Transactions of the Nigerian Association of Mathematical Physics
Volume 1, (November, 2015), **pp 237 – 254**

Novel State Results on Equipment Replacement Problems and Excel Solution Implementation Templates

Ukwu Chukwunenye

Department of Mathematics, University of Jos, P.M.B. 2084, Jos, Plateau State, Nigeria.

Abstract

This research article obtained the structure of the sets of feasible equipment ages corresponding to various decision periods, in equipment replacement problems, thereby obviating the need for network diagrams for such determination. The proofs were achieved by deft deployment of change of variables technique, maximum equipment operational age constraint and appropriate set addition definition. In the sequel, the article designed solution implementation templates for the corresponding dynamic programming recursions. These templates circumvent the inherent tedious and prohibitive manual computations associated with dynamic programming formulations and may be optimally appropriated for sensitivity analyses on such models.

Keywords: Ages, Excel, Equipment, Feasible, Dynamic, Optimal, Policy, Programming, Recursions, Replacement, Solution, Template.

1.0 Introduction

Consider the problem of researching an optimal Equipment Replacement policy over an $n-$ period planning horizon. At the start of each year a decision is made whether to keep the equipment in service an extra year or to replace it with a new one at some salvage value. As remarked by Taha[1] "the determination of the feasible values for the age of the machine at each stage is somewhat tricky". The determination process for the optimal replacement age is normally initiated with network diagrams with machine ages on the vertical axis and decision years on the horizontal axis. See also Winston [2]for alternative network representation. Then dynamic recursions are formulated as functions of the decision years, the corresponding feasible equipment ages, the problem data and the cash-flow profile. For the problem to be well-posed, it is pertinent that the cash-flow profile be provided for $\max\{m,n\}$ years, where m is the mandatory equipment replacement age. Unfortunately network diagrams are unwieldy, cumbersome and prone to errors, especially for large problem instances; consequently the integrity of the desired solution may be compromised. The need for analytic determination of sets of equipment ages corresponding to various decision years has therefore become imperative. Furthermore even after the sets of equipment ages have been optimally obtained, the manual computations involved in the dynamic recursions are for the most part quite prohibitive. Thus the design of electronic solution templates with minimal manual intervention would be a welcome development that should add to the existing body of knowledge. This issue has been effectively addressed in this research article.

2.0 Theoretical Analysis

In this section, the problem data, working definitions, elements of the DP model and the dynamic programming (DP) recursions are laid out as follows:

Equipment Starting age $= t_1$

Equipment Replacement age $= m$

S_i = The set of feasible equipment ages (states) in decision period i (say year i), $i \in \{1, 2, \ldots, n\}$

Corresponding author: Ukwu Chukwunenye, E-mail: cukwu@hotmail.com, Tel.: +2347052765751

$r(t)$ = annual revenue from a $t-$ year old equipment

$c(t)$ = annual operating cost of a $t-$ year old equipment

$s(t)$ = salvage value of a $t-$ year old equipment; $t = 0,1,\ldots,m$

I = fixed cost of acquiring a new equipment in any year

The elements of the DP are the following:

1. Stage i, represented by year $i,\ i \in \{1, 2, \ldots, n\}$
2. The alternatives at stage (year) i. These call for keeping or replacing the equipment at the beginning of year i
3. The state at stage (year) i, represented by the age of the equipment at the beginning of year i.

Let $f_i(t)$ be the maximum net income for years $i, i+1,\ldots, n-1, n$ given that the equipment is t years old at the beginning of year i.

Note: The definition of $f_i(t)$ starting from year i to year n implies that backward recursion will be used. Forward recursion would start from year 1 to year i.

The following theorem is applicable to the backward recursive procedure:

2.1 Theorem 1

$$f_i(t) = \max\begin{cases} r(t) - c(t) + f_{i+1}(t+1); \text{ IF KEEP} \\ r(0) + s(t) - I - c(0) + f_{i+1}(1); \text{ REPLACE} \end{cases}$$

$f_{n+1}(x) = s(x),\ \ i = 0,1,\ldots,n-1,\ x =$ age of machine at the start of period $n+1$

Proof

Decision: KEEP

$r(t)-c(t) =$ net revenue (income) from operating the equipment an extra year when it is already t years old. Then the equipment age advances to $t+1$ years and hence $f_{i+1}(t+1) =$ maximum income for years $i+1,\ldots,n$ given that the equipment is $t+1$ years old at the start of year $i+1$.

Decision: REPLACE

$r(0) = 1-$ year revenue from a new equipment (age 0)

$c(0)$ = cost of operating a new equipment for 1 year (from the start of year i to the end of year i)

I = fixed cost of a new equipment

$s(t)$ = salvage cost for a $t-$ year old equipment

$$\text{Net income} = \underbrace{r(0) - c(0)}_{\substack{\text{net income for operating} \\ \text{the new equipment from the} \\ \text{beginning of each feasible year to the} \\ \text{end of that year. The age of the} \\ \text{equipment then advances to 1 year}}} + s(t) - I + \underbrace{f_{i+1}(1)}_{\substack{\text{max net income for} \\ \text{years } i+1,\ldots,n \text{ given} \\ \text{that the machine is 1-year} \\ \text{old at the start of year } i+1}}$$

$f_{n+1}(x) = s(x)$ or $f_{n+1}(.) = s(.) \Rightarrow$ Sell off the equipment at the end of the planning horizon at price $s(.)$, regardless ofits age, with no further income realized from the beginning of year $n+1$, since the planning horizon length is n years. Therefore the recursive equation is correct. This completes the proof.

[2] provided the following alternative formulation and dynamic programming recursions of the equipment replacement problem:

The equipment is purchased and deployed at time 0. Let $g(t)$ be the minimum cost incurred from time t to time n, including the purchase cost and the salvage value for the newly purchased machine given that a new machine has been purchased at time t. Let c_{tx} be the net cost, including purchase cost and salvage value of purchasing a machine at time t and operating it until time x. Then the appropriate recursion is

$$g(t) = \min_x \{c_{tx} + g(x)\} \qquad (t = 0,1,\ldots,n-1)$$

Subject to: $t+1 \le x \le t+3$, $x \le n$ and the terminal condition $g(n) = 0$.

Verma [3] and Gupta & Hira [4], used average annual cost criteria to determine alternative optimal policies and the corresponding optimal rewards in a non-dynamic programming fashion.

