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Convergence Speed of Some Hybrid Schemes For The Class of Contractive-Type Maps In Locally Convex Spaces

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Abstract

In this paper, we study the convergence of Picard-multistep-type iterative schemes and use the schemes to approximate the fixed point of contractive-like map in complete metrizable locally convex spaces. We also investigate their convergence speed (using PYTHON 2.5.4) with others (Mann, Picard-Mann, Ishikawa, Picard-Ishikawa, Noor, Picard-Noor and multistep) for increasing and decreasing functions. The results show that Picard-multistep-SP and multistep-SP converges faster than the other schemes for the functions under this study. Our convergence results generalize and extend multitude of results in the literature, including the results ofBerinde (2004).

Keywords and Phrases: Strong convergence, hybrid iterative schemes, contractive-like map, convergence speed, metrisable locally convex spaces

1.0 Introduction

Metric fixed point theory has been widely studied by experts in the past decades, since fixed point theory plays a vital role in Mathematics and applied sciences, such as Optimization, Mathematical models and Economic theories. Different iterative schemes have been used to approximate the fixed point of various contractive operators in different spaces like Metric, Normed linear,Partial metric and Cone metric spaces (for details seeIshikawa (1974), Mann (1953), Noor (2000), Phuengrattana and Suantai (2011) and Rhoades and Soltuz (2004).

In this study, we shall introduce some modified hybrid schemes and use these schemes to approximate the fixed point of a class of contractive-type mapping in a locally convex space. We shall also investigate the convergence speed of these various schemes to fixed point of the map considered.

A locally Convex space (X, u) with topology u is a topological vector space which has of a local base of convex neighborhoods of zero [Schaffer (1999), Chap.7]. It is metrizable if it is Hausdorff and has countable zero basis. Consequently, it is metrisable if u can be described by a countable family of continuous seminorms [Schaffer (1999)]. X is Hausdorff if and only if for each non-zero x $\in$ X, there is some p $\in$Q with p(x)>0 [Olaleru (2006)].

Let X be a metrizable topological space and C be a closed convex nonempty subset of X and

T: C $\rightarrow$ C a self-map of C. Assume that $F_T = \{p \in C : Tp = p\}$ is the set of fixed points of T.

For $x_0 \in C,$ the Picard iterative scheme is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n \geq 0 \qquad (1)$$

For $x_0 \in C,$ the Mann iterative scheme [Mann (1953)] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ n \geq 0 \qquad (2)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$

Observe that, if $\alpha_n = 1$ for each n, then the Mann iterative scheme (2) reduces to the Picard iterative scheme (1).

Olaleru (2006) proved the convergence of Mann iterative process using the Zamfirescu operators [Zamfirescu (1972)] and generalized several results in literature to complete metrizable locally convex spaces.

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For $x_0 \in C,$ the Ishikawa iterative scheme [Ishikawa (1974)] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \geq 0 \qquad (3)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$

Observe that, if $\beta_n = 0$ for each n, then the Ishikawa iterative scheme (3) reduces to the Mann iterative scheme (2).

For $x_0 \in C,$ the Noor iterative scheme [Noor (2000)] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T z_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n,\ n \geq 0 \qquad (4)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$

Observe that, if $\gamma_n = 0$ for each n, then the Noor iterative scheme (4) reduces to the Ishikawa iterative scheme (3).

For $x_0 \in C,$ the multistep iterative scheme [Rhoades and Soltuz (2004)] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y^1{}_n,$$
$$y^1{}_n = (1-\beta^i{}_n)x_n + \beta^i{}_n T y_n{}^{i+1},\ (i = 1,2,3,\ldots,k-2)$$
$$y_n{}^{k-1} = (1-\beta_n{}^{k-1})x_n + \beta_n{}^{k-1} T x_n,\ k \geq 2,\ n \geq 0 \qquad (5)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n{}^i\}_{n=0}^{\infty}$, $(i = 1,2,3,\ldots,k-2)$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$

Observe that the multistep iterative scheme (5) is a generalization of Noor, Ishikawa and the Mann iterative schemes. In fact, if k=3 in (5), we have the Noor iterative scheme (4), if k=2 in (5), we have the Ishikawa iteration (3) if k=2 and $\beta_n{}^1$ =0 in (5), we have the Mann iterative scheme (2).

Phuengrattana and Suantai (2011) introduced SP-iterative scheme and used the scheme to approximate the fixed point of continuous functions on an arbitrary interval. They also compared the convergence speed of Mann, Ishikawa, Noor and SP-iterative processes and proved that the SP-iterative process converges faster than the others (Mann, Ishikawa and Noor schemes).

