments were described by a system of ordinary differential equations and the resulting differential equations were analyzed for existence and uniqueness of solution and were found to have a feasible solution. The equations were solved for both the disease free and the endemic equilibrium states. The analysis for stability was done for disease free equilibrium state. We used the method of characteristic equation of the Jacobian determinant to show the local asymptotic stability (LAS) of the model at the disease free equilibrium state.

We also established that the disease free equilibrium state for the model was globally asymptotically stable (GAS) whenever the effective reproduction number $R_0 < 1$.Numerical simulations were carried out with the help of Mathematical Software(Maple) using parameter values from published data as the base

Keywords: Existence of solution, stability analysis, disease free equilibrium state, epidemic equilibrium state, reproduction number, local asymptotic stability (LAS), global asymptotic stability (GAS), Numerical simulation

1.0 Introduction

The menace of infectious diseases is not only a major cause of death and misery to human but also has the potential to have major social and economic impact. Many infectious diseases are spread by direct contact between susceptibles and infectives. Other diseases are spread in the environment and are transmitted to the human population by insects or other vectors (Daley, D.J.and Gani, J. 2005; D'Agata, et al., 1993).

Many a time, the propagation of an infectious disease can be severe and catastrophic to the degree that within few days it spreads at an unimaginable rate. For an instance, in an English boarding school with a total of 763 boys, there was an epidemic of influenza from 22nd January to 4th February 1978. A total of 512 boys were put to bed during the epidemic that seems to have started from a single infected boy (Guillemo Abramson, 2011).

Controlling infectious disease has been an increasingly complex issue in recent years. A major strategy to control infectious diseases is vaccination (Alonso- Quesada, S.and De lasen, M. 2008;De lasen, Mand Alonso- Quesada, S. 2010). Vaccination has been established as an indispensable instrument to fight against the propagation of epidemic diseases (Trottier & Philippe, 2011; Nareshetal, R. 2008).

The various vaccination strategies campaigns allowed health authorities to achieve "herd immunity". The theory behind the development of "herd immunity" is: in disease that can be passed from person to person, it is more difficult to pass that disease easily when there are those who are immune to it. The more immune individuals there are, the less likely it is that a susceptible person will come into contact with someone who has the disease. For example, if "person A" had smallpox and contacted "person B" who was immune because of vaccination, "person B" would not get ill and could not pass on the disease to "person C" when he comes into contact with him. So even if "person C" is not vaccinated, he gets indirectly protection from the disease(Mohamadhassani, M. and Haveshki, M. 2011).

Studies also supported that vaccination had a longer lasting effect than originally thought. Data collected by Mack from

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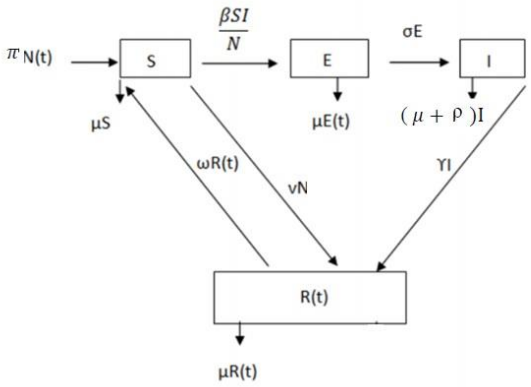
Western countries with importations of smallpox between 1950 and 1971 showed that the case fatality rate was 52% in never vaccinated individuals, 1.4% in those vaccinated 0 – 10 years before exposure and still only 11% in those vaccinated over 20 years before exposure. In general, vaccination has been regarded as one of the most powerful tools of fighting and eradicating epidemic diseases and the global eradication of smallpox ranks one of the greatest triumphs in medicine. The World Health Organisation officially certified that smallpox had been eradicated on December 9, 1979, 2 years after the last case in Somalia(Onyebuchi Chukwu, C.O. 2013).