2.2 Pertinent Remarks on the DP Recursions

For $i \in \{1, 2, \cdots, n\}$, $f_i(t)$ may be identified as $f_i(t) = \max\limits_{\{K,R\}} \left\{ f_i^K(t), f_i^R(t) \right\}$, where

$f_i^K(t) = r(t) - c(t) + f_{i+1}(t+1)$ and $f_i^R(t) = r(0) + s(t) - I - c(0) + f_{i+1}(1)$

For $i \in \{1, 2, \cdots, n\}$ and $t \in S_i$ the optimal decision may be identified as $D_i(t)$, where

$$D_i(t) = \underset{\{K,R\}}{\operatorname{argmax}}\, g_i(t, K, R);\ g_i(t, K, R) = \begin{cases} f_i^K(t), & \text{if Decision is KEEP} \\ f_i^R(t), & \text{if Decision is REPLACE} \end{cases}$$

Define

$$x_i = \begin{cases} 1, & \text{if decision is REPLACE in stage } i \text{ (start of decision year } i) \\ 0, & \text{if decision is KEEP in stage } i \text{ (start of decision year } i) \end{cases}$$

Then

$g_i(t, K, R) = f_i(t) = (1 - x_i) f_i^K(t) + x_i f_i^R(t),\ i \in \{1, 2, \cdots, n\}$

If the revenue profile is not given then $r(t)$ may be set identically equal to zero, in which case

$-f_i(t) =$ minimum cost associated with operating the equipment from the start of decision year i to the end of decision year n. If the variable cost profile is not given then $c(t)$ may be set identically equal to zero, in which case

$f_i(t) =$ maximum net revenue from the start of decision year i to the end of decision year n. If the cost profile is not given then $c(t)$ and I may be set identically equal to zero, in which case

$f_i(t) =$ maximum accrueable revenue from the start of decision year i to the end of decision year n. If the equipment is bought new at the beginning of the decision year 1, then the optimal return is $g_1(0) = -I + f_1(0)$ or $-g_1(0) = I - f_1(0)$, as appropriate.

3.0 Results and Discussion

3.1 Theorem 2 (First Main Result)

Let S_i denote the set of feasible equipment ages at the start of the decision year i. Let t_1 denote the age of the machine at the start of the decision year i, that is, $S_1 = \{t_1\}$. Then for $i \in \{1, 2, \ldots, n\}$,

$$S_i = \begin{cases} \left\{ \min\limits_{2 \le j \le i} \{ j - 1, m \} \right\} \cup \left\{ 1 + (i - 2 + t_1) \operatorname{sgn}\left(\max\{ m + 2 - t_1 - i, 0 \} \right) \right\}, & \text{if } t_1 < m \\ \left\{ \min\limits_{2 \le j \le i} \{ j - 1, m \} \right\}, & \text{if } t_1 \ge m \end{cases}$$

3.1.1 Corollary 1

If $t_1 < m$, then for $i \in \{2, \ldots, n\}$,

$$S_i = \begin{cases} \left\{ \min\limits_{2 \le j \le i} \{ j - 1, m \} \right\} \cup \{ i - 1 + t_1 \}, & \text{if } i \le m + 1 - t_1 \\ \left\{ \min\limits_{2 \le j \le i} \{ j - 1, m \} \right\}, & \text{if } i > m + 1 - t_1 \end{cases}$$

3.1.2 Corollary 2

If $t_1 \ge m$ and $m \ge n$, then for $i \in \{2, \ldots, n\}$,

$$S_i = \{1, \ldots, i-1\}$$

3.1.3 Corollary 3

$S_1 = \{t_1\}$. If $m \le n$ and $t_1 \in \{0,1\}$, then for $i \in \{2, \ldots, n\}$,

$$S_i = \left\{ \min_{2 \le j \le i+1} \{ j-1,\ m\} \right\}$$

3.1.4 Corollary 4

If $t_1 < m$ and $m > n$, then for $i \in \{2, \ldots, n\}$,

$$S_i = \begin{cases} \{1, \ldots, i-1, i-1+t_1\}, \text{ if } \ i \le m+1-t_1 \\ \\ \{1, \ldots, i-1\}, \qquad \text{if } i > m+1-t_1 \end{cases}$$

3.1.5 Corollary 5

If the mandatory replacement age restriction is waived, then

$S_i = \{1, 2, \cdots,\ i-1,\ i-1+t_1\}$, $i \in \{2, \cdots, n\}$.

3.2 Proof of Theorem 2

The sketches below represent the network diagrammatic insight for the proof, for chosen $\{m, n, t_1\}$ data set.

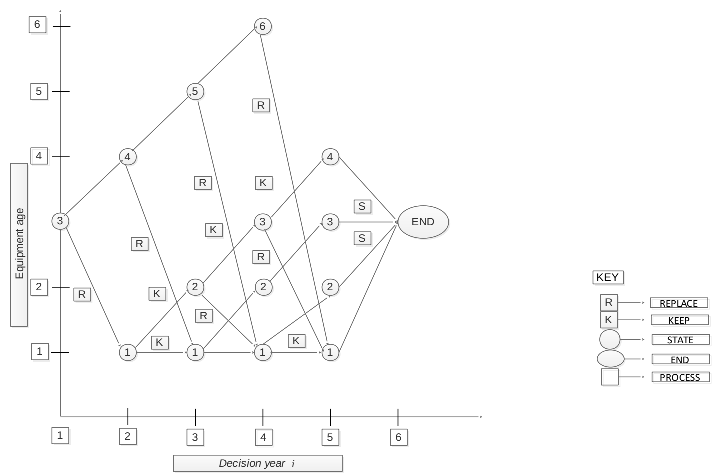

Figure 1: Network Diagram for Equipment Replacement Problem with Starting Age 3, Mandatory Replacement Age 6 for a 4-year Planning Horizon.

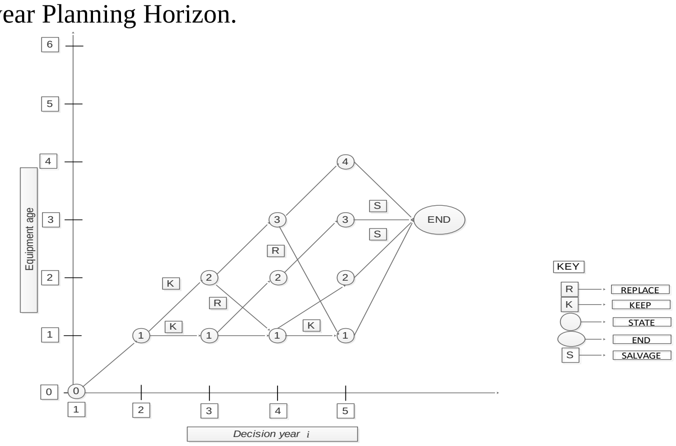

Figure 2: Network Diagram for Equipment Replacement Problem with Starting Age 0, Mandatory Replacement Age 6,for 4-year Planning Horizon.

Proof is by mathematical inductive principle, initiated with $S_1 = \{t_1\}$.

3.2.1 Case 1: $t_1 < m$

The starting node is always t_1; so, $S_1 = \{t_1\}$, a singleton. The feasible alternative decision set from agenode t_1 is $\{R, K\}$, leading to the decision year 2 and age set $\{1, 1+t_1\}$. So $S_2 = \{1, 1+t_1\}$, noting that $1+t_1 \le m$.