For $x_0 \in C,$ the SP iterative scheme [23] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)z_n + \beta_n T z_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n,\ n \geq 0 \qquad (6)$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$

Observe that, if $\gamma_n = 0$ for each n, then the SP iterative scheme (6) reduces to the Ishikawa iterative scheme (3). Also, if $\gamma_n, \beta_n = 0$ for each n then the SP- iteration process (6) reduces to the Mann iterative process (2).

For $x_0 \in C,$ the multistep-SP iterative scheme [G¨ursoy, Karakaya and Rhoades (2013)] is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)y_n{}^1 + \alpha_n T y^1{}_n,$$
$$y^1{}_n = (1-\beta^j{}_n)y_n{}^{j+1} + \beta^j{}_n T y_n{}^{j+1},\ (j = 1,2,3,\ldots,q-2),$$

$$y_n^{q-1} = (1-\beta_n^{q-1})x_n + \beta_n^{q-1}Tx_n,\ q \geq 2,\ n \geq 0 \tag{7}$$

Where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n^{j}\}_{n=0}^{\infty}$, $(j = 1,2,3,\ldots,q-2)$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Khan (2013), introduced the Picard-Mann hybrid iterative scheme for a single nonexpansive mapping T and showed that his new scheme converges faster than all of Picard (1), Mann (2) and Ishikawa (3) iterative schemes in the sense of Berinde (2004) for contractions. For any initial point $a_0 \in C,$ the Picard-Mann iterative scheme [Khan (2013)] is the sequence $\{a_n\}_{n=0}^{\infty}$ is defined by

$$a_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)a_n + \alpha_n Ta_n,\ n \geq 0 \tag{8}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in (0,1).

Motivated by the work of Khan [Khan (2013)], we shall introduce the following hybrid iterative schemes and prove their strong convergence results for contractive-like inequality operators [Imoru and Olatinwo (2013)] in locally convex spaces. Also, we will investigate their rate of convergence for this class of operators.

For any initial point $b_0 \in C,$ the Picard-Ishikawa hybrid iterative hybrid scheme is the sequence $\{b_n\}_{n=0}^{\infty}$ is defined by

$$b_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)b_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)b_n + \beta_n Tb_n,\ n \geq 0 \tag{9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in (0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty.$.

For any initial point $e_0 \in C,$ the Picard-Noor iterative hybrid scheme is the sequence $\{e_n\}_{n=0}^{\infty}$ is defined by

$$e_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)e_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)e_n + \beta_n Th_n,$$
$$h_n = (1-\beta_n)e_n + \beta_n Te_n,\ n \geq 0 \tag{10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in (0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

For any initial point $h_0 \in C,$ the Picard-SP iterative hybrid scheme is the sequence $\{h_n\}_{n=0}^{\infty}$ is defined by

$$h_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)z_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)x_n + \beta_n Tx_n,$$
$$x_n = (1-\beta_n)h_n + \beta_n Th_n,\ n \geq 0 \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in (0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

For any initial point $c_0 \in C,$ the Picard-AK iterative hybrid scheme is the sequence $\{c_n\}_{n=0}^{\infty}$ is defined by

$$c_{n+1} = Ty_n,$$
$$y_n = (1-\alpha_n)z_n + \alpha_n Tz_n,$$
$$z_n = (1-\beta_n)c_n + \beta_n Tc_n,\ n \geq 0 \tag{12}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in (0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

For any initial point $d_0 \in C$, the Picard-S iterative hybrid scheme is the sequence $\{d_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} d_{n+1} &= Ty_n, \\ y_n &= (1-\alpha_n)Td_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)d_n + \beta_n Td_n,\ n \geq 0 \end{aligned} \tag{13}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in (0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

For $x_0 \in C$, the Picard-multistep hybrid iterative scheme is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= Ty^1{}_n, \\ y^1{}_n &= (1-\alpha^i{}_n)x_n + \alpha^i{}_n Ty_n{}^{i+1},\ (i = 1,2,3,\ldots,k-2) \\ y_n{}^{k-1} &= (1-\alpha_n{}^{k-1})x_n + \alpha_n{}^{k-1}Tx_n,\ k \geq 2,\ n \geq 0 \end{aligned} \tag{14}$$

where $\{\alpha_n{}^i\}_{n=0}^{\infty}$, $(i = 1,2,3,\ldots,k-2)$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Observe that the Picard-multistep hybrid iterative scheme (14) is a generalization of Picard- Noor (10), Picard-Ishikawa (9) and Picard- Mann (8) hybrid iterative schemes. Infact, if k=4 in (13), we have Picard-Noor iterative scheme (10), if k=3 in (14), we have the Picard-Ishikawa iterative scheme (9) if k=3 and $\alpha_n{}^2 = 0$ in (14), we have Picard- Mann (8) iterative scheme.

