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On the Formulation of Finite Element Model for the Solution of Second Order Boundary Value Problems Using Garlerkin Method

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Abstract

In this paper, we derive a finite element model for solving one dimensional boundary value problems using Galerkin method. The mathematical brevity in the equal distribution of the differentiation among the weight function $w(x)$ and the dependent variable $u(x)$ yields the weak form of the differential equation. Upon the assembling of the element equation and imposition of boundary condition, we obtain exactly the same number of algebraic equation as the number of unknown primary and secondary degrees of freedom. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1.0 Introduction

Obviously, all problems that are describable by ordinary and partial differential equations can be solved by the finite element method. In this paper, we consider only steady state problems [1,2,3].the application of the finite element method to a given problem involves six steps namely (1) Discretization of the domain into a finite element mesh (2) Development of element equation (3) Assembly of elements to obtain the equation of the whole problem (4) Imposition of the boundary conditions of the problem (5) solution of the assembled equations (6) post-processing of the result.

1.1 Model Boundary Value Problem

Taking equation (1.1) as our model boundary value problem, let us consider the problem of finding the function $u(x)$ that satisfies the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-q=0 \qquad \textit{for} \quad 0<x<L$$

And the boundary conditions (1.1)

$$u(0)=u_0 \qquad , \qquad \left(a\frac{du}{dx}\right)\Big|_{x=L}=Q_0$$

Our ability to develop a numerical procedure by which equation (1.1) can be solved to all possible boundary conditions could be viewed as a procedure for solving all field problems ranging from conduction and convention heat transfer in a plane wall, Axial deformation of bars, transverse deflection of a cable etc.

2.0 Formulation of the Model

After discretizing the domain Ω=(0,L) into a set of line segments called finite elements of length located between points A and B (see Figure 1.1) we derive the algebraic equation that relates the primary variables to the secondary variables at the nodes of the elements. Usually, this involves three steps namely (1) construction of the weighted residual or weak form of the differential equation (2) Assume the form of the approximate solution over a typical finite element (3) Derive the finite element equations by substituting the approximate solution into the weak form.

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Step 1: WEAK FORM

We multiply the governing differential equation with a weight function w(x) and integrate over a typical element i.e

$$0 = \int_{X_A}^{X_B} w(x)\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)+Cu-q\right]dx \qquad (1.2)$$

$$0 = \int_{X_{AA}}^{X_B}\left[-w(x)\frac{d}{dx}\left(a\frac{du}{dx}\right)+cw(x)u(x)-w(x)q\right]dx \qquad (1.3)$$

But

$$-w(x)\frac{d}{dx}\left(a\frac{du}{dx}\right) = -\frac{d}{dx}\left(w(x)a\frac{du}{dx}\right)+a\frac{dw}{dx}\frac{du}{dx} \qquad (1.4)$$

Hence (1.3) becomes

$$0 = \int_{X_A}^{X_B}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wq\right)dx-\left[Wa\frac{du}{dx}\right]_{X_A}^{X_{BA}} \qquad (1.5)$$

As a rule in finite element method,

$$-Q_A = \left(a\frac{du}{dx}\right)\bigg|_{X_A}, \qquad (1.6)$$

$$Q_B = \left(a\frac{du}{dx}\right)\bigg|_{X_B} \qquad (1.7)$$

Where u is the primary variable and its specification constitute the essential boundary condition and $\left(a\frac{du}{dx}\right)=Q_0$ is the secondary variable and its specification constitutes the natural boundary condition.

Hence with notation in (1.6), (1.7), the variational (or weak) form becomes

$$0 = \int_{X_A}^{X_B}\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wq\right)dx-w(x_A)Q_A-W(x_B)Q_B \qquad (1.8)$$

Step 2: Approximation of the solution

Having gotten the weak form of the differential equation, we seek the approximation solution U^e over the element $\Omega^e = (x_A, x_B)$ on the form of algebraic polynomial [4,5]. For the variational statement at hand, the minimum polynomial order is linear. A complete polynomial is of the form

$$U^e = a + bx \qquad (1.9)$$

Where a and b are constant and can be determined using the nodal conditions

$$U^e = u_1^e \qquad at\ x = x_A, U^e = u_2^e \quad at \quad x = x_B$$

i.e

$$u_1^e = a + bx_A \quad , u_2^e = a + bx_B$$

In matrix form

$$\begin{pmatrix} u_1^e \\ u_2^e \end{pmatrix} = \begin{pmatrix} 1 & x_A \\ 1 & x_B \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Using crammer's rule, yields

$$a = \frac{1}{h_e}(u_1^e x_B - u_2^e x_A) \quad , \quad b = \frac{1}{h_e}(u_2^e - u_1) \qquad (1.10)$$

where $h_e = x_B - x_A$

Substitution of (1.10) into (1.9) yields

$$u(x) \approx U^e(x) = \psi_1^e(x)u_1^e + \psi_2^e(x)u_2^e = \sum_{j=1}^{2} \psi_j(x)u_j^e \tag{1.11}$$

Where $\psi_1^e(x) = \dfrac{x_B - x}{h_e} = 1 - \dfrac{\bar{x}}{h_e}$ (local coordinate)

$\psi_2^e(x) = \dfrac{x - x_A}{h_e} = \dfrac{\bar{x}}{h_e}$ (local coordinate) since $x = x_1 + \bar{x}$

Which are called the linear element approximation function (or shape functions).

