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The Convexity of the Total Cost Function for a Fixed Lifetime Inventory System

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Abstract

The total cost function for a fixed lifetime inventory system is the sum of all its cost parameters, which include the holding cost, shortage cost, outdate cost, ordering cost etc. Showing that these total cost functions are convex has being a problem. Many authors in the literature did not show that their total cost functions were convex. In this work, we develop a computer based method (using wolfram mathematical 8.0) for showing that, the total cost function for a fixed lifetime inventory system is convex.

Keywords and Phrase: Convex, cost function, cost parameters, determinant.

1.0 Introduction

The convexity of any total cost function come with a lot of advantages. Some of these advantages include;

1) Convex functions are guaranteed to have globally optimal solution.
2) Any local minimum for a convex function is also a global minimum.
3) Where a minimum exists for a convex function, the minimum is unique.

Several Authors [2-5] did not show that their total cost functions were convex.

To prove the convexity of any total cost function, it is necessary for the second order partial derivatives of the total cost function to be greater than or equal to zero, that is $f^{ii}(x, y) \geq 0$. where x is the state variable and y is the decision variable for the model. Where it is difficult or is not possible to show that $f^{ii}(x, y) \geq 0$, we propose a computer based method that can be used to show whether $f\ (x, y)$ is convex or not convex. The method involves the formation of the Hessian matrix for the total cost function. The Hessian matrix consists of the second order partial derivatives of the total cost function with respect the state and decision variables. We outline the steps for the method.

STEP1: Input the total cost function for the model.
STEP2: Obtain the first order partial derivatives of the total cost function with respect to the state and decision variables.
STEP3: From step 2, obtain the second order partial derivatives with respect to the state and decision variables.
STEP4: Form the Hessian matrix of the total cost function from the second order derivatives obtained in Step 3.
STEP5: Determine the determinant of the Hessian matrix.
STEP6: If the determinant of the Hessian matrix is greater than or equal to zero, then the total cost function is convex.

We illustrate the steps of the method on an arbitrary function $f(x, y)$.

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Given $f(x,y)$, *we have the following sequence of operations*

Input $f(x,y)$

obtain f_x *and* f_y.

obtain f_{xx} *and* f_{xy}

obtain f_{yx} *and* f_{yy}.

Next form the Hessian matrix ***

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Compute $\det H$

if $|H| \geq 0$

then f *is convex*

otherwise f *is not convex.*

Next ,we illustrate the method on a simple example and then use the method to show the convexity of a few known total cost functions in the literature.After showing that the determinant of a given total cost function is greater than or equal to zero, we than plot the graph for the total cost function.In plotting the graphs, we first plot the graphs of the functions in examples (2) to (5) over a range of positive and negative integers and thereafter discard the negative values and plot the graphs for the positive values only, since the negative values are not relevant to us.

Example 1: $f(x,y) = 3x^2 + 3y^2 - 6x + 2y + 20$

Where x *and* y are state and decision variables respectively. Using mathematical 8 , we input the function and run the programme. The result obtained and the graph for example 1 is shown below.

```
f=3x^2+3y^2-6x+2y+20
D[f,x]
D[D[f,x],x]
D[D[f,x],y]
D[f,y]
D[D[f,y],x]
D[D[f,y],y]
Det[{{D[D[f,x],x],D[D[f,x],y]},{D[D[f,y],x],D[D[f,y],y]}}]
```

$20 - 6x + 3x^2 + 2y + 3y^2$

$-6 + 6x$

6

0

$2 + 6y$

0

6

36

Plot $3D[3x^2 + 3y^2 - 6x + 2y + 20, \{x, -2, 2\}, \{y, -8, 8\}]$

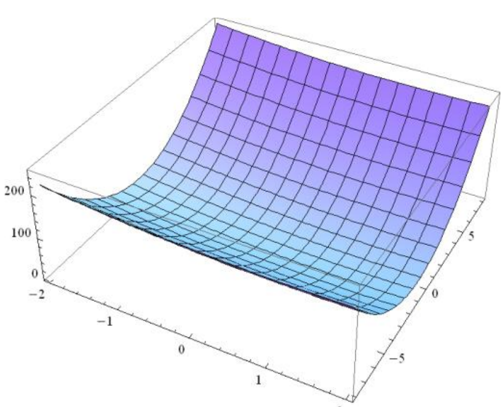

Figure 1: graph of example 1.