Also, for $x_0 \in C$, the Picard-multistep-SP hybrid iterative scheme is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= Ty^1{}_n, \\ y^1{}_n &= (1-\alpha^j{}_n)y_n{}^{j+1} + \alpha^j{}_n Ty_n{}^{j+1},\ (j = 1,2,3,\ldots,q-2) \\ y_n{}^{q-1} &= (1-\alpha_n{}^{q-1})x_n + \alpha_n{}^{q-1}Tx_n,\ q \geq 2,\ n \geq 0 \end{aligned} \tag{15}$$

where $\{\alpha_n{}^j\}_{n=0}^{\infty}$, $(j = 1,2,3,\ldots,q-2)$ are real sequences in [0,1] such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Several generalizations of the Banach fixed point theorem have been proved to date, (for example see [Chatterjea (1972), Kannan (1968) and Zamfirescu (1972)]). One of the most commonly studied generalizations hitherto is the one proved by Zamfirescu (1972), which is stated as thus:

Theorem 1.2[Zamfirescu (1972)]. Let X be a complete metric space and T:X $\rightarrow$X a Zamfirescu operator satisfying

$$d(Tx,Ty) \leq h\max\{d(x,y), \tfrac{1}{2}[d(x,Tx)+d(y,Ty)], \tfrac{1}{2}[d(x,Ty)+d(y,Tx)]\} \tag{16}$$

where $h \in [0,1)$. Then, T has a unique fixed point and the Picard iteration converges to p for any $x_0 \in X$.

Observe that in a Banach space setting, condition (16) implies

$$\| Tx - Ty \| \leq \delta \| x - y \| + 2\delta \| x - Tx \| \tag{17}$$

where $\delta \in [0,1)$ and $\delta = \max\{h, \frac{h}{2-h}\}$, for details of proof see[Berinde, V. (2004)].

The most commonly used methods of approximating the fixed points of the Zamfirescu operators are Picard, Mann [Mann (1953)], Ishikawa [Ishikawa (1974)] and Noor [Noor (2000)] iterative processes. Berinde (2004) proved the convergence of Mann and Ishikawa iterative schemes in the class of quasi-contractive operators in arbitrary Banach space while Rafiq (2006) proved the convergence of Noor iterative process (3) using the Zamfirescu operators defined by (17).

2.0 Results

We prove that multistep-type hybrid iterative schemeconverges strongly to the unique fixed point for contractive-like operators in the following theorem:

Theorem 1. Let (X, f_c) be a complete metrizable locally convex space, C a closed convex subset of X and $T : C \to C$ be an operator satisfying the condition

$$f_c(Tx - Ty) \le \delta \; f_c(x - y) + \varphi(f_c(x - Tx)) \tag{18}$$

for each $y \in C$ and $\delta \in [0,1)$. For $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the Picard-multistep hybrid iterative process defined by (14), where $\{\alpha_n\}_{n=0}^{\infty}$, i=1, 2, ...,k-1 are real sequences in [0,1]. Then

(i) T has a unique fixed point p;

(ii) Picard-multistep hybrid iterative scheme converges strongly to p.

Proof:

(i) We shall first establish that the mapping T satisfying the contractive condition (18) has a unique fixed point.

Suppose there exist $p_1, p_2 \in F_T$, and that $p_1 \ne p_2$, with $f_c(p_1 - p_2) > 0$, then

$$0 < f_c(p_1 - p_2) = f_c(Tp_1 - Tp_2) \le \delta \; f_c(p_1 - p_2) + \varphi(f_c(p_1 - Tp_1)) = \delta \; f_c(p_1 - p_2) + \varphi(0)$$

Thus, $(1-\delta) f_c(p_1 - p_2) \le 0.$

Since $\delta \in [0,1)$, then $(1-\delta) > 0$ and $f_c(p_1 - p_2) \le 0.$ Since norm is nonnegative, we have that $f_c(p_1 - p_2) = 0.$

That is, $p_1 = p_2$ (say). Thus, T has a unique fixed point p.

(ii) Next we shall establish that $\lim_{n\to\infty} x_n = p$. That is, we show that the Picard-multistep hybrid iterative process (14) converges strongly to p of T.

