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Variable Order Multi-Derivative Linear Multistep Methods with one Off-Step Point

R. I. Okuonghae, E. M. Ezeimo and M. N. O. Ikhile

Advanced Research Laboratory, Department of Mathematics, University of Benin, Benin City. Nigeria.

Abstract

This paper exploresmulti-derivative methods with one off-step point for stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The choice of the grid point gives $A(\alpha)$-stability property of the resultant hybrid methods. Though machine limitation has constrained the investigation of the hybrid linear multistep methods (LMM) beyond the step number $k = 14$,but the methods are $A(\alpha)$-stable for $k \le 14$ and are of order $p = k + 2$. Numerical results demonstrates the application of these methods on stiff problems.

Keywords: Backward differentiation formulas, General linear methods (GLM), MuIti-derivative LMM, Third derivative linear multistep methodsA/A(α)-stability.
AMS subject classification: 65L05, 65L06

1.0 Introduction

The purpose herein is to consider a subclass of multi-derivative hybrid linear multistep methods,

$$\begin{cases} y_{n+v_j} = \sum_{j=0}^{k} \Omega_j y_{n+j} + \sum_{i=1}^{s} h^i \left(\sum_{j=0}^{k} \phi_j^{(s)} f_{n+j}^{(i-1)} \right), \quad v_j \ne t; \quad j = 1(1)r, t = 1(1)k, \\ y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{i=1}^{s} h^i \left(\sum_{j=0}^{k} \beta_j^{(s)} f_{n+j}^{(i-1)} \right) + \sum_{i=1}^{s} h^i \left(\sum_{j=0}^{r} \lambda_j^{(s)} f_{n+v_j}^{(i-1)} \right), \end{cases}$$

for the stiff ordinary differential equations (ODEs)

$$\begin{cases} y' = f(x, y(x)), x \in [x_0, X], \\ y_0 = y(x_0), \quad y \in R^m, \ f \in R \times R^m, \ m \ge 1. \end{cases} \tag{1}$$

The constants Ω_j, $\phi_j^{(s)}$ α_j, β_j, $\lambda_j^{(s)}$ can be determined by Taylor series expansion and defines a method uniquely. An initial value problem which stiffness ratio $S >> 1$ is said to be stiff [11]. As has been noted in [4], stiffness often arises in the modelling of gas mixtures in which the time scales of the reactants are much smaller than the movement times. Such other areas include pollution models, nozzle design for rockets, combustion and nuclear reactions. Other areas include biological modelling, chemical kinetics and fluid flow. In this regard, this paper exploresmulti-derivative methods with an off-step point and the grid point is chosen as to ensure $A(\alpha)$-stability property. Multi-derivative methods have advantage of bypassing theDahlquist order barrier [7] in linear multistep method (LMM); see [1, 2, 3, 6, 7, 10, 11, 12, 13, 14, 17-24, 26].Infact, methods of this nature, naturally promotes high order with larger stability region than the conventional LMM of comparable step numberk. Indeed, methods with these desirable properties are essential for stiff problems in (1).

A simple example is the third derivative LMM in [26].

Corresponding author: R. I. Okuonghae, E-mail: okunoghae01@yahoo.co.uk, Tel.: +2348032342321 & 8023325312 (MNOI)

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k}\alpha_j f_{n+j} + h^2\beta_k g_{n+k} + h^3\eta_k l_{n+k}; \quad p = k+4, \tag{2}$$

where,

$\{y'_{n+j}\}_{j=0}^{k} = \{f_{n+j}\}_{j=0}^{k} = \{f(x_{n+j}, y_{n+j})\}_{j=0}^{k}$, $g_{n+k} = y''_{n+k} = f'(x_{n+k}, y_{n+k})$, $l_{n+k} = y'''_{n+k} = f''(x_{n+k}, y_{n+k})$. In (2), k is the step number, h is the step length and $\{\alpha_j\}_{j=0}^{k}$; β_k, and η_k are the discrete coefficients of the arising methods, see [26]. This algorithm in (2) is an extension of the second derivative LMM (SDLMM) in [8]. When (2) is applied to the linear problem

$$y' = \lambda y, \ \mathrm{Re}(\lambda) < 0, \tag{3}$$

it yields a stability polynomial,

$$\pi(w,z) = w^k - w^{k-1} - z\sum_{j=0}^{k}\alpha_j w^j - z^2\beta_k w^k - z^3\eta_k w^k; \quad z = \lambda h.$$

The scheme in (2) is A-stable for $k \le 3$ and $A(\alpha)$-stable for $k = 4, 5$. The stability graphs are in Fig. 1.

Another example is the third derivative backward differentiation formula (TDBDF),

$$y_{n+k} = \sum_{j=0}^{k-1}\alpha_j y_{n+j} + h\beta_k f_{n+k} + h^2\gamma_k g_{n+k} + h^3\eta_k l_{n+k} \tag{4}$$

in [24]. This is an extended version of the second derivative backward differentiation formulas (SDBDF) , see [11]. The SDBDF is $A(\alpha)$-stable up to order $k = 10$. As for the above methods, various modifications now exist; see [2, 6, 8, 11, 12, 17-24]. In like manner, the stability polynomial of the TDBDF (4) [24] is

$$\pi(w,z) = w^k - \sum_{j=0}^{k-1}\alpha_j w^j - z\beta_k w^k - z^2 g_k w^k - z^3\eta_k w^k; \quad z = \lambda h.$$

The coefficients of the methods are given therein. The algorithm in (4) is A-stable for $k = 2(1)3$ and $A(\alpha)$-stable for $k = 1$, k = 4(1)9, see Fig. 2.

