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New Eighth-Order ConvergenceIterative Method for Solving Nonlinear Equations

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Abstract

Modification of Newton's method with higher-order convergence is presented. The modification is based on Noor and Khan's fourth order method. In terms of computation cost, the newmethod requires four evaluation of the function and one evaluation of the first derivative per iteration. Analysis of convergence shown that the method has 8^{th}-order convergence with efficiency index 1.5157. Some numerical examples have shown that the proposed method performed very well compared with some existing methods.

Keywords: Nonlinear equation, Newton's method, Forward difference approximation, order of convergence

1.0 Introduction

Finding iterative methods for solving the nonlinear equation $f(x) = 0$ is an important area of research in numerical analysis. It has interesting application in several branches of pure and applied science. Newton's method is a well known and commonly used quadratically convergent iterative method for finding roots of nonlinear equation. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

In recent years, many researchers have made many modifications in this method to get higher order iterative methods. These methods developed include using various techniques by introducing some more steps to Newton's method. In this way, not only the convergence order but efficiency index of the method may also be increased.

Recently, researchers have introduced different three step iterative methods of various order for solving nonlinear equation (see [1-9] and some reference therein).

Motivated in this direction, we also introduce here a three-step method of convergence order eight with efficiency index 1.5157.

2.0 Development of Eighth-Order Method

In this section, we construct a three step cycles method of eight-order for solving nonlinear equations. Noor and Khan [10] developed a two cycle iterative method free from second derivative (see Algorithm 2.9 in [10]) with4^{th} order convergence, which is written as

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)}, \end{cases} \quad (2)$$

(See [10] for the derivation of (2))

To achieve an 8^{th} order method we add an additional cycle to (2) by applying Newton's iteration scheme

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad (3)$$

and we have an iterative scheme given as:

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$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{4}$$

We observed from this modification that, although (4) is of order eight with efficiency index $8^{\frac{1}{5}} = 1.5157$ it converges slowly with low precision. To overcome this setback we replace $f'(z_n)$ in the last cycle by forward-difference approximation $f[z_n, w_n]$ and we have

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, w_n]}, \tag{5}$$

where $w_n = z_n + f(z_n)$ and $f[\cdot,\cdot]$ denotes the first order divided difference. This adjustment reduced the number of derivative evaluation by one.

We therefore propose the following three-step iterative method:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n = x_n - \left[\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right]\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, w_n]} \end{cases} \tag{6}$$

3.0 Convergence Analysis

We now derive error equation of the proposed iterative method. We prove that the iterative method is of convergence order eight.

Theorem 1

Let $\alpha \in I$ be simple zero of a sufficiently differentiable function $f: I \rightarrow R$ for an open interval I. Then, the method that is defined by equation (6) has eight-order of convergence and satisfies the error equation as follows:

$$e_{n+1} = 2c_2(\, c_2^2c_3^2 - 2c_2^4c_3 + c_2^6)e_n^8 + O(e_n^8) \tag{7}$$

where $e_n = x_n - \alpha$; and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$, for $k = 2,3,4, \ldots$

Proof: Let α be a simple zero of the nonlinear equation $f(x) = 0$, and $x_n = \alpha + e_n$. By the Taylor expansion, we have the equation as follows:

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_7e_n^7 + c_8e_n^8 + c_9e_n^9 + O(e_n^9) \ldots] \tag{8}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + 10c_{10}e_n^9 + O(e_n^{10}) \ldots] \tag{9}$$

Now

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} = {} & e_n - c_2e_n^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ & -(-4c_5 + 10c_2c_4 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_n^5 \\ & -(17c_3c_4 - 33c_2c_3^2 + 52c_3c_2^3 - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 16c_2^5)e_n^6 - O(e_n^7)\end{aligned} \tag{10}$$

Substituting (10) in the second step of (6), we have

$$\begin{aligned}y_n = x_n - \frac{f(x_n)}{f'(x_n)} = {} & \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ & +(4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_n^5 \\ & +(-17c_3c_4 + 33c_2c_3^2 - 52c_3c_2^3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 16c_2^5)e_n^6 \\ & +O(e_n^7)\end{aligned} \tag{11}$$

$$\begin{aligned}f(y_n) = f'(\alpha)[& c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 \\ & -(4c_5 + 5c_2c_4 + 3c_3^2 - 12c_3c_2^2 + 6c_2^4)e_n^5 \\ & +(-17c_3c_4 + 37c_2c_3^2 - 73c_3c_2^3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 28c_2^5)e_n^6 \\ & +O(e_n^7)\end{aligned} \tag{12}$$