In view of (14) and (18), we have

$$f_c(x_{n+1} - p) = f_c(Ty_n{}^1 - Tp). \tag{19}$$

Using (18), with $y = y_n{}^1$, gives

$$f_c(x_{n+1} - p) = f_c(Ty_n{}^1 - Tp) \le \delta \; f_c(y_n{}^1 - p) + \varphi(f_c(p - Tp)). \tag{20}$$

Substituting (20) in (19), we have

$$f_c(x_{n+1} - p) \le \delta \; f_c(y_n{}^1 - p). \tag{21}$$

We note that $\alpha_n{}^i \in [0,1]$ for $n \ge 0$ and $1 \le i \le k-2.$

$$\begin{aligned}
& f_c(y_n{}^1 - p) \le (1-\alpha_n{}^1) f_c(x_n - p) + \alpha_n{}^1 f_c(Ty_n{}^2 - Tp) \\
& \le (1-\alpha_n{}^1) f_c(x_n - p) + \alpha_n{}^1[\delta \; f_c(y_n{}^2 - p) + \varphi(f_c(p - Tp))] \\
& = (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1 f_c(y_n{}^2 - p) \\
& \le (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1[(1-\alpha_n{}^2) f_c(x_n - p) + \alpha_n{}^2 f_c(Ty_n{}^3 - p)] \\
& \le (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1[(1-\alpha_n{}^2) f_c(x_n - p) + \alpha_n{}^2(\delta f_c(y_n{}^3 - p) + \varphi(f_c(p - Tp)))] \\
& = (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1(1-\alpha_n{}^2) f_c(x_n - p) + \delta^2\alpha_n{}^1\alpha_n{}^2 f_c(y_n{}^3 - p) \\
& \le (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1(1-\alpha_n{}^2) f_c(x_n - p) + \delta^2\alpha_n{}^1\alpha_n{}^2[(1-\alpha_n{}^3) f_c(x_n - p) \\
& + \alpha_n{}^3(\delta f_c(y_n{}^4 - p) + \varphi(f_c(p - Tp)))] \\
& = (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1(1-\alpha_n{}^2) f_c(x_n - p) + \delta^2\alpha_n{}^1\alpha_n{}^2(1-\alpha_n{}^3) f_c(x_n - p) \\
& + \delta^3\alpha_n{}^1\alpha_n{}^2\alpha_n{}^3 f_c(y_n{}^4 - p) \\
& \le (1-\alpha_n{}^1) f_c(x_n - p) + \delta\alpha_n{}^1(1-\alpha_n{}^2) f_c(x_n - p) + \delta^2\alpha_n{}^1\alpha_n{}^2(1-\alpha_n{}^3) f_c(x_n - p) \\
& + \delta^3\alpha_n{}^1\alpha_n{}^2\alpha_n{}^3 f_c(y_n{}^4 - p) + \ldots + \delta^{k-2}\alpha_n{}^1\alpha_n{}^2\alpha_n{}^3 \ldots \alpha_n{}^{k-3}\alpha_n{}^{k-2} f_c(y_n{}^{k-1} - p).
\end{aligned} \tag{22}$$

$$\begin{aligned}
& f_c(y_n{}^{k-1} - p) \le (1-\alpha_n{}^{k-1}) f_c(x_n - p) + \alpha_n{}^{k-1} f_c(Tx_n - Tp) \\
& \le (1-\alpha_n{}^{k-1}) f_c(x_n - p) + \alpha_n{}^{k-1}[\delta f_c(x_n - p) + \varphi(f_c(p - Tp))]
\end{aligned}$$

$$= (1-\alpha_n^{k-1}) f_c(x_n - p) + \delta\alpha_n^{k-1} f_c(x_n - p). \quad (23)$$

(22) and (23) hold since Tp=p and $\varphi(0) = 0$.

Substituting (23)and (22) in (18), we have

$$f_c(x_{n+1} - p) \le \delta[(1-\alpha_n^1) f_c(x_n - p) + \delta\alpha_n^1(1-\alpha_n^2) f_c(x_n - p) + \delta\alpha_n^1(1-\alpha_n^2) f_c(x_n - p)$$
$$+ \delta^2\alpha_n^1\alpha_n^2(1-\alpha_n^3) f_c(x_n - p) + \delta^3\alpha_n^1\alpha_n^2\alpha_n^3(1-\alpha_n^4) f_c(x_n - p) + \ldots +$$
$$+ \delta^{k-2}\alpha_n^1\alpha_n^2\alpha_n^3 \ldots \alpha_n^{k-3}\alpha_n^{k-2}(1-\alpha_n^{k-1}) f_c(x_n - p) + \delta^{k-1}\alpha_n^1\alpha_n^2\alpha_n^3 \ldots \alpha_n^{k-3}\alpha_n^{k-2}\alpha_n^{k-1} f_c(x_n - p)]$$
$$\le \delta[1-\alpha_n^1(1-\delta)] f_c(x_n - p). \quad (24)$$

By (24), we inductively obtain

$$f_c(x_{n+1} - p) \le \delta[\prod_{m=0}^{n}(1-(1-\delta)\alpha_m^1)] f_c(x_0 - p). \quad (25)$$

Using the fact that $\delta \in [0,1)$, $\alpha_m^1 \in [0,1]$ and $\sum_{m=0}^{\infty}\alpha_m^1 = \infty$,it result that

$$\lim_{n\to\infty}\prod_{m=0}^{n}[1-(1-\delta)\alpha_m^1)] = 0.$$

Hence, $\lim_{n\to\infty} f_c(x_{n+1} - p) = 0$.