Step 3: FINITE ELEMENT MODEL

Since we are using the Galerkin method, we substitute (1.11) for $u(x)$ and $\psi_1^e, \psi_2^e, ..., \psi_n^e$ for $w(x)$ into the weak form (1.8) to obtain n algebraic equations

$$0 = \int_{x_A}^{x_B} \left[a \frac{d\psi_1^e}{dx} \left(\sum_{j=1}^{n} \psi_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_1^e \left(\sum_{j=1}^{n} \psi_j^e \frac{\psi_j^e}{dx} \right) - \psi_1^e q \right] dx - \sum_{j=1}^{n} \psi_1^e(x_j) Q_j^e \tag{1.12}$$

$$0 = \int_{x_A}^{x_B} \left[a \frac{d\psi_2^e}{dx} \left(\sum_{j=1}^{n} \psi_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_2^e \left(\psi_j^e \psi_j^e(x) \right) - \psi_2^e q \right] dx - \sum_{j=1}^{n} \psi_2^e(x_j) Q_j^e \tag{1.13}$$

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$$0 = \int_{x_A}^{x_B} \left[a \frac{d\psi_i}{dx} \left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_i^e \left(\sum_{j=1}^{n} u_j^e \psi_j^e(x) \right) - \psi_i^e q \right] dx - \sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e \tag{1.14}$$

i^{th} equation

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$$0 = \int_{x_A}^{x_B} \left[a \frac{d\psi_n^e}{dx} \left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx} \right) + c\psi_n \left(\sum_{j=1}^{n} u_j^e \psi_j^e(x) \right) - \psi_n^e q \right] dx - \sum_{j=1}^{n} \psi_n^e(x_j^e) Q_j^e \tag{1.15a}$$

The i^{th} algebraic equation can be written as

$$0 = \sum_{j=1}^{n} k_{ij}^e u_j^e - f_i - Q_i \qquad (i = 1,2,\ldots,n) \tag{1.15b}$$

Where

$$k_{ij}^e = \int_{x_A}^{x_A} \left(a \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c\psi_i^e \psi_j^e \right) dx = B(\psi_i, \psi_j), \tag{1.15c}$$

$$f_i^e = \int_{x_A}^{x_B} q\psi_i^e dx = l(\psi_i^e)$$

Equation (1.5) can be expressed in terms of the coefficients $kij^e,\ f_i^e\ and\ Q_i^e$ as

$$
\begin{aligned}
&k_{11}\, u_1^e + k_{12}^e\, u_2^e + \ldots + k_{1n}^e\, u_n^e = f_1^e + Q_1^e \\
&k_{21}\, u_2^e + k_{22}^e\, u_2^e + \ldots + k_{2n}^e\, u_n^e = f_2^e + Q_2^e \\
&. \qquad\qquad . \qquad\qquad . \qquad . \\
&. \qquad\qquad . \qquad\qquad . \qquad . \\
&. \qquad\qquad . \qquad\qquad . \qquad . \\
&k_{n1}\, u_1^e + k_{n2}^e\, u_2^e + \ldots + k_{nn}^e\, u_n^e = f_n^e + Q_n^e
\end{aligned}
\tag{1.16a}
$$

In matrix notation, the linear algebraic equation (1.16a) can be written as

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \tag{1.16b}$$

The matrix [**k**^{e}] is called the coefficient matrix, or stiffness matrix. The column vector{**f**^{e}} is the source vector, or force vector.

For mesh of linear elements, the element Ω^e is located between the global nodes $x_A = x_e\ and\ x_B = x_{e+1}$

Hence

$$k_{ij}^e = \int_{x_e}^{x_{e+1}} \left(a_e \frac{d\psi_i}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i\, \psi_j^e \right) dx, \quad f_2^e = \int_{x_e}^{x_{e+1}} q_i\ \psi_i^e\ dx \tag{1.17}$$

or, in the local coordinate system $\bar{x}$

$$k_{ij}^e = \int_{e0}^{h} \left(a_e \frac{d\psi_i}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{0_e}^{h_{e1}} q_e\ \psi_i^e\ d\bar{x} \tag{1.18}$$