Example 2: Hark and Hahn [2], considered the case of an optimal policy for products with an inventory level-dependent demand rate and fixed lifetime. They obtained their total cost function $L(S,T)$ as

$$L(S,T) = \frac{1}{T}\{P(Q-W) - C_P Q - C_h H - C_o - C_W W\}$$

Where

$S =$ *order up to level*

$p =$ *selling price*

$T =$ *cycle lenght*

$Q =$ *order size*

$W = i_T - (n-1)Q$ *outdate quantity*

$H = \{1/\alpha(2-\beta)\}\{S^{2-\beta} - i_T^{2-\beta}\}$ *on hand inventory*

$C_p =$ *unit purchasing cost*

$C_o =$ *procurement cost per order*

$C_h =$ *inventory holding cost*

$C_w =$ *disposal cost per unit outdate*

Applying our method to example 2, we input the total cost function, run the sequence of operations in *** above and plot the graph of the total cost function, first over a range of positive and negative integers and then over a range of positive integers. The result is shown below.

```
W=iT-(n-1)Q
H=(1/∝(2-β))(S2-β-Subscript[i, T]2- β)
f=(1/T)(P(Q-W)-CP*Q-Ch*H-C0-C*W)
D[f,S]
D[D[f,S],S]
D[D[f,S],Q]
D[f,Q]
D[D[f,Q],S]
D[D[f,Q],Q]
Det[{{D[D[f,S],S],D[D[f,S],Q]},{D[D[f,Q],S],D[D[f,Q],Q]}}
]
```

$$-(-1+n)Q + i_T$$

$$\frac{(2-\beta)(S^{2-\beta} - i_T^{2-\beta})}{\alpha}$$

$$\frac{-C_0 - QC_P + P(Q + (-1+n)Q - i_T) - C(-(-1+n)Q + i_T) - \frac{(2-\beta)C_h(S^{2-\beta} - i_T^{2-\beta})}{\alpha}}{T}$$

$$-\frac{S^{1-\beta}(2-\beta)^2 C_h}{T\alpha}$$

$$-\frac{S^{-\beta}(1-\beta)(2-\beta)^2 C_h}{T\alpha}$$

$$0$$

$$\frac{-C(1-n) + nP - C_P}{T}$$

$$0$$

$Plot\ 3D[(1/20)(40Q^2 - 400 - (S/5) - 1, \{Q, -8, 8\}, \{S, -4, 4\}]$

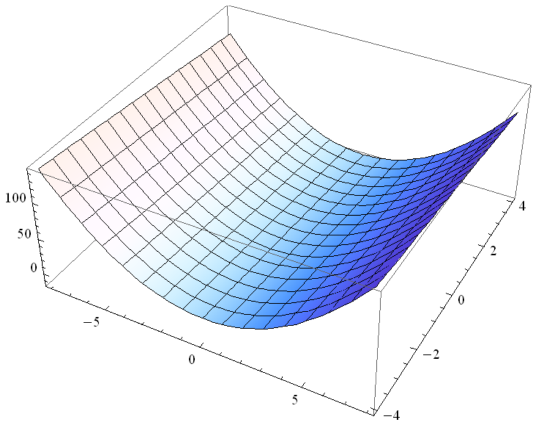

Figure 2: The graph of example 2 over negative and positive integers.
Plot3D$[(1/20)(40Q^2 - 400(S/5) - 1), \{Q, 0,28\}, \{S, 0,4\}]$