That is $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Theorem 1 leads to the following corollary:

Corollary 1. Let (X, f_c) be a complete metrizable locally convex space, C a closed convex subset of X and $T: C \to C$ be an operator satisfying the condition

$$f_c(Tx - Ty) \le \delta\ f_c(x - y) + \varphi(f_c(x - Tx)) \quad (26)$$

for each $y \in C$ and $\delta \in [0,1)$. For $e_0, b_0, a_0, x_0 \in C$, let $\{e_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$ be the Picard-Noor hybrid, Picard-Ishikawa hybrid, Picard-Mann hybrid and Picard iterative schemes defined by (10), (9), (8), (1) respectively where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1]. Then

(i) T has a unique fixed point p;
(ii) Picard-Noor hybrid iterative scheme (10) converges strongly to p;
(iii) Picard-Ishikawa hybrid iterative scheme (9) converges strongly to p;
(iv) Picard-Mann hybrid iterative scheme (8) converges strongly to p;
(v) Picard iterative scheme (1) converges strongly to p.

Remark: Theorem 1 improves several known results in literature including the results of Berinde (2004) and Rhoades' theorem2 [Rhoades (1974)] by considering iterative schemes and contractive operators that are more general than those in literature.

Theorem 2. Let (X, f_c) be a complete metrizable locally convex space, C a closed convex subset of X and $T: C \to C$ be an operator satisfying the condition

$$f_c(Tx - Ty) \le \delta\ f_c(x - y) + \varphi(f_c(x - Tx)) \quad (27)$$

for each $y \in C$ and $\delta \in [0,1)$. For $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the Picard-multistep-SP hybrid iterative process defined by (15), where $\{\alpha_n\}_{n=0}^{\infty}$, j=1, 2, ..., q-1 are real sequences in [0,1]. Then

(i) T has a unique fixed point p;
(ii) Picard-multistep-SP hybrid iterative scheme converges strongly to p.

Proof:

(i) We shall first establish that the mapping T satisfying the contractive condition (27) has a unique fixed point.

Suppose there exist $p_1, p_2 \in F_T$, and that $p_1 \ne p_2$, with $f_c(p_1 - p_2) > 0$, then

$$0 < f_c(p_1 - p_2) = f_c(Tp_1 - Tp_2) \le \delta\ f_c(p_1 - p_2) + \varphi(f_c(p_1 - Tp_1)) = \delta\ f_c(p_1 - p_2) + \varphi(0)$$

Thus, $(1-\delta) f_c(p_1 - p_2) \leq 0.$

Since $\delta \in [0,1),$ then $(1-\delta) > 0$ and $f_c(p_1 - p_2) \leq 0.$ Since norm is nonnegative, we have that $f_c(p_1 - p_2) = 0.$ That is, $p_1 = p_2$ (say). Thus, T has a unique fixed point p.

(ii) Next we shall establish that $\lim_{n\to\infty} x_n = p$. That is, we show that the Picard-multistep-SP hybrid iterative process (15) converges strongly to p of T.

In view of (15) and (27), we have

$$f_c(x_{n+1} - p) = f_c(Ty_n^{\ 1} - Tp). \tag{28}$$

Using (2.10), with $y = y_n^{\ 1}$, gives

$$f_c(Ty_n^{\ 1} - Tp) \leq \delta\ f_c(y_n^{\ 1} - p) + \varphi(f_c(p - Tp)). \tag{29}$$

Substituting (29) in (28), we have

$$f_c(x_{n+1} - p) \leq \delta\ f_c(y_n^{\ 1} - p). \tag{30}$$

An application of (15) and (27), with $y = y_n^{\ 1}$ give $f_c(y_n^{\ 1} - p) \leq (1-\alpha_n^{\ 1}) f_c(y_n^{\ 2} - p) + \alpha_n^{\ 1} f_c(Ty_n^{\ 2} - Tp)$

$$\leq (1-\alpha_n^{\ 1}) f_c(y_n^{\ 2} - p) + \alpha_n^{\ 1}[\delta\ f_c(y_n^{\ 2} - p) + \varphi(f_c(p - Tp))]$$