2.0 Model Description

The SEIR is partitioned into compartments. S(t), E(t), I(t) and R(t) where S(t) is used to represent the number of individuals not yet infected with the disease at time t, or those "prone to disease"; E(t) denotes 'infected' or 'exposed' which stands for the number of individuals who have been infected with the disease but who do not still have any disease symptoms; I(t) denotes the number of individuals who have been infected with the disease and are capable of spreading the disease to those in the Susceptible category; R(t) is the compartment used for those individuals who have been infected and then recovered from the disease.

In the model,πN is the recruitment rate into the population which was as a result of birth or loss of acquired immunity. v is the rate of vaccination. $\rho$is the rate of death from disease related cases while μ is the rate of death from causes unrelated to the infection, β is the transmission constant (with the total number of infections per unit of time at time tbeing$\beta \frac{S(t)I(t)}{N(t)}$), σ is the rate of moving from exposed stage to infectious stage though death due to the disease during the latent stage is neglected. γ is the recovery rate and ω is the rate of losing immunity. N is the total population and all parameters are non-negative.



The flow chart showed that those who were successfully vaccinated would receive immunity and moved straight to the recovered class though some of them would die naturally. Besides, some of those who were not vaccinated or not successfully vaccinated would also die naturally. The remaining people who were not vaccinated or not successfully vaccinated would contact the disease and became exposed. As a result of the fact that the exposed individuals were totally ignorant of their status, some of them would die naturally while the remaining people would become infectious. Some of the infectious individuals would die either naturally or as a result of the infection while the remaining individuals would be cured of the disease and moved to the recovered class at a rate γ.Some of those who recovered from the disease would die naturally while the remaining people would become susceptible again and the flow would go like that.

2.1 Model Equations

$$\frac{dS}{dt} = -\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t)I(t)}{N(t)} - vN(t) \quad (1)$$

$$\frac{dE}{dt} = \frac{\beta S(t)I(t)}{N(t)} - (\mu + \sigma)E(t) \quad (2)$$

$$\frac{dI}{dt} = \sigma E(t) - (\mu + \gamma + \rho)I(t) \quad (3)$$

$$\frac{dR}{dt} = -(\omega + \mu)R(t) + \gamma I(t) + vN(t) \quad (4)$$

Assume total population is N = S+E+I+R

2.2 The Existence and Uniqueness of Solution for the Model

Here, the system of equations representing the model is analyzed for existence and uniqueness of solutions. The validity and implementation of any mathematical model depend on whether the given system of equations has a solution, and if it has, we check if the solution is unique. This subsection is thus concerned with finding the existence and uniqueness of solution of the model using Lipschitz criteria.

Let

$$f_1 = -\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t) \quad (1)$$

$$f_2 = \frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma) E(t) \quad (2)$$

$$f_3 = \sigma E(t) - (\mu + \gamma + \rho) I(t) \quad (3)$$

$$f_4 = -(\omega + \mu) R(t) + \gamma I(t) + vN(t) \quad (4)$$

Theorem 1: **(Derrick and Grossman, 1976)**

Let D^1 denotes the region

$|t - t_0| \leq a$, $\| x - x_0 \| \leq b$, $x = (x_1, x_2, \ldots\ldots x_n)$, $x_0 = (x_{10}, x_{20}, \ldots\ldots x_{n0})$

and suppose that f(t,x) satisfies the Lipschitz condition

$\|(t,x_1) - f(t,_2)\| \leq k\|x_1 - x_2\|$.

Whenever the pairs (t, x_1) and (t, x_2) belong to D^1, where k is a positive constant, Then, there is a constant $\delta > 0$ such that there exists a unique continuous vector solution x (t) of the system in the interval $t - t_0 \leq \delta$. It is important to note that the condition is satisfied by requirement that $\frac{\partial f_i}{\partial x_j}$; i=1,2,…, be continuous and bounded in D^1

We now return to our model equations (1) – (5). We are interested in the region

$$0 \leq \alpha \leq R$$

We look for a bounded solution in this region and whose partial derivatives satisfy

$\delta \leq \alpha \leq 0$, where α and δ are positive constants

Theorem 2

Let D' denote the region $0 \leq \alpha \leq R$. Then, equations (1) – (5) have a unique solution. We show that $\frac{\partial f_i}{\partial x_j}$, $i, j = 1, 2, 3, 4$ are continuous and bounded inD'.