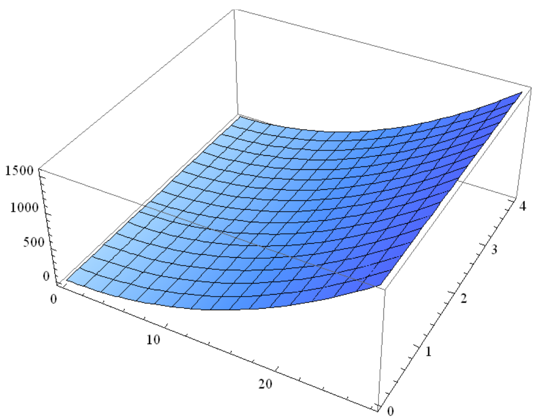

Figure 3: The graph of example 2 over positive integers only

Example 3: Nahmias and Pierskalla [3] , considered the case of optimal policy for a product that perishes in two periods subject to stochastic demand and obtained the total cost function $L(x, y)$ as

$$L(x, y) = cy + \int_0^{x+y} (x + y - t)\, f(t)dt + r \int_{x+y}^{\infty} (t - (x + y))\, f(t)dt + \theta \int_0^{y} F(t + x)F(y - t)dt$$

where

$x = amount\ of\ one\ period\ old\ \ products$

$y = ordered\ \ products$

$c = ordering\ \cos t$

$h = holding\ \cos t$

$r = shortage\ \ \cos t$

$\theta = outdate\ \cos t$

Applying our method , we input the total cost function, run the sequence of operations in *** above and plot the graph of the total cost function, fisrt over a range of positive and negative intergers and than over a range of positive integers only.The result is shown below.

```
p=c*y+h*Integrate[(x+y-t)f(t),{t,0,x+y}]+r*Integrate[(t-(x+y))f(t),{t,x+y,infinity}]+□*Integrate[F(t+x)F(y-t),{t,0,y}]
D[p,x]
D[D[p,x],x]
D[D[p,x],y]
D[p,y]
D[D[p,y],x]
D[D[p,y],y]
Det[{{D[D[p,x],x],D[D[p,x],y]},{D[D[p,y],x],D[D[p,y],y]}}]
```

$$cy + h(\frac{fx^3}{6} + \frac{1}{2}fx^2y + \frac{1}{2}fxy^2 + \frac{fy^3}{6}) + r(\frac{f\text{infinity}^3}{3} - \frac{1}{2}f\text{infinity}^2x + \frac{fx^3}{6} - \frac{1}{2}f\text{infinity}^2y + \frac{1}{2}fx^2y + \frac{1}{2}fxy^2 + \frac{fy^3}{6}) + (\frac{1}{2}F^2xy^2 + \frac{F^2y^3}{6})\theta$$

$$h(\frac{fx^2}{2} + fxy + \frac{fy^2}{2}) + r(-\frac{f\text{infinity}^2}{2} + \frac{fx^2}{2} + fxy + \frac{fy^2}{2}) + \frac{1}{2}F^2y^2\theta$$

$$h(fx + fy) + r(fx + fy)$$

$$h(fx + fy) + r(fx + fy) + F^2y\theta$$

$$c + h(\frac{fx^2}{2} + fxy + \frac{fy^2}{2}) + r(-\frac{f\text{infinity}^2}{2} + \frac{fx^2}{2} + fxy + \frac{fy^2}{2}) + (F^2xy + \frac{F^2y^2}{2})\theta$$

$$h(fx + fy) + r(fx + fy) + F^2y\theta$$

$$h(fx + fy) + r(fx + fy) + (F^2x$$

```
A=Integrate[(x+y-t),{t,0,x+y}]
B=Integrate[(t-(x+y)),{t,x+y,infinity}]
T=Integrate[F(t+x)F(y-t),{t,0,y}]
h=1
r=1
F=1
θ =0.05
c=1
Plot3D[c*y+h*A+r*B+θ*T,{x,-3,3},{y,-40,40}]
```