$$= [1-\alpha_n^{\ 1}(1-\delta)] f_c(y_n^{\ 2} - p). \tag{31}$$

(31) holds since Tp=p and $\varphi(0) = 0.$

Also an application of (15) and (27), with $y = y_n^{\ 2}$ give

$$f_c(y_n^{\ 2} - p) \leq (1-\alpha_n^{\ 2}) f_c(y_n^{\ 3} - p) + \alpha_n^{\ 2} f_c(Ty_n^{\ 3} - Tp)$$

$$\leq (1-\alpha_n^{\ 2}) f_c(y_n^{\ 3} - p) + \alpha_n^{\ 2}[\delta\ f_c(y_n^{\ 3} - p) + \varphi(f_c(p - Tp))]$$

$$= [1-\alpha_n^{\ 2}(1-\delta)] f_c(y_n^{\ 3} - p). \tag{32}$$

Similarly, an application of (15) and (27) with $y = y_n^{\ 3}$ give

$$f_c(y_n^{\ 3} - p) \leq (1-\alpha_n^{\ 3}) f_c(y_n^{\ 4} - p) + \alpha_n^{\ 3} f_c(Ty_n^{\ 4} - Tp)$$

$$\leq (1-\alpha_n^{\ 3}) f_c(y_n^{\ 4} - p) + \alpha_n^{\ 3}[\delta\ f_c(y_n^{\ 4} - p) + \varphi(f_c(p - Tp))]$$

$$= [1-\alpha_n^{\ 3}(1-\delta)] f_c(y_n^{\ 4} - p). \tag{33}$$

Continuing the above process with $y = y_n^{\ q-2}$, we have:

$$f_c(y_n^{\ q-2} - p) \leq [1-\alpha_n^{\ q-2}(1-\delta)] f_c(y_n^{\ q-1} - Tp). \tag{34}$$

Finally, an application of (15) and (27) with $y = y_n^{\ q-1},$ give

$$f_c(y_n^{\ q-1} - p) \leq (1-\alpha_n^{\ q-1}) f_c(x_n - p) + \alpha_n^{\ q-1} f_c(Tx_n - Tp)$$

$$\leq (1-\alpha_n^{\ q-1}) f_c(x_n - p) + \alpha_n^{\ q-1}[\delta\ f_c(x_n - p) + \varphi(f_c(p - Tp))]$$

$$= [1-\alpha_n^{\ q-1}(1-\delta)] f_c(x_n - p). \tag{35}$$

Substituting (35) in (34), (34) in (33), (33) in (32), (32) in (31) and (31) in (30) respectively, we have

$$f_c(x_{n+1} - p) \leq \delta\ [[1-\alpha_n^{\ 1}(1-\delta)][1-\alpha_n^{\ 2}(1-\delta)][1-\alpha_n^{\ 3}(1-\delta)]\ldots$$

$$[1-\alpha_n^{\ q-2}(1-\delta)][1-\alpha_n^{\ q-1}(1-\delta)]] f_c(x_x - p).$$

$$\leq \delta[1-\alpha_n^{\ 1}(1-\delta)] f_c(x_n - p)$$

$$\leq \delta[\prod_{m=0}^{n}\ (1-\alpha_m^{\ 1}(1-\delta))] f_c(x_0 - p)$$

$$\leq \delta[e^{-(1-\delta)\sum_{m=0}^{n}\alpha_m{}^1}] f_c(x_0 - p). \qquad (36)$$

Since $\delta \in [0,1)$, $\alpha_m{}^1 \in [0,1]$ and $\sum_{m=0}^{\infty}\alpha_m{}^1 = \infty$, it result that

So $e^{-(1-\delta)\sum_{m=0}^{n}\alpha_m{}^1} \to 0$ as $n \to \infty.$

Thus $\lim_{n\to\infty} f_c(x_{n+1} - p) = 0.$ That is $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. This ends the proof.

Theorem 2 leads to the following corollary:

Corollary 2. Let (X, f_c) be a complete metrizable locally convex space, C a closed convex subset of X and $T: C \to C$ be an operator satisfying the condition

$$f_c(Tx - Ty) \leq \delta\ f_c(x - y) + \varphi(f_c(x - Tx)) \qquad (37)$$

for each $y \in C$ and $\delta \in [0,1).$ For $h_0, a_0, x_0 \in C$, let $\{h_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$ be the Picard-SP hybrid, Picard-Mann hybrid and Picard iterative schemes defined by (11), (8), (1) respectively, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, are real sequences in [0,1]. Then

(i) T has a unique fixed point p;
(ii) the Picard-SP hybrid iterative scheme (11) converges strongly to p;
(iii) the Picard-Mann hybrid iterative scheme (8) converges strongly to p;
(iv) the Picard iterative scheme (1) converges strongly to p.