Recall

$$f_1 = -\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t) \quad (1)$$

$$f_2 = \frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma) E(t) \quad (2)$$

$$f_3 = \sigma E(t) - (\mu + \gamma + \rho) I(t) \quad (3)$$

$$f_4 = -(\omega + \mu) R(t) + \gamma I(t) + vN(t) \quad (4)$$

Using equation (1), we have the partial derivatives below

$\left|\frac{\partial f_1}{\partial S}\right| = \left|-\mu - \frac{\beta I}{N}\right| < \infty$; $\left|\frac{\partial f_1}{\partial E}\right| = 0 < \infty$; $\left|\frac{\partial f_1}{\partial I}\right| = \left|-\frac{\beta S}{N}\right| < \infty$; $\left|\frac{\partial f_1}{\partial R}\right| = \omega < \infty$

These partial derivatives exist, continuous and are bounded. Similarly, for the rest equations, we have that

$\left|\frac{\partial f_2}{\partial S}\right| = \left|\frac{\beta I_1}{N}\right| < \infty$ (Since we are dealing with a finite population)

$\left|\frac{\partial f_2}{\partial E}\right| = |-\mu - \sigma| < \infty$; $\left|\frac{\partial f_2}{\partial I}\right| = \left|\frac{\beta S}{N}\right| < \infty$; $\left|\frac{\partial f_2}{\partial R}\right| = 0 < \infty$

And

$\left|\frac{\partial f_3}{\partial S}\right| = 0 < \infty$; $\left|\frac{\partial f_3}{\partial E}\right| = |\sigma| < \infty$; $\left|\frac{\partial f_3}{\partial I}\right| = |-\mu - \gamma - \rho| < \infty$; $\left|\frac{\partial f_3}{\partial R}\right| = 0 < \infty$

Lastly,

$\left|\frac{\partial f_4}{\partial S}\right| = 0 < \infty$; $\left|\frac{\partial f_4}{\partial E}\right| = 0 < \infty$; $\left|\frac{\partial f_4}{\partial I}\right| = |\gamma| < \infty$; $\left|\frac{\partial f_4}{\partial R}\right| = |-\omega - \mu| < \infty$

Since all the partial derivatives exist and are finite (bounded and defined), the system of equations has a feasible solution in R^4

2.3 Equilibrium States of the Model

2.3.1 The Disease Free Equilibrium State (DFE)

Diseases free equilibrium state is the equilibrium in the absence of infection and is such that I(t)=0. At equilibrium state, $\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$

$$-\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t) = 0 \quad (5)$$

$$\frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma) E(t) = 0 \quad (6)$$

$$\sigma E(t) - (\mu + \gamma + \rho) I(t) = 0 \quad (7)$$

$$-(\omega + \mu) R(t) + \gamma I(t) + vN(t) = 0 \quad (8)$$

From equation (7), $E(t) = \frac{(\mu+\gamma)I(t)}{\sigma}$

But E(t) = 0 since I(t)=0

From equation (8);

$$R = \frac{vN(t)}{\omega + \mu}$$

From equation (5);

$$S = \frac{1}{\mu}\left\{\pi N(t) + vN(t)\left[\frac{\omega}{\omega+\mu} - 1\right]\right\}$$

Hence the Disease Free Equilibrium State is given as

$$(S, E, I, R) = \left(\frac{N}{\mu}\left\{\pi - \frac{\mu V}{\omega+\mu}\right\}, 0, 0, \frac{vN(t)}{\omega+\mu}\right)$$

2.3.2 The endemic equilibrium state (EE)

At endemic equilibrium state, $I(t) \neq 0$; we recall the system of equations at equilibrium state;

$-\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t) = 0$ (5)

$\frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma)E(t) = 0$ (6)

$\sigma E(t) - (\mu + \gamma + \rho)I(t) = 0$ (7)