From the formula,

$S_2 = \{1\} \cup \{1+(t_1)\operatorname{sgn}(\max\{m-t_1, 0\})\} = \{1\} \cup \{1+t_1\} = \{1, 1+t_1\}$, since $1+t_1 \le m$. Therefore the theorem is valid for $i \in \{1,2\}$.

Assume the validity of the theorem for $2 \le i \le k$, for some integer $k \le n-1$. Then

$S_k = \left\{\left\{\min_{2\le j\le k}\{j-1, m\}\right\} \cup \{1+(k-2+t_1)\operatorname{sgn}(\max\{m+2-t_1-k, 0\})\}\right\}$, by induction hypothesis.

$$\operatorname{sgn}(\max\{m+2-t_1-k, 0\}) = \begin{cases} 1, \text{ if } k \le m+1-t_1 \\ 0, \text{ if } k > m+1-t_1 \end{cases}$$

$$\Rightarrow \quad S_k = \begin{cases} \left\{\min_{2\le j\le k}\{j-1, m\}\right\} \cup \{1+(k-2+t_1)\}, \text{ if } k \le m+1-t_1 \\ \left\{\min_{2\le j\le k}\{j-1, m\}\right\}, \text{ if } k > m+1-t_1 \end{cases}$$

$$S_{k+1} = \{1\} \cup \{1+t_k : t_k \in S_k \text{ and } t_k \le m-1\} = \{1\} \bigcup_{t_k \le m-1} \{1+t_k : t_k \in S_k\} = \{1\} \bigcup_{\substack{t_k \in S_k \\ t_k \le m-1}} \{1+t_k\} = \{1\} \bigcup_{\substack{t_k \in S_k \\ 1+t_k \le m}} \{1+t_k\}$$

$$= \{1\} \cup \left\{\min_{2\le j\le k}\{j, m\}\right\} \cup \begin{cases} 1+1+k-2+t_1, \text{ provided } k+1 \le m+1-t_1 \text{ (for feasibility)} \\ 1, \text{ if } k+1 > m+1-t_1, \text{ since } 1 \in S_{k+1} \end{cases}$$

Above decomposition of S_{k+1}, combined with the fact that $\left\{\min_{2\le j\le k}\{j, m\}\right\} = \left\{\min_{3\le j\le k+1}\{j-1, m\}\right\}$

automatically yields

$$S_{k+1} = \{1\} \cup \left\{\min_{3\le j\le k+1}\{j-1, m\}\right\} \cup \{1+(k+1-2+t_1)\operatorname{sgn}(\max\{m+2-t_1-(k+1), 0\})\}$$

$$= \left\{\left\{\min_{2\le j\le k+1}\{j-1, m\}\right\} \cup \{1+(k+1-2+t_1)\operatorname{sgn}(\max\{m+2-t_1-(k+1), 0\})\}\right\}$$

Therefore the theorem is valid for $i = k+1$ and hence valid for all $i \in \{1, 2, \ldots, n\}$. Note the application of change of variables technique.

3.2.2 Case 2: $t_1 \ge m$

If $t_1 \ge m$ the first decision is "Replace"; this transforms the machine's age to 1, corresponding to the decision year 2. For year 2, age 1 "Keep" transforms age 1 to age 2 and correspondingly decision year 2 to decision year 3. "Replace" preserves the age 1 and transforms decision year 2 to decision year 3. Any subsequent "Keep" from age 2 leads to decision year > 3. Therefore $S_2 = \{1\}$ and $S_3 = \{1, 2\}$, so the theorem is valid for $i \in \{2, 3\}$ when $t_1 \ge m$.

Assume that the theorem is valid for $4 \le i \le k$ for some integral $k \le n-1$. Then

$S_k = \left\{\left\{\min_{2\le j\le k}\{j-1, m\}\right\}\right\}$, by the induction hypothesis and

$$S_{k+1} = \{1\} \cup \{1+t_k : t_k \in S_k \text{ and } t_k \le m-1\} = \{1\} \bigcup_{t_k \le m-1} \{1+t_k : t_k \in S_k\}$$

$$= \{1\} \bigcup_{\substack{t_k \in S_k \\ t_k \le m-1}} \{1+t_k\} = \{1\} \bigcup_{\substack{t_k \in S_k \\ 1+t_k \le m}} \{1+t_k\} = \{1\} \cup \left\{\min_{2\le j\le k}\{j, m\}\right\}$$

$$=\{1\}\cup\left\{\min_{3\le j\le k+1}\{j-1,m\}\right\}=\left\{\left\{\min_{2\le j\le k+1}\{j-1,m\}\right\}\right\},$$

since $1\in S_i$ for each $i\in\{1,\ldots,n\}$ and $m\ge 1$. Therefore the theorem is valid for $i=k+1$ and hence valid for all $i\in\{1,2,\ldots,n\}$. This completes the proof.Note the application of change of variables technique.

3.2.3 Interpretation of Case 2: $t_1\ge m$

The case $t_1\ge m$ translates to trading in any equipment of such age. The policy here is to replace each equipment of age $\ge m$ with a new one, even if it was received as a gift. It follows immediately that if a new equipment is purchased and placed in service at the start of the decision year 1, then

$$t_1\ge m\Rightarrow f_1(t_1)=s(t_1)+f_1(0).$$

3.2.4 Proof of Corollary 1

$$\operatorname{sgn}\left(\max\{m+2-t_1-i,0\}\right)=\begin{cases}1, & \text{if } i\le m+1-t_1\\ 0, & \text{if } i>m+1-t_1\end{cases}$$

Therefore, for $i\in\{2,\cdots,n\}$,

$$S_i=\left\{\min_{2\le j\le i}\{j-1,m\}\right\}\cup\left\{1+\left(i-2+t_1\right)\operatorname{sgn}\left(\max\{m+2-t_1-i,0\}\right)\right\},\text{if } t_1<m$$

$$\Rightarrow\quad S_i=\begin{cases}\left\{\min_{2\le i\le i}\{j-1,m\}\right\}\bigcup\{1+(i-2+t_1)\}, & \text{if } i\le m+1-t_1\\ \left\{\min_{2\le j\le i}\{j-1,m\}\right\}, & \text{if } i>m+1-t_1.\end{cases}$$

The validity of corollary 1 is established.

3.2.4 Proof of Corollary 2

If $t_1\ge m\ge n$, in theorem 2, then $2\le j\le i\le m\Rightarrow\left\{\min_{2\le j\le i}\{j-1,m\}\right\}=\{1,\cdots,i-1\}$

$$\Rightarrow S_i=\{1,\cdots,i-1\},\text{ as desired.}$$

3.2.5 Proof of Corollary 3

$t_1\in\{0,1\}\Rightarrow i-1+t_1\le i\le n;\ 2\le j\le i\Rightarrow 1\le j-1\le i-1;\ \min_{2\le j\le i+1}\{j-1,m\}\le m,$ for each j.