3.0 Numerical Examples

We investigate the performance of our schemes by comparing our modified iterative schemes (multstep-SP and Picard-multstep-SP hybrid iterative schemes) with others (Mann, Picard-Mann, Ishikawa, Picard-Ishikawa, Noor and Picard-Noor, multistep and Picard-multistep iterative schemes) using numerical examples with the help computer programs in PYTHON 2.5.4. The
results are shown in Tables1-2, 3-4, 5-6, by taking initial approximation

$a_0 = b_0 = c_0 = e_0 = h_0 = d_0 = 0.8$ and $\alpha_n = \beta_n = \gamma_n = \alpha_n{}^i = \frac{1}{(1+n)^{\frac{1}{2}}}$, (for i =1,2,3,...,k-2) for all the iterative schemes.

Example of Increasing Function

Let $f: [0,8] \to [0,8]$ be defined by $f(x) = \frac{x^2 + 9}{10}$. Then f is an increasing function. The comparison of these iterative schemes to the fixed point $p = 1$ is shown in Tables 1-2.

Example of Decreasing Function

Let $f: [0,1] \to [0,1]$ be defined by $f(x) = (1-x)^m$, m=7, 8,.... Then f is a decreasing function. By taking m=8,the comparison of these iterative schemes to the fixed point p=0.18834768 is shown in Tables 3-4.

Table 1: Numerical Examples for Increasing Function

N(number of iterations)	Mann iteration	Picard-Mann iteration	Ishikawa iteration	Picard-Ishikawa iteration	Noor iteration	Picard-Noor iteration
0	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000
1	0.89468544	0.98004620	0.90394972	0.98171251	0.90491177	0.90394972
2	0.94396850	0.99786857	0.95364871	0.99823630	095463374	0.95364871
3	0.97002960	0.99977079	0.97757941	0.99982908	0.97831976	0.97757941
4	0.98392420	0.99997533	0.98914252	0.99998343	0.98963086	0.98914252
5	0.99136421	0.99999735	0.99473919	0.99999839	0.99503879	0.99473919
6	0.99535722	0.99999971	0.99745028	0.99999984	0.99762582	0.99745028
7	0.99750287	0.99999997	0.99940088	0.99999998	0.99886374	0.99876408
8	0.99865661	0.99999999	0.99940088	1.00000000	0.99945618	0.99940088
9	0.99927720	1.00000000	0.99970956	-	0.99973972	0.99970956
10	0.99961108	-	0.99985920	-	0.99987542	0.99985920
11	0.99979072	-	0.99993174	-	0.99994037	0.99993174
12	0.99988739	-	0.99996691	-	0.99997146	0.99996691
.	.	.	.	.	.	.
24	.	-	1.00000000	-	1.00000000	1.00000000
.	.	.	.	.	.	.
28	1.00000000	-	-	-	-	-

Table 2: Numerical Example for Increasing Function

N(number of iterations)	SP iteration	Picard-SP iteration	Multistep iteration	Picard-multistep iteration	Multistep-SP iteration	Picard-multistep-SP iteration
0	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000
1	0.97002960	0.97002960	0.90502396	0.90502396	0.99998245	1.00000000
2	0.99535722	0.99535722	0.95475180	0.95475180	1.00000000	-
3	0.99927720	0.99927720	0.97840974	0.97840974	-	-
4	0.99988739	0.99988739	0.98969059	0.98969059	-	-
5	0.99998245	0.99998245	0.99507550	0.99507550	-	-
6	0.99999727	0.99999727	0.99764732	0.99764732	-	-
7	0.99999957	0.99999957	0.99887591	0.99887591	-	-
8	0.99999993	0.99999993	0.99946290	0.99946290	-	-
9	0.99999999	0.99999999	0.99974337	0.99974337	-	-
10	1.00000000	1.00000000	0.99987738	0.99987738	-	-
11	-	-	0.99994141	0.99994141	-	-
12	-	-	0.99997200	0.99997200	-	-
13	-	-	.	0.99998662	-	-
.	.	.	.	.	.	.
.	.	.	.	.	.	.
24	-	-	1.00000000	1.00000000	-	-