$-(\omega + \mu)R(t) + \gamma I(t) + vN(t) = 0$ (8)

Let S(t) = k, E(t) = x, I(t) = y, R(t) = z

Therefore, the system of equations becomes:

$-\mu k + \omega z + \pi N(t) - \frac{\beta k y}{N} - vN = 0$ (9)

$\frac{\beta k y}{N} - (\mu + \sigma)x = 0$ (10)

$\sigma x - (\mu + \gamma + \rho)y = 0$ (11)

$-(\omega + \mu)z + \gamma y + vN = 0$ (12)

From equation (11);

$$x = \frac{(\mu+\gamma+\rho)}{\sigma} y \quad (13)$$

Putting equation (13) into (10) to obtain:

$$\frac{\beta k y}{N} - (\mu + \sigma)\frac{(\mu + \gamma + \rho)}{\sigma} y = 0$$

$$y\left[\frac{\beta k}{N} - (\mu + \sigma)\frac{(\mu + \gamma + \rho)}{\sigma}\right] = 0$$

$$y \neq 0 \;\; \gg \;\; \frac{\beta k}{N} - (\mu + \sigma)\frac{(\mu + \gamma + \rho)}{\sigma} = 0$$

Hence; $k = \frac{N(\mu+\sigma)(\mu+\gamma+\rho)}{\sigma\beta}$ (14)

From equation (12);

$-(\omega + \mu)z + \gamma y + vN = 0$

$z = \frac{1}{\omega+\mu}(vN + \gamma y)$ (15)

From equation (9); $-\mu k + \omega z + \pi N(t) - \frac{\beta k y}{N} - vN$

$z = \frac{1}{\omega}\left(\frac{\beta k y}{N} + \mu k - \pi N(t) + vN\right)$ (16)

Equating (15) and (16), we have:

$$\frac{1}{\omega+\mu}(vN + \gamma y) = \frac{1}{\omega}\left(\frac{\beta k y}{N} + \mu k - \pi N(t) + vN\right)$$

$$\left(\frac{\gamma}{(\omega+\mu)} - \frac{\beta k}{\omega N}\right)y = \frac{\mu}{\omega}k - \frac{N}{\omega}(\pi - v) - \frac{vN}{\omega+\mu}$$

$$\left(\frac{\omega N\gamma - (\omega+\mu)\beta k}{\omega N(\omega+\mu)}\right)y = \frac{(\omega+\mu)\mu k - N(\omega+\mu)(\pi - v) - \omega v N}{\omega(\omega+\mu)}$$

$$y = \frac{\omega N(\omega+\mu)}{\omega N\gamma - (\omega+\mu)\beta k}\left(\frac{(\omega+\mu)\mu k - N(\omega+\mu)(\pi - v) - \omega v N}{\omega(\omega+\mu)}\right)$$

Using the value of k as obtained in equation (14);

$$y = N\left(\frac{(\omega+\mu)\left\{\mu\frac{N(\mu+\sigma)(\mu+\gamma+\rho)}{\sigma\beta} - N(\pi-v)(\omega+\mu)\right\} - \omega vN}{\omega N\gamma - (\omega+\mu)\beta\frac{N(\mu+\sigma)(\mu+\gamma+\rho)}{\sigma\beta}}\right)$$

$$y = \frac{N}{\beta}\left(\frac{\sigma\beta(\omega+\mu)(v-\pi)+\mu(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)-\omega v\beta\sigma}{\sigma\omega\gamma-(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)}\right) \quad (17)$$

From equation (13); $x = \frac{(\mu+\gamma+\rho)}{\sigma}y$

Using equation (17) for the value of y in equation (13), we have;

$$x = \frac{N(\mu+\gamma+\rho)}{\sigma\beta}\left(\frac{\sigma\beta(\omega+\mu)(v-\pi)+\mu(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)-\omega v\beta\sigma}{\sigma\omega\gamma-(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)}\right) \quad (18)$$

From equation (15); $z = \frac{1}{\omega+\mu}(vN+\gamma y)$

$$z = \frac{N}{\omega+\mu}\left(\frac{\gamma}{\beta}\left(\frac{\sigma\beta(\omega+\mu)(v-\pi)+\mu(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)-\omega v\beta\sigma}{\sigma\omega\gamma-(\omega+\mu)(\mu+\sigma)(\mu+\gamma+\rho)}\right)+v\right) \quad (19)$$

Hence, the endemic equilibrium states are given as in equations number (14), (17), (18) and (19).