Therefore, if $m\le n$ and $t_1\in\{0,1\}$ in theorem 2, then for $i\in\{2,\ldots,n\}$,

$$S_i=\left\{\min_{2\le j\le i+1}\{j-1,m\}\right\},\text{ as stated.}$$

3.2.6 Proof of Corollary 4

$m>n\Rightarrow$ the set of ages $\{1,\cdots,i-1\}$ is feasible, since $i\in\{2,\cdots,n\}$. So in theorem 2 we need only examine the feasibility of $i-1+t_1$; this is assured if $i-1+t_1\le m$, that is, if $i\le m+1-t_1$; $i-1+t_1$ is deleted if $i>m+1-t_1$.

Thus corollary 4 is established.

3.2.7 Proof of Corollary 5

The validity of corollary 5 is established by setting $m=\infty$.

3.3 Application Problems

A company needs to determine the optimal replacement policy for a current t_1- year old equipment over the next four years. The following table gives the data of the problem. The company requires that a 6 – year old equipment be replaced. The cost of a new machine is \$100,000.

Table 1: Pertinent Data for Optimal Policy and Return Determination

Age: t yrs.	**Revenue: $r(t)$ (\$)**	**Operating cost: $c(t)$ (\$)**	**Salvage value: $s(t)$ (\$)**
0	20,000	200	-
1	19,000	600	80,000
2	18,500	1,200	60,000
3	17,200	1,500	50,000
4	15,500	1,700	30,000
5	14,00	1,800	10,000
6	12,200	2,200	5,000

Solve the above problems for $t_1 \in \{0,1,\cdots,6\}$, using dynamic programming recursions.

An Exceltemplate will now be designed and deployed to solve the problem with starting age zero.In the sequel an exposition on the template will be given using the above problem as an illustrative example.
The template has the capacity for automatic generation of optimal solutions as soon as the relevant data and the states at the start of each decision year have been appropriately typed in – an exercise that should consume no more than five minutes. The problems corresponding to $t_1 \in \{1,\cdots,6\}$, will be solved thereafter followed by yet another problem that at first does not quite seem to fit the structure, but turns out to be amenable to it, after all.

Table 2: Template Solution of the Equipment Replacement Problem with Starting Age 0

Equipment Replacement Problem Solution Template: (new) 0- year Starting age Implementation

Replacement Age = 6 yrs

Given Data	I =	100000		Stage 4 Computations in \$1000 V(0) = r(0)-c(0)-I	-80.2	Optimal Solution		
Age, t (yr)	**Revenue, r(t) (\$)**	**Operating cost, c(t) (\$)**	**Salvage value, s(t) (\$)**	**K r(t)+s(t+1)-c(0)**	**R)+s(t) + s(1)-c(0) - I**	**f4(t)**	**Decision**	**t**
0	20000	200						
1	19000	600	80000	**78.4**	**79.8**	**79.8**	**R**	**1**
2	18500	1200	60000	**67.3**	**59.8**	**67.3**	**K**	**2**
3	17200	1500	50000	**45.7**	**49.8**	**49.8**	**R**	**3**
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data	I =	100000		Stage 3 Computations in \$1000		Optimal Solution		
Age, t (yr)	**Revenue, r(t) (\$)**	**Operating cost, c(t) (\$)**	**Salvage value, s(t) (\$)**	**K r(t)-c(t)+f4(t+1**	**R V(0)+s(t)+f4(1)**	**f3(t)**	**Decision**	**t**
0	20000	200						
1	19000	600	80000	**85.7**	**79.6**	**85.7**	**K**	**1**
2	18500	1200	60000	**67.1**	**59.6**	**67.1**	**K**	**2**
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data	I =	100000		Stage 2 Computations in \$1000		Optimal Solution		
Age, t (yr)	**Revenue, r(t) (\$)**	**Operating cost, c(t) (\$)**	**Salvage value, s(t) (\$)**	**K r(t)-c(t)+f3(t+1)**	**R V(0)+s(t)+f3(1)**	**f2(t)**	**Decision**	**t**
0	20000	200						
1	19000	600	80000	**85.5**	**85.5**	**85.5**	**K/R**	**1**
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data	I =	100000		Stage 1 Computations in \$1000		Optimal Solution		
Age, t (yr)	**Revenue, r(t) (\$)**	**Operating cost, c(t) (\$)**	**Salvage value, s(t) (\$)**	**K r(t)-c(t)+f2(t+1)**	**R V(0)+s(t)+f2(1)**	**f1(t)**	**Decision**	**t**
0	20000	200		**105.3**	**5.3**	**105.3**	**K**	**0**
1	19000	600	80000					
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = \$105,300.00

OPTIMAL SOLUTION: 0K1K2K3R1S; 0K1R1K2K3S (Alternate Optima)

(Decision is initiated from stage 1: from the starting age 0)

If the new equipment is paid for prior to the beginning of decision year 1 then the maximum net income is $g_1(0) = f_1(0) - I = \$(105,300.00 - 100,000.00) = \$5,300.00$

3.4 An Exposition on the Solution Template

Use Excel references A1:I2 for documentation. Save the problem data in the indicated cells using the Copy and Paste functionality. Under the decision R, save the fixed value $V(0) = r(0) - c(0) - I$ under the fixed cell reference $F$4

3.4.1 Stage n Computations (Here n = 4):

For $t = 1$, under REPLACE, type the following code in the cell reference F10:

=If($I10 = “”,””,$F$4+ 0.001*($D$10+D10)) <ENTER> to secure $f_4^R(1)$.

Click back on cell F10, position the cursor at the right edge of the cell until a crosshair appears. Then drag the crosshair vertically down to the last required F15 corresponding to $t = m = 6$, to secure

$f_4^R(2)$, and $f_4^R(3)$ and the blank spaces.

Henceforth, the act of clicking back on a specified cell, positioning the cursor at the right edge of the cell until a crosshair appears and the crosshair-dragging routine will be referred to as clerical routine/duty.

For $t = 1$, under KEEP, type the following code in the cell reference E10:

=If($I10 =$D$2,”Must Replace”, if($I10= “”,””,0.001*(B10 +D11-C10)))<ENTER> to secure $f_4^K(1)$.

Perform the clerical duty to secure $f_4^K(2)$ and $f_4^K(3)$. To secure $f_4(t)$, for $t \in \{1,2,3\}$, type the following code in the cell reference G10:

=If ($I10 = “ ”,” ”,if (E10 = “Must Replace”, F10, max(E10,F10)))<ENTER> to secure $f_4(1)$.

Then perform the clerical routine to secure $f_4(2)$ and $f_4(3)$.

3.4.2 Remarks

In Excel, the max and min functions return values for only numeric expressions, ignoring string constants; for example if the number 5 is saved in B2 and the string constant “**Must**” is saved in C2, Then in D2, the code: = max(B2, C2) <Enter> returns 5. In E2, the code: = max(B2,C2) <Enter> also returns 5. Therefore the code segment involving “ if (E10 = “Must Replace”, F10 “ may be dispensed with throughout the template.