By considering (12) and (8) we obtain

$$\begin{aligned}f(y_n) - f(x_n) = f'(\alpha)[& e_n + (2c_2^2 - c_3)e_n^3 + (-2c_4 + 7c_2c_3 - 5c_2^3)e_n^4 \\ & +(5c_5 + 5c_2c_4 + 3c_3^2 - 12c_3c_2^2 + 6c_2^4)e_n^5 \\ & +(-37c_2c_3^2 + 73c_3c_2^3 - 28c_2^5 - 34c_4c_2^2 + 17c_3c_4 + 13c_2c_5 - 4c_6)e_n^6 \\ & +O(e_n^7)\end{aligned} \tag{13}$$

and

$$f(y_n) - 2f(x_n) = f'(\alpha)[e_n - c_2e_n^2 + (4c_2^2 - 3c_3)e_n^3 + (-5c_4 + 14c_2c_3 - 10c_2^3)e_n^4 + (9c_5 + 10c_2c_4 + 6c_3^2 - 24c_3c_2^2 + 12c_2^4)e_n^5 + (-9c_2c_3^2 + 146c_3c_2^3 - 74c_2c_3^2 - 56c_2^5 - 68c_4c_2^2 + 34c_3c_4 - 26c_2c_5)e_n^6 + O(e_n^7) \quad (14)$$

Using (13) and (14) we have

$$\frac{f(y_n) - f(x_n)}{f(y_n) - 2f(x_n)} = 1 + c_2e_n + (2c_3 - c_2^2)e_n^2 + (3c_4 - 2c_2c_3)e_n^3 + (-4c_5 + 3c_2c_4 + 3c_3^2 - 15c_3c_2^2 + 8c_2^4)e_n^4 + (5c_6 - 22c_3c_2^3 + 10c_4c_2^2 + 2c_3c_4 - 26c_2c_5 + 14c_2^5)e_n^5 + O(e_n^5) \quad (15)$$

Thus;

$$\frac{f(y_n) - f(x_n)}{f(y_n) - 2f(x_n)} \cdot \frac{f(x_n)}{f'(x_n)} = e_n + (c_2c_3 - c_2^3)e_n^4 + (-13c_2c_4 - 7c_3^2 + 20c_3c_2^2 - 6c_2^4)e_n^5 + (10c_6 + 18c_2^5 + 42c_2c_3^2 - 58c_3c_2^3 + 34c_4c_2^2 - 27c_3c_4 - 31c_2c_5)e_n^6 + O(e_n^7) \quad (16)$$

Put (16) in the second step of (6), we obtain

$$z = \alpha + (-c_2c_3 + c_2^3)e_n^4 + (13c_2c_4 + 7c_3^2 - 20c_3c_2^2 + 6c_2^4)e_n^5 + O(e_n^6) \quad (17)$$

$$f(z) = f'(\alpha)[(-c_2c_3 + c_2^3)e_n^4 + (13c_2c_4 + 7c_3^2 - 20c_3c_2^2 + 6c_2^4)e_n^5 + O(e_n^6) + (-10c_6 - 18c_2^5 - 42c_2c_3^2 + 58c_3c_2^3 - 34c_4c_2^2 + 27c_3c_4 + 31c_2c_5)e_n^6 + O(e_n^7) \quad (18)$$

Using (18) and the Taylor series expansion of the existing divided differences in the denominator of the third step of equation (6) we have:

$$x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,w_n]} = \alpha + 2c_2(c_2^2c_3^2 - 2c_2^4c_3 + c_2^6)e_n^8 + O(e_n^9) \quad (19)$$

or, in the final form

$$e_{n+1} = 2c_2(c_2^2c_3^2 - 2c_2^4c_3 + c_2^6)e_n^8 + O(e_n^9) \quad (20)$$

which shows that the proposed iterative method (6) has eighth-order convergence.■

Remark I

The order of convergence of iterative method (6) is 8. Per iteration of the proposed method requires four function evaluation and one evaluation of first derivative. We take into account the definition of efficiency index by Ostrowski in [11] as $p^{\frac{1}{d}}$; where p is the order of convergence and d is the number of functional evaluation per step. If we assumed that the functions evaluation are equal as the first evaluation, we have that the efficiency index of the proposed method as $8^{\frac{1}{5}} \approx 1.5157$ which is better than $2^{\frac{1}{2}} \approx 1.4142$ of classical Newton's method.