Table 3: Numerical Example for Decreasing Function

N(number of iterations)	Mann iteration	Picard-Mann iteration	Ishikawa iteration	Picard-Ishikawa iteration	Noor iteration	Picard-Noor iteration
0	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000
1	0.51715819	0.00295419	0.51820175	0.00290350	0.51818383	0.51818383
2	0.33535963	0.03298199	0.34834693	0.89680408	0.34647661	0.34647661
3	0.23025531	0.06336713	0.26593384	0.00096679	0.25404936	0.25404936
4	0.19242188	0.09970826	0.23021568	0.91021186	0.21039925	0.21039925
5	0.18835385	0.14122038	0.21279037	0.00081906	0.19421433	0.19421433
6	0.18834762	0.17479994	0.20326713	0.91123700	0.18970710	0.18970710
7	0.18834768	0.18710496	0.19769124	0.00080862	0.18864833	0.18864833
8	-	0.18831670	0.19429009	0.91130946	0.18841341	0.18841341
9	-	0.18834710	0.19216311	0.00080789	0.18836201	0.18836201
10	-	0.18834767	0.19081216	0.91131455	0.18835080	0.18835080
11	-	0.18834768	0.18994564	-	0.18834836	0.18834836
12	-	-	-	.	0.18834783	0.18834783
13	-	-	-	.	0.18834771	0.18834771
14	-	.	.	.	.	.
15	-	-	-	-	0.18834768	0.18834768
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
37	-	-	0.18834770	-	-	-

Table 4: Numerical Example for Decreasing Function

N(number of iterations)	SP iteration	Picard-SP iteration	multistep iteration	Picard-multistep iteration	Multistep-SP iteration	Picard-multistep-SP iteration
0	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000	0.80000000
1	0.23025530	0.18287087	0.51818413	0.51818413	0.18834768	0.18834768
2	0.18834762	0.18847109	0.34670870	0.34670870	-	-
3	0.18834768	0.18834505	0.25697475	0.25697671	-	-
4	-	0.18834774	0.21638616	0.21643215	-	-
5	-	0.18834768	0.19949734	0.19960662	-	-
6	-	-	0.19273171	0.19284285	-	-
7	-	-	0.19006371	0.19014196	-	-
8	-	-	0.18901819	0.18906402	-	-
9	-	-	0.18860949	0.18863371	-	-
10	-	-	0.18844988	0.18846189	-	-
11	-	-	0.18838757	0.18839329	-	-
12	-	-	0.18836325	0.18836589	-	-
13	-	-	0.18835376	0.18835495	-	-
14	-	-	0.18835005	0.18835058	-	-
15	-	-	0.18834861	.0.18834884	-	-
.	.	.	.	.	.	.
21	-	-	0.18834768	0.18834768	-	-
.	.	.	.	.	.	.
37	-	-	-	-	-	-

4.0 Discussion

Increasing Function $f(x) = \dfrac{x^2 + 9}{10}$

The Mann iterative scheme converges to a fixed point in 28 iterations, Picard-Mann hybrid scheme converges in 9 iterations, Ishikawa scheme converges in 24 iterations, Picard-Ishikawa scheme converges in 8 iterations, Noor scheme converges in 24 iterations, Picard-Noor scheme converges in 24 iterations, SP scheme converges in 10 iterations, Picard-SP scheme converges in 10 iterations, multistep scheme converges in 24 iterations, Picard- multistep scheme converges in 24 iterations, multistep-SP scheme converges in 2 iterations, Picard-multistep-SP scheme converges in 1 iteration.

Decreasing Function $f(x)=(1-x)^m$, m=8.

The Mann iterative scheme converges to a fixed point in 7 iterations, Picard-Mann hybrid scheme converges in 11 iterations, Ishikawa scheme converges in iterations greater than 37, Picard-Ishikawa scheme oscillates between 0 and 1 (it never converges), Noor scheme converges in 15 iterations, Picard-Noor scheme converges in 15 iterations, SP scheme converges in 3 iterations, Picard-SP scheme converges in 5 iterations, multistep scheme converges in 21 iterations, Picard- multistep scheme converges in 21 iterations, multistep-SP scheme converges in 1 iteration, Picard-multistep-SP scheme converges in 1 iteration.

5.0 Remark

(1) The order of decreasing rate of convergence in the case of increasing functions are: Picard-multistep-SP, multistep-SP, Picard-Ishikawa, Picard-Mann, (SP and Picard-SP), (Ishikawa, Noor, Picard-Noor and multistep) iterative schemes.

(2) The order of decreasing rate of convergence in the case of decreasing functions are: (Picard-multistep-SP and multistep-SP), SP, Picard-SP, Mann, Picard-Mann, (Noor and Picard-Noor), (multistep and Picard-multistep) and Ishikawa iterative schemes.

(3) The Picard-Ishikawa scheme does not converge for decreasing functions.

6.0 Conclusion

(1). Our Picard-multistep-SP hybrid scheme is faster than the others (multistep-SP, multistep, Picard-SP, SP, Picard-Noor, Noor, Picard-Ishikawa, Ishikawa, Picard-Mann and Mann iterative schemes) for increasing functions.

(2). Picard-multistep-SP and multistep-SP scheme converges in 1iteration and faster than Picard-multistep, multistep, SP, Picard-SP, Picard-Noor, Noor, Picard-Mann and Mann iterative schemes for decreasing functions.

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