2.4 Dynamical Behavior of the model

2.4.1 Stability analysis of the disease free equilibrium state

It has already been established that the system of equations (1) – (4) has disease free equilibrium state

$$E_1 = (S,E,I,R) = \left(\frac{N}{\mu}\left\{\pi - \frac{\mu v}{\omega+\mu}\right\}, 0, 0, \frac{vN(t)}{\omega+\mu}\right)$$

Again, the general variational matrix corresponding to the system is given as

$$J = \begin{bmatrix} -\frac{\beta I}{N}-\mu & 0 & -\frac{\beta S}{N} & \omega \\ \frac{\beta I}{N} & -(\mu+\sigma) & \frac{\beta S}{N} & 0 \\ 0 & \sigma & -(\mu+\gamma+\rho) & 0 \\ 0 & 0 & \gamma & -(\mu+\omega) \end{bmatrix} \quad (20)$$

At the disease free equilibrium state, using the expression E_1, we obtain;

$$J = \begin{bmatrix} -\mu & 0 & -\frac{\beta}{\mu}(\pi-\frac{\mu v}{\omega+\mu}) & \omega \\ 0 & -(\mu+\sigma) & \frac{\beta}{\mu}(\pi-\frac{\mu v}{\omega+\mu}) & 0 \\ 0 & \sigma & -(\mu+\gamma+\rho) & 0 \\ 0 & 0 & \gamma & -(\mu+\omega) \end{bmatrix} \quad (21)$$

The characteristics equation is obtained from the Jacobian determinant with the values λ.

$$|J-\lambda I| = \begin{vmatrix} -\mu-\lambda & 0 & -\frac{\beta}{\mu}(\pi-\frac{\mu v}{\omega+\mu}) & \omega \\ 0 & -(\mu+\sigma)-\lambda & \frac{\beta}{\mu}(\pi-\frac{\mu v}{\omega+\mu}) & 0 \\ 0 & \sigma & -(\mu+\gamma+\rho)-\lambda & 0 \\ 0 & 0 & \gamma & -(\mu+\omega)-\lambda \end{vmatrix} = 0$$

The above matrix reduces to

$$(-\mu-\lambda)(-\mu-\omega-\lambda)\begin{vmatrix} -(\mu+\sigma)-\lambda & \frac{\beta}{\mu}(\pi-\frac{\mu v}{\omega+\mu}) \\ \sigma & -(\mu+\gamma+\rho)-\lambda \end{vmatrix} = 0$$

Clearly the first two eigen values are:

$$\lambda_1 = -\mu$$
$$\lambda_2 = -(\mu+\omega)$$

The remaining characteristic equation is given as:

$$\lambda^2 + (2\mu + \sigma + \gamma + \rho)\lambda + (\mu + \sigma)(\mu + \gamma + \rho) - \frac{\sigma\beta(\pi(\omega + \mu) - \mu v)}{\mu(\omega + \mu)} = 0$$

We use criteria to check for the roots of the characteristics equation above.

Theorem (Routh-Hurwitz condition for roots of quadratic equation): The roots of the characteristics equation of a quadratic equation

$a_2\lambda^2 + a_1\lambda + a_0 = 0, will$have negative roots, if and only if the entire coefficients satisfy$a_n > 0$

The resulting quadratic equation:

$$\lambda^2 + (2\mu + \sigma + \gamma + \rho)\lambda + (\mu + \sigma)(\mu + \gamma + \rho) - \frac{\sigma\beta(\pi(\omega + \mu) - \mu v)}{\mu(\omega + \mu)} = 0$$

Will have all negative roots if

$$(\mu + \sigma)(\mu + \gamma + \rho) > \frac{\sigma\beta(\pi(\omega + \mu) - \mu)}{\mu(\omega + \mu)}$$

Hence in the absence of the disease (DFE), β is zero, hence it follows that the above inequality holds, and therefore the system of equations is said to be locally asymptotically stable.