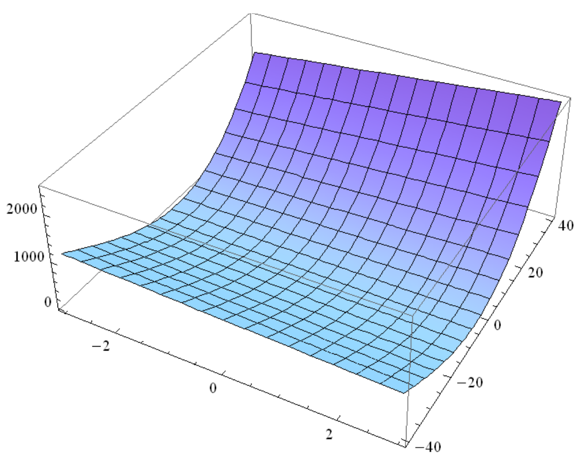

Figure 4: Graph of example3 over a range of positive and negative integers.

```
A=Integrate[(x+y-t),{t,0,x+y}]
B=Integrate[(t-(x+y)),{t,x+y,infinity}]
T=Integrate[F(x+y)F(y-t),{t,0,y}]
infinity=0
h=1
r=1
θ =0.05
c=1
Plot3D[c*y+h*A+r*B+ θ *T,{x,0,4},{y,0,40}]
```

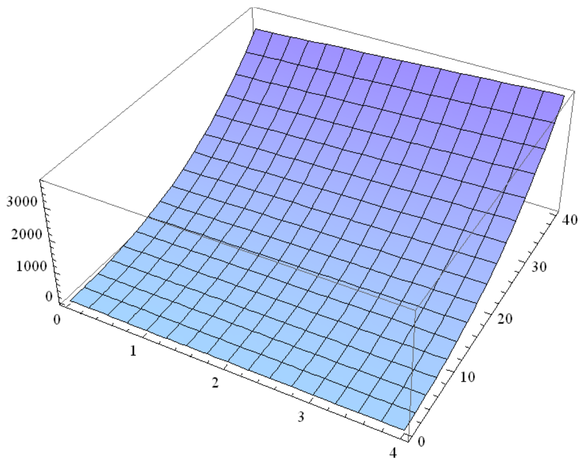

Figure 5: graph of example 3 over a range of positive integers only.

Example 4: Chiu [1] analyzed a continuous review inventory model based on approximations to the expected outdating , expected shortage and expected inventory level. The total cost function $EAC(Q,r)$ for the model was given as

$$EAC(Q,r) = \{(k + cQ + pES + wER)/ET\} + hOH$$

where

$$ER = \int_0^{r+Q} (r + Q - d_{m+l}) f_{m+l}(d_{m+l}) d_{m+l} - \int_0^{r} (r - d_{m+l}) f_{m+l}(d_{m+l}) d_{m+l}$$

$$ES = \int_r^{\infty} (d_l - r) f_l(d_l) d_l$$

$$ET = \frac{Q + ES - ER}{D}$$

$$OH = r - Dl + \frac{Q}{2}$$

$k =$ *fixed ordering* cost

$w =$ *outdate* cost

$p =$ *shortage* cost

$h =$ *holding* cost

$c =$ *replenishment* cost

$l =$ *order leadtime*

$m =$ *lifetime*

Applying our method, we input the total cost function, run the sequence of operations in *** above and plot the graph of the function, first over a range of positive and negative integers and then over a range of positive integers. The result is shown below.

```
ER=Integrate[(r+Q-t),{t,0,r+Q}]-Integrate[(t-r),{t,r,infinity}]
ES=Integrate[(t-r),{t,r,infinity}]
ET=(Q+ES-ER)/d
OH=r-(d*l)+(Q/2)
f=((k+c*Q+p*ES+w*ER)/ET)+h*OH
D[f,Q]
D[D[f,Q],Q]
D[D[f,Q],r]
D[D[f,r],Q]
D[D[f,r],r]
Det[{{D[D[f,Q],Q],D[D[f,Q],r]},{D[D[f,r],Q],D[D[f,r],r]}}]
```

Next we plot the graph for example 4.

```
ER=Integrate[(r+Q-t),{t,0,r+Q}]-Integrate[(r-t),{t,0,r}]
ES=Integrate[(t-r),{t,r,infinity}]
```

```
ET=2
H=r-10+(Q/2)
infinity=0
k=10
c=1
p=2
w=0.05
h=1
f=((k+c*Q+p*ES+w*ER)/ET)+h*H
Plot3D[f,{r,-8,8},{Q,-24,24}]
```