To obtain the optimal decision for each of the stage 4 states $t \in S_4 = \{1,2,3\}$, type the following code in the cell reference H10:

=If($I10 = “ ”,” ”,if(A10 = $D$2, “R”, if(F10 > E10,”K”, “K/R”)))) <ENTER> to secure $D_4(1)$.

Then perform the clerical routine to secure $D_4(2)$ and $D_4(3)$.

3.4.3 Stage (n - 1) Computations (Here n - 1 = 3):

For $t = 1$, under REPLACE, type the following code in the cell reference F24:

=If($I24 = “”,””, $F$4+ 0.001*D24 +$G$10) <ENTER> to secure $f_3^R(1)$.

Perform the clerical duty to secure $f_3^R(2)$.

For $t = 1$, under KEEP, type the following code in the cell reference E24:

=If($I24 =$D$2,”Must Replace”, if ($I24=0, “”,””, 0.001*(B24-C24+1000*$G$11))<ENTER>to secure $f_3^K(1)$. Perform the clerical duty to secure $f_3^K(2)$.

To secure $f_3(t)$, for $t \in \{1,2\}$, type the following code in the cell reference G24:

= If($I24 = “ ”,” ”,max(E24,F24)) <ENTER> to secure $f_3(1)$. Then perform the clerical routine to secure $f_3(2)$.

To obtain the optimal decision for each of the stage 3 states $t \in S_3 = \{1,2\}$, type the following code in the cell reference H24:

= If($I24 = “ ”,” ”,if(A24 =$D$2, “R”, if(F24> E24,”K”, “K/R”)))) <ENTER> to secure $D_3(1)$.

Then perform the clerical routine to secure $D_3(2)$ and the blanks.

3.4.4 Stage (n - 2) Computations (Here n - 2 = 2)

For $t=1$, under REPLACE, type the following code in the cell reference F38:

= If($I38 = “”,””, $F$4+ 0.001*D38 +$G$24) <ENTER> to secure $f_2^R(1)$.

Perform the clerical duty down to F43 to secure $f_2^R(2)$ and the blank spaces

For $t=1$, under KEEP, type the following code in the cell reference E38:

= If($I38 =$D$2,”Must Replace”,if ($I38= “”,””, 0.001*(B38-C38+1000*$G$25))<ENTER>to secure $f_2^K(1)$. Perform the clerical duty down to E43 to secure $f_2^K(2)$ and the blanks.

To secure $f_2(t)$, for $t \in S_2 = \{1\}$, type the following code in the cell reference G38:

=If($I38 = “ ”,” ”,max(E38,F38)) <ENTER> to secure $f_2(1)$.

Then perform the clerical routine down to G43 to secure the blank spaces.

To obtain the optimal decision for the only stage2 states $S_2 = \{1\}$, type the following code in the cell reference H38:

= If($I38 = “ ”,” ”,if(A38 = $D$2, “R”, if(F38> E38,”K”, “K/R”)))) <ENTER> to secure $D_2(1)$.

Then perform the clerical routine down to H43 to secure the blank spaces.

3.4.5 Stage 1 Computations

Here $S_1 = \{0\}$; so the rows are incremented by $5-1=4$ from the preceding stage computations.

For $t=0$, under REPLACE, type the following code in the cell reference F42:

=If($I42 = “”,””, $F$4+ 0.001*D42 +$G$38) <ENTER> to secure $f_1^R(0)$.

Perform the clerical duty down to F48 to secure the six blank spaces.

For $t=0$, under KEEP, type the following code in the cell reference E42:

=If($I42 =$D$2,”Must Replace”, if ($I42= “”,””, 0.001*(B42-C42+1000*$G$39))<ENTER>to secure $f_1^K(0)$. Perform the clerical duty down to E48 to secure the six blank spaces. To secure $f_1(t)$, for $t \in S_1 = \{0\}$, type the following code in the cell reference G42:

=If ($I42 = “ ”,” ”,max(E42,F42)) <ENTER> to secure $f_1(0)$.

Then perform the clerical routine down to G48 to secure the six blank spaces.To obtain the optimal decision for the only stage 1 state $S_1 = \{0\}$, type the following code in the cell reference H42:

= If ($I42 = “ ”,” ”,if(A42 = $D$2, “R”, if(F42> E42,”K”, “K/R”)))) <ENTER> to secure $D_1(0)$.

Then perform the clerical routine down to H48 to secure the blank spaces.

3.5 Observations on the Template, and its Implementation Scope

(1) For $i \in \{n-1\cdots,2\}$, the code for the determination of , $\left(f_i(t), D_i(t)\right)$ and $\left(f_{i-1}(t), D_{i-1}(t)\right)$ are Excel equivalent in the sense that they are the same except for a constant row reference difference (14 here).

The results for the pair $\left(f_i(t), D_i(t)\right)$ are automatic in stages $i \in \{n-2,\cdots,1\}$ following the implementation of the pair $\left(f_{n-1}(t), D_{n-1}(t)\right)$ in stage $n-1$ and the required copy and paste operations. It must be stressed that the Option “Keep Source Formatting” should be used in the paste operation.

(2) In particular, the template automatically solves any Equipment Replacement problem of the same structure, with the same data set, but different starting ages $t_1 \in \{1,2,\cdots,6\}$. All that is required is simply to type in the appropriate S_i values in column I, as in the above example.

(3) The template automatically solves any Equipment Replacement problem of the same structure, with different data set and different starting ages $t_1 \in \{1,2,\cdots,6\}$. All that is required is simply to type in the appropriate S_i values in column I and the data set, as in the above example.

(4) The template can solve any Equipment Replacement problem of the same structure, with a different data set and different starting ages $t_1 \le m$, the mandatory replacement age. In this case the age range in column A should be adjusted to $t \in \{0,1,\cdots,m\}$ and the appropriate S_i values assigned in column I, as in the above example.

(5) In general, the template can solve any Equipment Replacement problem of the same structure, with the different data set and different starting ages t_1, mandatory replacement age m and horizon length n. In this case the age range in column

A should still be adjusted to $t \in \{0,1,\cdots,m\}$ and the appropriate S_i values assigned in column I, as in the above example. The code for replacement should assign blank for all infeasible (blank) ages under column I.

The developed Excel solution template will now be deployed to solve the remaining seven Equipment Replacement problems.

3.6 More Illustrative Electronic Solution Examples

Let the Data set be $\{m,n,t_1\}$; $\{m,n\}=\{6,4\}$, $t_1 \in \{1,...,6\}$ and $I, c(t), r(t), s(t)$ invariant. Then

$t_1 = 1 \Rightarrow S_1 = \{1\}$, $S_2 = \{1,2\}$, $S_3 = \{1,2,3\}$, $S_4 = \{1,2,3,4\}$, Ending state $S_5 = \{1,2,3,4,5\}$.

Mandatory salvage is from S_5. Invoke Theorem 2 and corollary 3 to obtain S_i, for $t_1 \in \{1,2,\cdots,6\}$, $i \le 4$

$t_1 = 2 \Rightarrow S_1 = \{2\}$, $S_2 = \{1,3\}$, $S_3 = \{1,2,4\}$, $S_4 = \{1,2,3,5\}$, Terminal states $S_5 = \{1,2,3,4,6\} \rightarrow \text{Salvage}(S)$.