Where $x = x_e + \bar{x}\ and\ dx = d\bar{x},\ \dfrac{d\psi_i^e}{dx} = \dfrac{d\psi_i^e}{d\bar{x}}$

But $\quad \psi_1^e(\bar{x}) = 1 - \dfrac{\bar{x}}{h_e}, \quad \psi_2^e(\bar{x}) = \dfrac{\bar{x}}{h_e}$

We can compute k_{ij}^e and f_i^e by evaluating the integral to have

$$k_{11}^e = \frac{a_e}{h_e} + \tfrac{1}{3} c_e h_e$$

$$k_{12}^e = k_{21}^e = -\frac{a_e}{h_e} + \tfrac{1}{6} c_e h_e \qquad \text{(by symmetry)}$$

$$k_{22}^e = \frac{a_e}{h_e} + \tfrac{1}{3} c_e h_e$$

Similarly

$$f_1^e = \tfrac{1}{2} q_e h_e, \qquad f_{21}^e = \tfrac{1}{2} q_e h_e$$

Hence

$$[k^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{1.19a}$$

$$\{f^e\} = \frac{q_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

When a and q are element constant c=0, the finite element equations corresponding to the linear element are

$$\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{q_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \qquad (1.20)$$

$$\frac{a_e}{h_e}u^e - \frac{a_e}{h_e}u_2^e = \tfrac{1}{2}q_e h_e + Q_2^e$$

$$-\frac{a_e}{h_e}u_1{}^e + \frac{a_e}{h_e}u_2^e = \tfrac{1}{2}q_e h_e + Q_2^e \qquad (1.20b)$$

3.0 Connectivity of Elements

The assembly of element is carried out by imposing the following two conditions

(1) Continuity of primary variables at connecting nodes

$$u_n^e = u_1^{e+1}$$

(2). Balance of secondary variables at connecting nodes

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \textit{if no external point source is applied} \\ Q_0 & \textit{if an external point source of magnitude } Q_0 \textit{ is applied} \end{cases}$$

The interelement continuity of the primary variables is imposed by renaming the two variables $u_n^e = u_1^{e+1}$ at $x = x_N$ as one and the same.

To enforce balance of the secondary variables Q_i^e, it is clear that we set $Q_n^e + Q_1^{e+1}$ equal to zero or a specified value only if we have such expression in our equations. To obtain such expressions, we must add the n[th] equation of the element Ω^e to the first equation of the element Ω^{e+1} that is

$$\sum_{j=1}^{n} k_{nj}^e u_j^e = f_n^e + Q_n^e \quad \text{and} \quad \sum_{j=1}^{n} k_{1j}^{e+1} u_j^{e+1} = f_1^{e+1} + Q_1^{e+1}$$

To give

$$\sum_{j=1}^{n}\left(k_{nj}^e U_j^e + k_{1j}^{e+1} U_j^{e+1}\right) = f_n^e + f_1^{e+1} + \left(Q_n^e + Q_1^{e+1}\right)$$

$$= f_n^e + f_1^{e+1} + Q_0$$

Thus for a mesh of G linear elements (n=2), we have

$$\begin{bmatrix} k^1{}_{11} & k^1{}_{12} & & & \\ k^1{}_{21} & k_{22}^1 + k_{11}^2 & k_{12}^2 & 0 & \\ & k_{21}^2 & k_{22}^2 + k_{11}^3 & & \begin{matrix} \dots \\ k_{12}^G \end{matrix} \\ & & & \dots & \\ .. & .. & \dots & k_{22}^{G-1} + k_{11}^G & \\ 0 & & & k_{21}{}^G & k^G{}_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ u_G \\ u_{G+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \cdot \\ f_2^{G-1} + f_1^G \\ f_2^G \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \cdot \\ Q_2^{G-1} + Q_1^G \\ Q_2^G \end{Bmatrix} \qquad (1.21)$$

We then apply the boundary condition to the system of equations and solve to obtain the desired number of degree of freedoms.

The solution of the finite element equations give the nodal values of the primary unknown (e.g, displacement, velocity or temperature). Having established the finite element model for solving dimension boundary value problems, when then consider some applications.

4.0 Numerical Examples and Results

We apply this method on some special problems.

Problem 1: Consider a slab of thickness Land constant thermal conductivity K(wm^{-1} ^{0}c^{-1}). Suppose that the energy at a uniform rate of q_0 (wm^{-3}) is generated in the wall. Evaluate the temperature distribution in the wall when the boundary surfaces of the wall are subject to the boundary conditions.

T (0)=, and T(L)=T_2

Solution: The governing differential equations for this problem is given by

$$-\frac{d}{dx}(KA\frac{dT}{dx}) + c^nT = Aq + c^nT\infty \qquad , c^n = \rho B \qquad (1.22)$$

Where k is the thermal conductivity of the material, A is the cross sectional area, T is the temperature, q is the heat energy generated per unit volume. ρ is the density, c is the specific heat of the surrounding medium (the ambient temperature), β is the convection heat transfer coefficient.