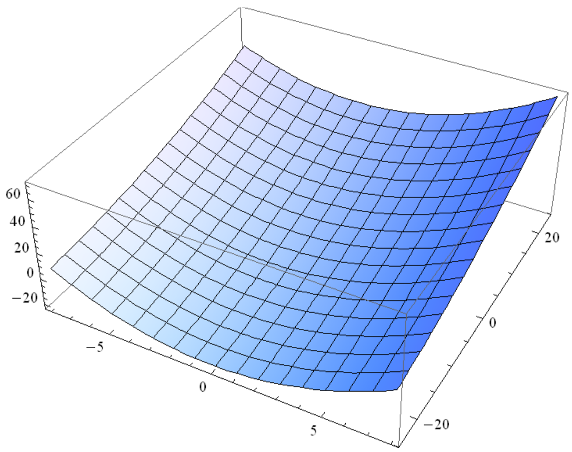

Figure 6: The graph of example 4 over a range of positive and negative integers.

```
Plot3D[f,{0,4},{Q,0,48}]
```

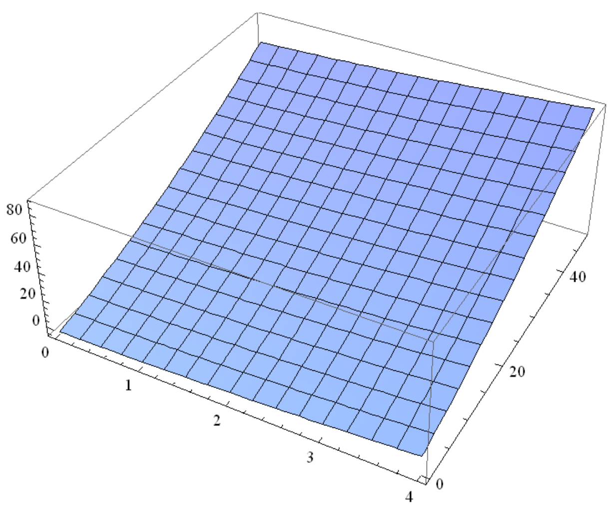

Figure 7: The graph of example 4 over a range of positive integers.

Chaaben [6], also used a different method to show that Chiu's total cost function was convex.

EXAMPLE 5: Pavee [5]: considered the optimal ordering policy for a perishable inventory system and obtained the total cost function $E(Q)$ as

$$E(Q) = \frac{kD}{Q} + h\{\frac{Q}{2} + k\sigma\sqrt{L}\} + w(\int_0^{r+Q}(r+Q-d_{m+L})f_{m+L}(d_{m+L}) - \int_0^{r}(r-d_{m+L})f(d_{m+L})d_{m+L})$$

where

$Q = order \quad size$

$r = reorder \quad point$

$L = leadtime$

$w = outdate \; cost$

$m = lifetime$

$k = ordering \; cost$

$\sigma = demand \quad rate$

He was able to show that the total cost function was convex. Applying our method and following the same steps in *** above, we verify that the function is convex. We cannot put the computer result for the derivatives and the determinant here because of the complex form. The graph for the function is shown below.

Plot3D$[(20/Q) + 2((Q/2) + 6) + 0.5(\text{Integrate}[(r + Q - t), \{t, 0, r + Q\}] - \text{Integrate}[(r - t), \{t, 0, r\}]), \{Q, -8,8\}, \{r, -3,3\}]$

k=2
d=10
h=2
σ =3
L=1
w=0.5

Plot3D[((k*d/Q))+h*((Q/2)+k* σ *Sqrt[L])+w*(Integrate[(r+Q-t),{t,0,r+Q}]-Integrate[(r-t),{t,0,r}]),{r,-5,5},{Q,-40,60}]

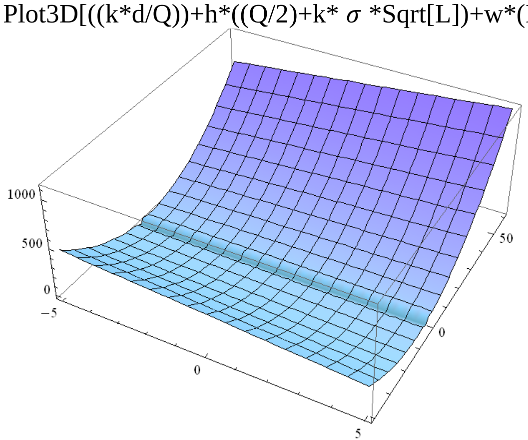

Figure 8: The graph of example 5 over a range of positive and negative integers.