Remark II

The error term in equation (16) can be regarded as correction to the wrongNoor and Khan [10] fourth-order method error term given as $e_{n+1} = (10c_4 - 4c_2c_3 + 5c_2^3)e_n^4$.

4.0 Numerical Examples

In this section, we present some numerical examples to illustrate the efficiency and the accuracy of the new developed iterative method. We also compare the performance of our method with some existing eighth-order iterative methods.

We compare (6) with some existing three-step, order eight methods in Matinfar et al (MAA) [12] which is given by:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{(f(z_n))^2}{f\big(z_n + f(z_n)\big) - f(z_n)} \quad n = 0,1,2, \ldots$$

and in Wang and Liu [13] which are respectively represented by XIA 1, XIA 2 and XIA 3.
XIA 1 is:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left[\frac{1}{2} + \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}\left(\frac{1}{2} + \frac{f(z_n)}{f'(y_n)}\right)\right].$$

XIA 2 is:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left[\frac{5f(x_n)^2 - 2f(x_n)f(y_n) + f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)} + \left(1 + 4\frac{f(y_n)}{f(x_n)}\right)\left(\frac{f(z_n)}{f(y_n)}\right)\right].$$

XIA 3 is:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f(x_n)^2 - 5f(x_n)f(y_n) - f(y_n)^2}{4f(x_n)^2 - 9f(x_n)f(y_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}\left[1 + 4\frac{f(z_n)}{f(x_n)}\right]\left[\frac{8f(y_n)}{4f(x_n) - 11f(y_n)} + 1 + \frac{f(z_n)}{f(y_n)}\right].$$

All computations on examples have been performed using the programming package *Python 2.7.10*with precision$\epsilon = 10^{-15}$ and the following stopping criteria is used for computer programs:

$i.\quad |x_{n+1} - x_n| < \epsilon \qquad ii.\quad |f(x_{n+1})| < \epsilon.$

The following tests functions used were obtained from [12] , where x_* is the actual root of the function.

$f_1(x) = x^3 + 4x^2 - 15,\qquad x_* = 1.63198055660636,$

$f_2(x) = xe^{x^2} - sin^2(x) + 3\cos(x) + 5,\qquad x_* = -1.207647827130919$

$f_3(x) = 10xe^{-x^2} - 1,\qquad x_* = 1.67963061042845,$

$f_4(x) = sin^2(x) - x^2 + 1,\qquad x_* = 1.404491648215341,$

$f_5(x) = x^5 + x^4 + 4x^2 - 15,\qquad x_* = 1.347428098968305.$

For comparison, we have shown in Table 1 the test results obtained from the absolute values of the function ($|f(x_n)|$)and ($|x_n - x_*|$)for the new method and the cited existing methods.