$t_1 = 3 \Rightarrow S_1 = \{3\}$, $S_2 = \{1,4\}$, $S_3 = \{1,2,5\}$, $S_4 = \{1,2,3,6\}$, Terminal states $S_5 = \{1,2,3,4\} \rightarrow \text{Salvage}(S)$.

$t_1 = 4 \Rightarrow S_1 = \{4\}$, $S_2 = \{1,5\}$, $S_3 = \{1,2,6\}$, $S_4 = \{1,2,3\}$, Terminal states $S_5 = \{1,2,3,4\} \rightarrow \text{Salvage}(S)$.

$t_1 = 5 \Rightarrow S_1 = \{5\}$, $S_2 = \{1,6\}$, $S_3 = \{1,2\}$, $S_4 = \{1,2,3\}$, Terminal states $S_5 = \{1,2,3,4\} \rightarrow \text{Salvage}(S)$.

$t_1 = 6 \Rightarrow S_1 = \{6\}$, $S_2 = \{1\}$, $S_3 = \{1,2\}$, $S_4 = \{1,2,3\}$, Terminal states $S_5 = \{1,2,3,4\} \rightarrow \text{Salvage}(S)$.

$t_1 > 6 \Rightarrow S_1 = \{t_1\}$, $S_2 = \{1\}$, $S_3 = \{1,2\}$, $S_4 = \{1,2,3\}$, Ending state $S_5 = \{1,2,3,4\} \rightarrow \text{Salvage}(S)$.

Perform the following sequence of simple operations with respect to the indicated cases:

Simply copy and paste the template in a suitable blank area and plug in S_1, S_2, S_3 and S_4 into the appropriate cells in column I to obtain the solution in one fell swoop, as seen below in tables 1 to 6. Terminal states S_5 are not part of the input data, but they feature implicitly in the computations at the terminal stage n.

Table 3: Template Solution of the Equipment Replacement Problem with Starting Age 1.

Equipment Replacement Problem Solution Template: 1- year Starting age Implementation

Replacement Age = 6 yrs

Given Data				Stage 4 Computations in $1000				
	I =	100000		V(0) = r(0)-c(0)-I	-80.2	Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f4(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)+s(t+1)-c(0)	)+s(t) + s(1)-c(0) - I			t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000	23.8	29.8	29.8	R	4
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 3 Computations in $1000				
	I =	100000				Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f3(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f4(t+1	V(0)+s(t)+f4(1)			t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000	45.5	49.6	49.6	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 2 Computations in $1000				
	I =	100000				Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f2(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f3(t+1)	V(0)+s(t)+f3(1)			t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000	66.9	65.5	66.9	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 1 Computations in $1000				
	I =	100000				Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f1(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f2(t+1)	V(0)+s(t)+f2(1)			t
0	20000	200						
1	19000	600	80000	85.3	85.3	85.3	K/R	1
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = $85,300.00

OPTIMAL SOLUTION: 1R1R1K2K3S; 1R1K2K3R1S; 1K2K3R1R1S **(Alternate Optima)**

(Decision is initiated from stage 1: from the starting age 1)

Table 4: Template Solution of the Equipment Replacement Problem with Starting Age 2.

Equipment Replacement Problem Solution Template: 2- year Starting age Implementation

Replacement Age = 6 yrs

Given Data				Stage 4 Computations in $1000				
I =	100000			V(0) = r(0)-c(0)-I	-80.2	Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f4(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)+s(t+1)-c(0)	)+s(t) + s(1)-c(0) - I			t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000					
5	14000	1800	10000	17.2	9.8	17.2	K	5
6	12200	2200	5000					

Given Data				Stage 3 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f3(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f4(t+1	V(0)+s(t)+f4(1)			t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000					
4	15500	1700	30000	31	29.6	31	K	4
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 2 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f2(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f3(t+1)	V(0)+s(t)+f3(1)			t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000					
3	17200	1500	50000	46.7	55.5	55.5	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 1 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f1(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f2(t+1)	V(0)+s(t)+f2(1)			t
0	20000	200						
1	19000	600	80000					
2	18500	1200	60000	72.8	65.3	72.8	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = $72,800.00
UNIQUE OPTIMAL SOLUTION: 2K3R1K2K3S
(Decision is initiated from stage 1: from the starting age 2)

Table 5: Template Solution of the Equipment Replacement Problem with Starting Age 3.

Equipment Replacement Problem Solution Template: 3- year Starting age Implementation

Replacement Age = 6 yrs

Given Data				Stage 4 Computations in $1000				
I =	100000			V(0) = r(0)-c(0)-I	-80.2	Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f4(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)+s(t+1)-c(0)	)+s(t) + s(1)-c(0) - I			t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000	Must Replace	4.8	4.8	R	6

Given Data				Stage 3 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f3(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f4(t+1	V(0)+s(t)+f4(1)			t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000	17	9.6	17	K	5
6	12200	2200	5000					

Given Data				Stage 2 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f2(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f3(t+1)	V(0)+s(t)+f3(1)			t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000	30.8	35.5	35.5	R	4
5	14000	1800	10000					
6	12200	2200	5000					

Given Data				Stage 1 Computations in $1000				
I =	100000					Optimal Solution		
Age, t	Revenue,	Operating	Salvage	K	R	f1(t)	Decision	
(yr)	r(t) ($)	cost, c(t) ($)	value, s(t) ($)	r(t)-c(t)+f2(t+1)	V(0)+s(t)+f2(1)			t
0	20000	200						
1	19000	600	80000					
2	18500	1200	60000					
3	17200	1500	50000	51.2	55.3	55.3	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = $55,300.00
OPTIMAL SOLUTION: 3R1K2K3R1S ; 3R1R1K2K3S (Alternate Optima)
(Decision is initiated from stage 1: from the starting age 3)

Table 6: Template Solution of the Equipment Replacement Problem with Starting Age 4.

Equipment Replacement Problem Solution Template: 4- year Starting age Implementation

Replacement Age = 6 yrs

Given Data — I = 100000 — Stage 4 Computations in $1000 — V(0) = r(0)-c(0)-I = -80.2 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)+s(t+1)-c(0)	R)+s(t) + s(1)-c(0) - I	f4(t)	Decision	t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — I = 100000 — Stage 3 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1	R V(0)+s(t)+f4(1)	f3(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000	Must Replace	4.6	4.6	R	6

Given Data — I = 100000 — Stage 2 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f3(t+1)	R V(0)+s(t)+f3(1)	f2(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000	16.8	15.5	16.8	K	5
6	12200	2200	5000					

Given Data — I = 100000 — Stage 1 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f2(t+1)	R V(0)+s(t)+f2(1)	f1(t)	Decision	t
0	20000	200						
1	19000	600	80000					
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000	30.6	35.3	35.3	R	4
5	14000	1800	10000					
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = $35,300.00
OPTIMAL SOLUTION: 4R1K2K3R1S ; 4R1R1K2K3S (Alternate optima)
(Decision is initiated from stage 1: from the starting age 4)

Table 7: Template Solution of the Equipment Replacement Problem with Starting Age 5.