k=2
d=10
h=2
σ=3
L=1
w=0.5

Plot3D[((k*d/Q))+h*((Q/2)+k* σ *Sqrt[L])+w*(Integrate[(r+Q-t),{t,0,r+Q}]-Integrate[(r-t),{t,0,r}]),{r,0,5},{Q,0,60}]

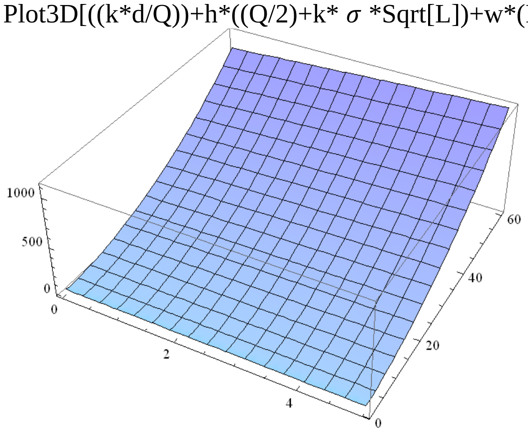

Figure 9: Graph of example 5 over a range of positive integers.

Example 6: Zhau and Yang [4] considered the case of an optimal replenishment policy for items with inventory-level-dependent demand and fixed lifetime under the LIFO policy and obtained the total cost function $C(S,T)$ as

$$C(S,T) = \frac{1}{T}\{(p-c)(S-i_T) - A - \frac{1}{n}(c+d)i_T - h\frac{(S^{2-\beta} - i_T{}^{2-\beta})}{\alpha(2-\beta)}$$

where

$A = order\ \cos t\ \ per\ order$

$c = unit\ \cos t$

$d = disposal\ \cos t$

$S = order\ up\ to\ level$

$T = order\ cycle\ lenght$

$i = inventory\ level$

$i_T = \{S^{1-\beta} - \alpha(1-\beta)T\}^{1/1-\beta}$

Applying our method, the determinant obtained from the Hessian matrix was negative, hence the total cost function by our condition cannot be convex. The graph is shown below.

$$\text{Plot3D}[(1/T)(10(S - (S^{1/2} - (1/2)T)^2) - (20) - 15(S^{1/2} - (1/2)T)^2 - (S^{3/2} - (S^{1/2} - (1/2)T)^{3/2}/(3/2))), \{S, -8, 8\}, \{T, -4, 4\}]$$

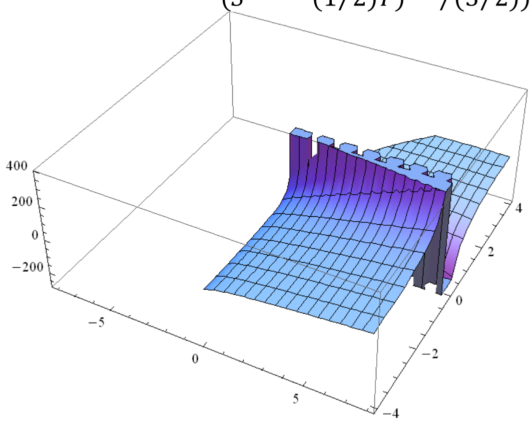

Figure 10: The graph of example 6 over a range of positive and negative integers.

2.0 Conclusion

We have presented an easy to use method, for showing that the total cost function for a fixed lifetime inventory system is convex. We have demonstrated its use on some cost functions in the literature. We have also shown that , when our condition is not satisfied, then the total cost function cannot be convex, as shown in example 6. We must also report that the complex nature of some of these cost functions made it difficult to differentiate manually . The use of a mathematical software(like mathematical 8) made it easy for us to differentiate and compute the determinant of the resulting Hessian matrices.

3.0 Reference

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