Table 1: Comparison of iterative methods results

Functions	**Method**	$\|x_k - x_*\|$	$f(x_k)$
$f_1(x), x_0 = 2$	(6)	0	3.552713678800501e − 015
	MAA	2.489597032973023e − 007	3.552713678800501e − 015
	XIA 1	2.489597032973023e − 007	3.552713678800501e − 015
	XIA 2	2.489597032973023e − 007	3.552713678800501e − 015
	XIA 3	2.489597032973023e − 007	3.552713678800501e − 015
$f_2(x), x_0 = -1$	(6)	0	$2.664535259100376e - 015$
	MAA	0	$2.664535259100376e - 015$
	XIA 1	$6.661338147750939e - 016$	$1.509903313490213e - 014$
	XIA 2	0	$2.664535259100376e - 015$
	XIA 3	$1.798561299892754e - 014$	$3.632649736573512e - 013$
$f_3(x), x_0 = 1.5$	(6)	0	$2.220446049250313e - 016$
	MAA	0	$2.220446049250313e - 016$
	XIA 1	0	$2.220446049250313e - 016$
	XIA 2	0	$2.220446049250313e - 016$
	XIA 3	0	$2.220446049250313e - 016$
$f_4(x), x_0 = 1.5$	(6)	0	$3.330669073875469e - 016$
	MAA	0	$3.330669073875470e - 016$
	XIA 1	0	$3.330669073875470e - 016$
	XIA 2	$2.220446049250313e - 016$	$3.330669073875470e - 016$
	XIA 3	$2.220446049250313e - 016$	$3.330669073875470e - 016$
$f_5(x), x_0 = 1.2$	(6)	0	$1.776356839400251e - 015$
	MAA	$4.280989683049796e - 004$	$1.776356839400251e - 015$
	XIA 1	$4.280989683049796e - 004$	$1.776356839400251e - 015$
	XIA 2	$4.280989683049796e - 004$	$1.776356839400251e - 015$
	XIA 3	$4.280989683049796e - 004$	$1.776356839400251e - 015$

From table 1 we can see that method (6) have better precision and performance at same number of iteration compare to other cited existing methods of equal order of convergence. In addition the proposed method is simpler and involves less number of computation compare to other existing method cited.

5.0 Conclusion

In this paper, a new three step iterative method based on Newton's method and forward difference approximation is proposed. Analysis has shown that the method has eight-order convergence with efficiency index 1.5157. Comparison of its performance was made with other existing methods with same order of convergence. It is noteworthy that the new method contains less computational evaluation and performs better in terms of precision than the cited existing methods [12] and [13].

7.0 References

[1] I. A. Al-subaihi, A. J. Alqarni. Higher-Order Iterative Methods for Solving Nonlinear Equations, Life Science Journal, 11, 12, 85-91, 2014.

[2] I. A. Al-subaihi, H. I. Siyyam. Efficient Techniques for Constructing Optimal Iterative Methods with Eighth and Sixteenth Order of Convergence for Solving Non-linear Equations, SYLWAN journal, 158, Issue. 5, 2014.

[3] M. A. Noor, K. I. Noor. Improved iterative methods for solving nonlinear equations, Appl. Math. Comput., 184, 270-275, 2007.

[4] H.I. Siyyam. An Iterative Method with Fifth-Order Convergence for Nonlinear Equations, Journal of Applied Mathematical Sciences, Vol. 3, No. 41, 2041-2053,2009.

[5] J. H. Yun. A note on three-step iterative method fornonlinear equations, Appl. Math. Comput., 202, 401-405, 2008.

[6] R. F. Kung. A family of fourth order methods for nonlinear equations, SIAM Journal on Numerical Analysis 10, pp. 876-879, 1973.

[7] J. P.. Jaiswal and S. Panday. An efficient optimal eight-order iterative method for solving nonlinear equations, Universal Journal of Computational Mathematics 1(3):83-95, 2013.

[8] J. Kou, Y. Li, X. Wang. Some variants of Ostrowski's method with seventh-order convergence. J. Comput. Appl. Math. 209:153-159, 2007. http://dx.doi.org/10.1016/j.cam2006.10.073.

[9] B. Ghanbari. Three-Step Iterative Methods with Sixth-Order Convergence for Solving Nonlinear Equations. Walailak Journal of Sci. and Tech; 9(3):249-253, 2012.

[10] M. A. Noor and W. A. Khan. Fourth-order Iterative method free from second derivative for solving Nonlinear Equation; Applied Mathematical Science, Vol.,6, no. 93,4617-4625, 2012.

[11] A. M. Ostrowski. Solutions of Equations and System of Equations; Academic Press: Newyork, NU, USA, 1960.

[12] M. Matinfar, M. Aminzadeh & S. Asadpour. A New Three-step Iterative Method for Solving Nonlinear Equations. Journal of Mathematical Extension, Vol. 6, No. 1, (2012), 19-28

[13] L. Liu and X. Wang. Eight-order methods high efficiency index for solving nonlinear equations, Applied Mathematics and Computation, 215(2010), 3449-3454