Since we are dealing with a plane wall, we set c^n=0.

The equation reduces to

$$-\frac{d}{dx}(KA\frac{dT}{dx}) = Aq \qquad (1.23)$$

The finite element model is

$$\left[k^e\right]\left\{T^e\right\} = \left\{f^e\right\} + \left\{Q^e\right\} \qquad (1.24)$$

Where

$$k_{ij}^e = \int_{x_A}^{x_B}\left(KA\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\right)dx \qquad (1.25a)$$

$$f_i^e = \int_{x_A}^{x_B}\psi_i\, Aqdx \qquad (1.25b)$$

We must select order of approximation (or type of elements) to evaluate the coefficients k_{ij} and f_i^e in (1.25).For the choice of linear elements and the data $a = KA = cons\tan t\ and\, q = Aq_0 = cons\tan t$, (1.24) taken the form

$$\frac{KA}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^2 \end{Bmatrix} = \frac{Aq_0h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \qquad (1.26)$$

For a minimum of two elements, N=2, (h= $\frac{1}{2}L$) and applying the boundary condition

$$\frac{KA}{h}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{Bmatrix} T_1 \\ u_2 \\ T_2 \end{Bmatrix} = \frac{Aq_0}{2}\begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_2^2 \end{Bmatrix} \qquad (1.27)$$

The solutions are

$$u_2 = \frac{q_0 h^e}{2k} + 1/2 (T_1 + T_2) \qquad (1.28)$$

$$(Q_1^1)_{equil} = -q_0 hA + \frac{KA}{2h}(T_1 + T_2) \qquad (1.29)$$

$$(Q_2^2)_{equil} = -q_0 hA + \frac{KA}{2h}(T_2 - T_1) \qquad (1.30)$$

Problem 2: Consider the temperature distribution of an insulated rod length L=1 and thermal diffusivity D=10. A constant heat is also being generated at the rate of Q=10. The boundary conditions are T(0)=5 and $\left(\frac{dT}{dx}\right)\Big|_{x=1} = 10 = q$

Solution: Suppose we use four elements (n=4) to illustrate the finite element solution. The nodes are x_1, x_2, x_3, x_4 and x_5
The differentiation equation is governed by;

$$D\frac{d^2T}{dx^2} + Q = 0 \qquad (1.31)$$

Where D is the thermal diffusion, T is the temperature and Q is the heat generation. Equation (1.31) becomes

$$10\frac{d^2T}{dx^2} + 10 = 0 \qquad (1.32)$$

Following the procedures, the finite element model is

$$\left[k^e\right]\left\{T^e\right\} = \left\{f^e\right\} \qquad (1.33)$$

Where

$$k_{ij} = \frac{10}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

$$f_i = -\frac{10(0.25)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The prescribe boundary condition at the end where the temperature is fixed T_1=5 and at the opposite end where the flux boundary applied q_5 =-10. The assembled final system equations for elements becomes

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 5.03125 \\ 0.0625 \\ 0.0625 \\ -0.1875 \end{Bmatrix}$$

The finite element solution for the temperature profile produces the values
T_2=4.97, T_3=4.91, T_4=4.78 and T_5=4.60

5.0 Conclusion

In this paper, we have formulated a finite element model that could be used to solve differential equations. We have illustrated how finite element method utilizes discrete elements to obtain the approximate solution of the governing differential equation. In addition, we showed how the final system equation is constructed from the discrete element equation.

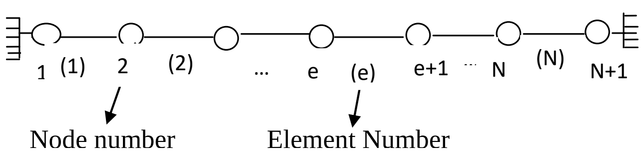

Figure 1.1: Finite Element Discretization of a One-Dimensional Domain

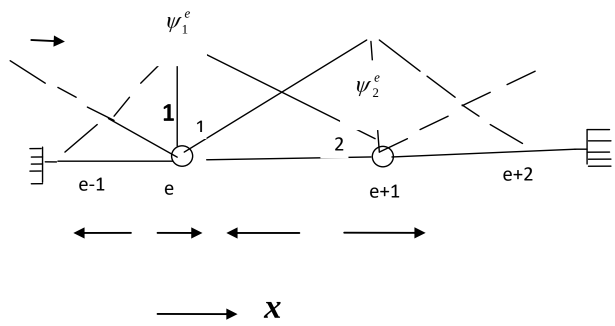

Figure 1.2: Local interpolation functions for two-node linear element $x_A = x_{e,} x_B = x_{e+1}$

6.0 References

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