2.5 Reproduction Number

The global asymptotic stability of the system of equations at the disease free equilibrium (DFE) can be established using the reproduction number of the system. By definition, the reproduction number of an infectious disease is the number of secondary case obtained from the introduction of an individual with a disease into a population of susceptible individuals (Abderrahman et al., 2007). The numerical value a basic reproduction numberR_0 gives an insight to the sustainability or otherwise of a disease in a population. If the computed$R_0 < 1$ then we conclude that the system of the equations at the disease free equilibrium is globally asymptotically stable and the disease dies out with the adequate vaccination strategy otherwise the system is unstable and the disease is maintained in the population. We introduced the technique due to Diekmann (1990) to constructs an n x n matrix from the system of equations of the model by considering only the infective classes in order to establish the basic reproduction number of the system of equations at the disease free equilibrium (DFE).

2.5.1 The Next Generation Matrix

Define $X_s$to be the set of all Disease Free states, that is

$X_s = \{x \geq 0 \mid x_i = 0, i = 1, 2, 3, \ldots\}$

In order to compute R_0; it is important to distinguish new infections from all other changes in the population.

Let $F_i(x)$ be the rate of appearance of new infections in compartments *i*,

V_i^+be the rate of transfer of individuals into compartment *i* by all other means.

V_i^-be the rate of transfer of individuals out of compartment *i:*

It is assumed that each function $(F_i(x), V_i^+, V_i^-)$ is continuously differentiable at least twice with respect to each variable.

The transmission model consists of the non-negative initial conditions together with the following system of equations

$\dot{x}_i = f_i(x) = F_i(x) - V_i(x), \quad i = 1, 2, 3, \ldots, n$

Where $V_i(x) = V_i^- - V_i^+$ and the functions satisfy the following conditions:

(a) If $x \geq 0$, then $F_i(x), V_i^-(x), V_i^+(x) \geq 0$ for$i = 1, 2, 3, \ldots, n$. That is, if the compartment is empty, there will be no transfer of individuals out of the compartment by death, infection nor other means

(b) If $x_i = 0$, then $V_i^- = 0$ (that is, nobody leaves the compartment). In particular if $x \in X_s$, then $V_i^- = 0$ for $i = 1, 2, 3, \ldots, m$

(c) $F_i = 0$, for $i > m$ (m is the number of infective classes)

(d) If $x \in X_s$, then $F_i = 0$ and $V_i^- = 0$ for all $i = 1, 2, 3, \ldots, m$

(e) If $F_i(x)$ is set to zero, then all the eigenvalues of $DF(x_0)$ have negative real parts.

Lemma 1.1

If x_0 is a disease free equilibrium (DFE) of the system of equations and $f_i(x)$ satisfies conditions (a) – (e), then the derivatives $DF(x_0)$ and $DV(x_0)$ are partitioned as

$$DF(x_0) = \begin{pmatrix} f_{11} & \cdots & f_{1j} \\ \vdots & \ddots & \vdots \\ f_{i1} & \cdots & f_{ij} \end{pmatrix}, i = 1, 2, 3, \ldots, m; \; j = 1, 2, 3, \ldots, m$$

$$DV(x_0) = \begin{pmatrix} v_{11} & \cdots & v_{1j} \\ \vdots & \ddots & \vdots \\ v_{i1} & \cdots & v_{ij} \end{pmatrix}, i = 1, 2, 3, \ldots, m; \; j = 1, 2, 3, \ldots, m$$

Where F and V are the m x m matrix defined by:

$$F = \frac{\partial F(x_0)}{\partial x_j} \; and \; V = \frac{\partial V(x_0)}{\partial x_j}$$

F is non-negative and V is a non-singular matrix.