Equipment Replacement Problem Solution Template: 5 - year Starting age Implementation

Replacement Age = 6 yrs

Given Data — I = 100000 — Stage 4 Computations in $1000 — V(0) = r(0)-c(0)-I = -80.2 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)+s(t+1)-c(0)	R)+s(t) + s(1)-c(0) - I	f4(t)	Decision	t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — I = 100000 — Stage 3 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1	R V(0)+s(t)+f4(1)	f3(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — I = 100000 — Stage 2 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f3(t+1)	R V(0)+s(t)+f3(1)	f2(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000	Must Replace	10.5	10.5	R	6

Given Data — I = 100000 — Stage 1 Computations in $1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f2(t+1)	R V(0)+s(t)+f2(1)	f1(t)	Decision	t
0	20000	200						
1	19000	600	80000					
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000	22.7	15.3	22.7	K	5
6	12200	2200	5000					

(Maximum net income for years 1 to 4) = $22,700.00
OPTIMAL SOLUTION: 5K6R1K2K3S
(Decision is initiated from stage 1: from the starting age 5)

Table 8: Template Solution of the Equipment Replacement Problem with Starting Age 6.

Equipment Replacement Problem Solution Template: 6 - year Starting age Implementation

Replacement Age = 6 yrs

Given Data — Stage 4 Computations in $1000

I = 100000 V(0) = r(0)-c(0)-I -80.2

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)+s(t+1)-c(0)	R)+s(t) + s(1)-c(0) - I	Optimal Solution f4(t)	Decision	t
0	20000	200						
1	19000	600	80000	78.4	79.8	79.8	R	1
2	18500	1200	60000	67.3	59.8	67.3	K	2
3	17200	1500	50000	45.7	49.8	49.8	R	3
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — Stage 3 Computations in $1000

I = 100000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1	R V(0)+s(t)+f4(1)	Optimal Solution f3(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.7	79.6	85.7	K	1
2	18500	1200	60000	67.1	59.6	67.1	K	2
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — Stage 2 Computations in $1000

I = 100000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f3(t+1)	R V(0)+s(t)+f3(1)	Optimal Solution f2(t)	Decision	t
0	20000	200						
1	19000	600	80000	85.5	85.5	85.5	K/R	1
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000					

Given Data — Stage 1 Computations in $1000

I = 100000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f2(t+1)	R V(0)+s(t)+f2(1)	Optimal Solution f1(t)	Decision	t
0	20000	200						
1	19000	600	80000					
2	18500	1200	60000					
3	17200	1500	50000					
4	15500	1700	30000					
5	14000	1800	10000					
6	12200	2200	5000	Must Replace	10.3	10.3	R	6

(Maximum net income for years 1 to 4) = $10,300.00

OPTIMAL SOLUTION: 0K1K2K3RS ; 0K1R1K2K3S (Alternate optima)

The determination of the alternate optimal policies is initiated from stage 1: from the starting age 6

Observe that $f_1(6) = s(6) + f_1(0),$ an instance of the relation: $f_1(t_1) = s(t_1) + f_1(0),$ for $t_1 \geq m.$

3.6.1 Further Equipment Replacement Problem

An auto repair shop always needs to have an engine analyzer available. A new engine analyzer costs $1000. The costs m_i of maintaining an engine analyzer during its i^{th} year of operation is as follows:
$m_1 = \$60, m_2 = \$80, m_3 = \$120.$ An analyzer may be kept for 1, 2, or 3 years, and after i years of use, $(i = 1, 2, 3)$ it may be traded in for a new one. If an i - year - old engine analyzer is traded in, a salvage value s_i is obtained, where $s_1 = \$800, s_2 = \$600,$ and $s_3 = \$500.$ Given that a new machine must be purchased at the present time (time 0), the shop wishes to determine a replacement and trade - in policy that minimizes
Net cost = (Maintenance Cost) + (Replacement Cost) + (Salvage Value received)
during the next 5 years. How can this be achieved?

Solution

Time 0 corresponds to $i = 1,$ the beginning of the decision year 1 in theorem1. Also note the following correspondences:
$c(j-1) = m_j,\ j \in \{1, 2, 3\} \Rightarrow c(0) = m_1 = \$60, c(1) = m_2 = \$80, c(2) = m_3 = \$120,$

$s(t) = s_t, t \in \{1, 2, 3\} \Rightarrow s(1) = s_1 = \$800, s(2) = s_2 = \$600,$ and $s(3) = s_3 = \$500,\ m = 3,\ n = 5,\ t_1 = 0,\ r(t) \equiv 0.$

Our Excel solution implemetation template will be easily updated to $n = 5$. Below are the tabulated data for the solution implementation.

Table 9: Pertinent Data for Machine Replacement Problem 3.6.1.

Age: *t* yrs.	**Revenue: *r*(*t*) ($)**	**Operating cost: *c*(*t*) ($)**	**Salvage value: *s*(*t*) ($)**
0	0	60	-
1	0	80	800
2	0	120	600
3	0	0	500

It is optimal to preserve the age range 0 – 6 in the template to obviate the need for non-cosmetic editing in decision years 1 to 4. Clearly, corollary 3 is applicable with $m = 3,\ n = 5,\ t_1 = 0,$

$S_1 = \{0\}, S_2 = \{1\}, S_3 = \{1, 2\}, S_4 = \{1, 2, 3\},\ S_5 = \{1, 2, 3\},$ with ending states $S_6 = \{1, 2, 3\}$.

Since a new machine must be purchased at the beginning of the decision year1, the minimum net cost incurred is given by $g_1(0) = I - f_1(0)$; equivalently the maximum net returnfrom adopting an optimal replacement and trade - in policy is $-g_1(0) = -I + f_1(0)$.

The next page yields the solution template and the alternative optimal policies.

Table 10: Template Solution of the Equipment Replacement Problem 3.6.1 with Starting Age 0.