From Diekmann et al. (2000), the product of the matrix FV^{-1} is called the next generation matrix for the model and we shall set the reproduction number (R_0) as equal to the spectral radius $\rho(FV^{-1})$ $i.e.$

$$R_0 = \rho(FV^{-1})$$

We shall now apply the method to find the reproduction number of the model

$$\frac{dS}{dt} = -\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t)$$
$$\frac{dE}{dt} = \frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma) E(t)$$
$$\frac{dI}{dt} = \sigma E(t) - (\mu + \gamma + \rho) I(t)$$
$$\frac{dR}{dt} = -(\omega + \mu) R(t) + \gamma I(t) + vN(t)$$

Re-arranging the equations such that we start with the infective classes;

$$\frac{dE}{dt} = \frac{\beta S(t) I(t)}{N(t)} - (\mu + \sigma) E(t)$$
$$\frac{dI}{dt} = \sigma E(t) - (\mu + \gamma + \rho) I(t)$$
$$\frac{dR}{dt} = -(\omega + \mu) R(t) + \gamma I(t) + vN(t)$$
$$\frac{dS}{dt} = -\mu S(t) + \omega R(t) + \pi N(t) - \frac{\beta S(t) I(t)}{N(t)} - vN(t)$$

There are two infective classes in the system of equations above which are the Exposed compartment (E) and the Infected compartment (I), hence the next generational matrix is as given below:

$$F_i = \begin{pmatrix} \frac{\beta S(t) I(t)}{N(t)} \\ 0 \end{pmatrix}, V_i = \begin{pmatrix} (\mu + \sigma) E(t) \\ (\mu + \gamma + \rho) I(t) - \sigma E(t) \end{pmatrix},$$

We get F and V by getting the partial derivatives of the above matrix with respect to the infective compartment.

$$F = \begin{pmatrix} 0 & \frac{\beta S}{N} \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} (\mu + \sigma) & 0 \\ -\sigma & (\mu + \gamma + \rho) \end{pmatrix}$$

The inverse of the matrix V is obtained as:

$$V^{-1} = \begin{pmatrix} \frac{1}{(\mu + \sigma)} & 0 \\ \frac{\sigma}{(\mu + \sigma)(\mu + \gamma + \rho)} & \frac{1}{(\mu + \gamma + \rho)} \end{pmatrix}$$

The product of matrix F and V^{-1} is:

$$FV^{-1} = \begin{pmatrix} 0 & \frac{\beta S}{N} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\mu + \sigma)} & 0 \\ \frac{\sigma}{(\mu + \sigma)(\mu + \gamma + \rho)} & \frac{1}{(\mu + \gamma + \rho)} \end{pmatrix}$$
$$FV^{-1} = \begin{pmatrix} N\frac{\beta S \sigma}{(\mu + \sigma)(\mu + \gamma + \rho)} & \frac{\beta S}{N(\mu + \gamma + \rho)} \\ 0 & 0 \end{pmatrix}$$

We recall the disease free equilibrium state as:

$$(S, E, I, R) = \left(\frac{N}{\mu} \left\{ \pi - \frac{\mu v}{\omega + \mu} \right\}, 0, 0, \frac{vN(t)}{\omega + \mu} \right)$$

Using the value of S as above in the matrix for FV^{-1}, we have:

$$FV^{-1} = \begin{pmatrix} \frac{\beta \sigma}{\mu(\mu + \sigma)(\mu + \gamma + \rho)} \left\{ \pi - \frac{\mu v}{\omega + \mu} \right\} & \frac{\beta}{\mu(\mu + \gamma)} \left\{ \pi - \frac{\mu v}{\omega + \mu} \right\} \\ 0 & 0 \end{pmatrix}$$

The reproduction number is thus obtained as the spectral radius (the largest eigenvalue) of the above matrix, which is readily seen from the matrix above as:

$$R_0 = \rho(FV^{-1}) = \frac{\beta\sigma}{\mu(\mu+\sigma)(\mu+\gamma+\rho)}\left\{\pi - \frac{\mu v}{\omega+\mu}\right\}$$