Equipment Replacement Problem Solution Template: (New) 0 - year Starting age Implementation

Replacement Age = 3 yrs

Given Data — Stage 5 Computations in $1000

I = 1000 — V(0) = r(0)-c(0)-I -1.06 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)+s(t+1)-c(0)	R)+s(t) + s(1)-c(0) - I	f5(t)	Decision	t
0		60						
1		80	800	0.52	0.54	0.54	R	1
2		120	600	0.38	0.34	0.38	K	2
3			500	Must Replace	0.24	0.24	R	3
4								
5								
6								

Given Data — Stage 4 Computations in $1000

I = 1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f5(t+1	R V(0)+s(t)+f5(1)	f4(t)	Decision	t
0		60						
1		80	800	0.3	0.28	0.3	K	1
2		120	600	0.12	0.08	0.12	K	2
3			500	Must Replace	-0.02	-0.02	R	3
4								
5								
6								

Given Data — Stage 3 Computations in $1000

I = 1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1)	R V(0)+s(t)+f4(1)	f3(t)	Decision	t
0		60						
1		80	800	0.04	0.04	0.04	K/R	1
2		120	600	-0.14	-0.16	-0.14	K	2
3			500					
4								
5								
6								

Given Data — Stage 2 Computations in $1000

I = 1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f3(t+1)	R V(0)+s(t)+f3(1)	f2(t)	Decision	t
0		60						
1		80	800	-0.22	-0.22	-0.22	K/R	1
2		120	600					
3			500					
4								
5								
6								

Given Data — Stage 1 Computations in $1000

I = 1000 — Optimal Solution

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f2(t+1)	R V(0)+s(t)+f2(1)	f1(t)	Decision	t
0		60		-0.28	-1.28	-0.28	K	0
1		80	800					
2		120	600					
3			500					

(Maximum net return for years 1 to 5) = -$1,280.00 : (Minimum cost) = $1,280.00

Alternate Optima: 0K1K2K3R1R1S ; 0K1R1K2K3R1S , 0K1R1R1K2K3S

(Decision is initiated from stage 1: from the starting age 0)

Alternate Optima: 0K1K2K3R1R1S ; 0K1R1K2K3R1S , 0K1R1R1K2K3S

Therefore the optimal net cost is $g_1(0) = I - f_1(0) = \$\left(1000 - (-0.280) * 10^3\right) = \$1,280.00.$

For m and n fixed at 3 and 5 respectively, the above template can be deployed to perform sensitivity analysis for the determination of the optimal policy in a matter of minutes. The determination of the optimalpoliciesis automatic for any effected changes.

3.6.2 Equipment Replacement Problem with Non-Specification of Replacement Age

Suppose that a new car costs $10,000 and that the annual operating cost and resale value of the car are as shown in Table 11. If you have a new car now, determine a replacement policy that minimizes the net cost of owning and operating a car for the next six years.

Table 11: Pertinent Data for Problem 3.6.2.

AGE OF CARS (years)	RESALE VALUE	OPERATING COST
1	$7000	$300 (year 1)
2	$6000	$500 (year 2)
3	$4000	$800 (year 3)
4	$3000	$1200 (year 4)
5	$2000	$1600 (year 5)
6	$1000	$2200 (year 6)

Table 12: Template Solution of the Equipment Replacement Problem 3.6.2 with Starting Age 0.

Equipment Replacement Problem Solution Template: (new) 0- year Starting age Implementation

Replacement Age = 20 yrs

Given Data: I = 10000 | Stage 6 Computations in $1000 | V(0) = r(0)-c(0)-I = -10.3

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)+s(t+1)-c(0)	R)+s(t) + s(1)-c(0) - I	Optimal Solution f6(t)	Decision	t
0		300						
1		500	7000	5.5	3.7	5.5	K	1
2		800	6000	3.2	2.7	3.2	K	2
3		1200	4000	1.8	0.7	1.8	K	3
4		1600	3000	0.4	-0.3	0.4	K	4
5		2200	2000	-1.2	-1.3	-1.2	K	5
6		0	1000					

Given Data: I = 10000 | Stage 5 Computations in $1000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f6(t+1	R V(0)+s(t)+f6(1)	Optimal Solution f5(t)	Decision	t
0		300						
1		500	7000	2.7	2.2	2.7	K	1
2		800	6000	1	1.2	1.2	R	2
3		1200	4000	-0.8	-0.8	-0.8	K	3
4		1600	3000	-2.8	-1.8	-1.8	R	4
5		2200	2000					
6		0	1000					

Given Data: I = 10000 | Stage 4 Computations in $1000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1)	R V(0)+s(t)+f5(1)	Optimal Solution f4(t)	Decision	t
0		300						
1		500	7000	0.7	-0.6	0.7	K	1
2		800	6000	-1.6	-1.6	-1.6	K/R	2
3		1200	4000	-3	-3.6	-3	K	3
4		1600	3000					
5		2200	2000					
6		0	1000					

Given Data: I = 10000 | Stage 3 Computations in $1000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f4(t+1)	R V(0)+s(t)+f4(1)	Optimal Solution f3(t)	Decision	t
0		300						
1		500	7000	-2.1	-2.6	-2.1	K	1
2		800	6000	-3.8	-3.6	-3.6	R	2
3		1200	4000					
4		1600	3000					
5		2200	2000					
6		0	1000					

Given Data: I = 10000 | Stage 2 Computations in $1000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f3(t+1)	R V(0)+s(t)+f3(1)	Optimal Solution f2(t)	Decision	t
0		300						
1		500	7000	-4.1	-5.4	-4.1	K	1
2		800	6000					
3		1200	4000					
4		1600	3000					
5		2200	2000					
6		0	1000					

Given Data: I = 10000 | Stage 1 Computations in $1000

Age, t (yr)	Revenue, r(t) ($)	Operating cost, c(t) ($)	Salvage value, s(t) ($)	K r(t)-c(t)+f2(t+1)	R V(0)+s(t)+f2(1)	Optimal Solution f1(t)	Decision	t
0		300		-4.4	-14.4	-4.4	K	0
1		500	7000					
2		800	6000					
3		1200	4000					
4		1600	3000					
5		2200	2000					
6		0	1000					

Minimum total cost ($): 14400

Unique optimal policy: OK1K2R1K2R1K2S

Minimum total cost: $g_1(0) = I - f_1(0) = \$\left(10000 - 1000(-4.4)\right) = \$14,400.$

4.0 Conclusion

This research article developed computational formulas for the states corresponding to each decision year in in a certain class of equipment Replacement problems, thereby eliminating the drudgery and errors associated with the drawing of network for such determination. The article went further to design prototypical solution templates for optimal solutions to such problems, complete with an exposition on the interface and solution process. Finally the article solved nine illustrative examples. Manual solutions to these nine problems, starting with the determination of the feasible ages for each decision year via network diagrams would, in the best case scenario, consume no less than five hours. These would contrastquite sharply with the analytic feasible age'sdetermination and Excel implementations that took at most thirty minutes, demonstrating the efficiency, power and utility of the solution template prototype. In general, the template could be deployed to solve each equipment replacement problem in less than 10 percent of the time required for the manual generation of the alternate optima.

5.0 References

[1] Taha, H.A. Operations Research: An Introduction. Seventh Edition. Prentice-Hall of India, New Delhi. (2006). Pp. 418-421.

[2] Winston, L.W. Operations Research: Applications and Algorithm, Duxbury Press, Boston. (1994). Pp. 798-802.

[3] Verma, A.P. Operations Research. S.K. Kataria & Sons. New Delhi (2010). pp. 882-885.

[4] Gupta, P.K. and Hira, D.S. Problems in Operations Research (Principles and Solutions). S. Chand & Company Ltd. New Delhi (2012). Pp. 844-854.