Table 1: Source of Initial Value of the Parameter

Parameter	**Interpretation**	**Value**	**Source**
β	Transmission constant	0.091	Fred et al., (2014)
σ	Latent period	0.125	Fred et al., (2014)
μ	Natural death rate	0.005	Hypothetical value
π	Recruitment rate	0.45	Edward et al., (2014)
ω	Rate of losing immunity	0.36	Edward et al., (2014)
ρ	Disease induced death	0.009	Fred et al., (2014)
γ	Recovery rate	0.6	Edward et al., (2014)
v	Rate of vaccination	0.7	Edward et al., (2014)

The table above gives the source of initial value of each parameter except natural death rate μ which cannot be quantified. The values of some parameters are then varied to determine the effect on the reproduction number and the result is presented in the table below.

2.6 Numerical Simulation of Reproduction Number

Table 2: Numerical Simulation of Reproduction number.

π	β	σ	μ	γ	v	ω	ρ	R_0	Remark
0.45	0.091	0.125	0.005	0.60	0.70	0.36	0.009	1.0019	Unstable
0.45	0.191	0.125	0.005	0.60	0.70	0.36	0.009	2.0866	Unstable
0.45	0.291	0.125	0.005	0.60	0.70	0.36	0.009	3.1791	Unstable
0.45	0.391	0.125	0.005	0.60	0.70	0.36	0.009	4.2715	Unstable
0.45	0..081	0.125	0.005	0.60	0.70	0.36	0.009	0.8918	Stable
0.45	0.071	0.125	0.005	0.60	0.70	0.36	0.009	0.7817	Stable
0.45	0.061	0.125	0.005	0.60	0.70	0.36	0.009	0.6716	Stable
0.45	0.051	0.125	0.005	0.7	0.75	0.36	0.009	0.5607	Stable
0.45	0.041	0.125	0.005	0.7	0.80	0.36	0.009	0.4500	Stable
0.45	0.031	0.125	0.005	0.7	0.85	0.36	0.009	0.3397	Stable
0.45	0.021	0.125	0.005	0.7	0.90	0.36	0.009	0.2298	Stable
0.45	0.011	0.125	0.005	0.06	0.95	0.36	0.009	0.1202	Stable

2.7 Results and Discussion

This work analyzed the stability condition for an SEIR model of an infectious disease. We established that the system of equations has a unique solution using the Lipchitz condition. The disease free and the endemic equilibria states were obtained. The Jacobian matrix was evaluated for the DFE state and the roots of the resulting matrix were all negative which established the local asymptotic stability of the model. The reproduction number at the DFE was computed and simulated and the results show that it is globally stable at DFE for certain parameter values. The transmission rate β of an infection is a vital parameter that determines the effect of an infectious disease on a population of susceptible individuals. Its increment (that is,β) has a significant effect on the reproduction number of the model as can be seen above. The greater the transmission rates of an infection, the speedy the spread of the disease across the population. This is why the rate of spread of certain epidemic diseases like Ebola is quite outrageous compared to other diseases. From the simulation table, we deduce that at constant rate of loosing immunity and decreasing rate of disease transmission due to immunity from vaccination, the effective reproduction number at disease free equilibrium (DFE) tends to stable state. Also, increase in vaccination rate tends to move the model to a more stable disease free equilibrium because the effective reproduction number decreases with increase in vaccination rate.

2.8 Conclusion

In this work, we developed an SEIR epidemic model to show the efficacy of vaccination on the control of epidemic diseases. The disease free equilibrium state of the model was established and found out to be stable.

The disease free equilibrium state was stable because all the eigenvalues were negative. Stability of the disease free equilibrium state implied that when an epidemic manifested in a population, it died out with the adequate vaccination strategy.More so, the computation of reproduction number R_0 and the numerical simulation of it give an insight into the spread of an infectious disease within a population. The lower the contact rate between an infected and non-infected individuals, the lower the spread of the disease in the population and vice-versa; the vaccination parameter v reduces the effect of β on the reproduction number of the model, hence adequate and effective vaccination of individuals is an effective way of reducing the outbreak of disease in a population.

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