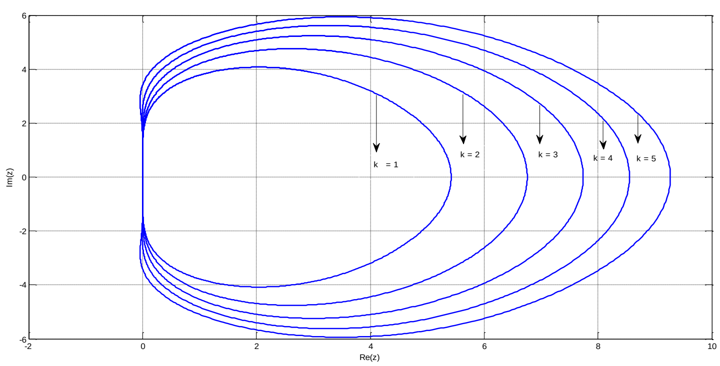

Fig. 1: The stability region of the method in (2) is the exterior of the closed curves.

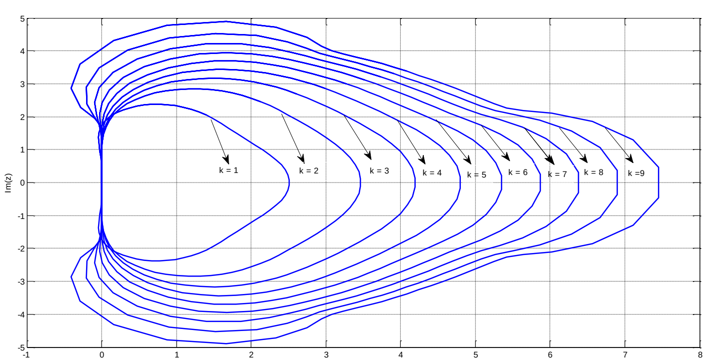

Fig. 2: The stability region of the method in (4) is the exterior of the closed curves.

The multi-derivative hybrid LMM for the solution of (1) of our interest is,

$$\begin{cases} y_{n+v} = \sum_{j=0}^{k} \Omega_j y_{n+j} + h\Phi_k f_{n+k} + h^2 \varphi_k g_{n+k}, \quad v = k - \frac{1}{2}, \quad q = k+2, & (5a) \\ y_{n+k} = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_v^{(1)} f_{n+v} + h^2 \beta_v^{(2)} g_{n+v} + h^3 \beta_k^{(3)} l_{n+k}, \quad p = k+2, & (5b) \end{cases}$$

where $\{y_{n+j}\}_{j=0}^{k}$, $y_{n+v}^{/} = f(x_{n+v}, y_{n+v})$, $g_{n+v} = y_{n+v}^{//} = f^{/}(x_{n+v}, y_{n+v})$, $l_{n+k} = y_{n+k}^{///} = f^{//}(x_{n+k}, y_{n+k})$,

$\{y_{n+j}^{/}\}_{j=0}^{k} = \{f_{n+j}\}_{j=0}^{k} = \{f(x_{n+j}, y_{n+j})\}_{j=0}^{k}$, $g_{n+k} = y_{n+k}^{//} = f^{/}(x_{n+k}, y_{n+k})$, $l_{n+k} = y_{n+k}^{///} = f^{//}(x_{n+k}, y_{n+k})$,

The hybrid method in (5),is based on the formalism of the TDBDF in (4). Replacing f_{n+k} and g_{n+k} in (4) with f_{n+v} and g_{n+v} yields (5b) while the predicted values of y_{n+v} in (5a) is used to obtain an approximation to the derivatives f_{n+v} and g_{n+v} at point x_{n+v} in the right-hand side of (5b) to yield the corrected value of y_{n+k}. The y_{n+k} denotes an approximation of the theoretical solution $y(x_{n+k})$ at the output point $x = x_{n+k}$ with a constant step-size h. For simplicity with no loss of generality assume $h > 0$ and that the method (5) is used to find $y(x)$ only when $x > x_0$. The scheme (5) is a subclass of the general linear method (GLM) in [5] which extension can be found in [2, 6, 19, 21].Applying the scheme in (5) to the linear problem in (3) yields

$$\Pi_k(w, z) = w^k - \sum_{j=0}^{k-1} \alpha_j w^j - z\beta_v^{(1)} \left(\sum_{j=0}^{k} \Omega_j w^j + z\Phi_k w^k + z^2 \varphi_k w^k \right) - z^2 \beta_v^{(2)} \left(\sum_{j=0}^{k} \Omega_j w^j + z\Phi_k w^k + z^2 \varphi_k w^k \right)$$
$$- z^3 \beta_k^{(3)} w^k, \quad v = k - \frac{1}{2}, \quad z = \lambda h, \quad w = e^{i\theta}, \quad \theta \in [0, 2\pi]. \qquad (6)$$

This is a polynomial of degree three in z which gives rise to three root locus curves which together, describe the stability domain of the hybrid method in (5). The corresponding stability characteristics are given in Table 3.
The error constants $C_{q+1}^{(1)}$, $C_{p+1}^{(2)}$ and the order $q = p = k+2$,arising from the hybrid scheme (5) are obtain from the local truncation error operator

$$L_1[y(x);\, h] = y(x_n + vh) - \sum_{j=0}^{k} \Omega_j y(x_n + jh) - h\Phi_k y'(x_n + kh) - h^2 \varphi_k y''(x_n + kh)$$
$$= C_{q+1} h^{q+1} y^{(q+1)}(x_n) + 0(h^{k+3}),$$
$$L_2[y(x);\, h] = y(x_n + kh) - \sum_{j=0}^{k-1} \alpha_j y(x_n + jh) - h\beta_v^{(1)} y'(x_n + vh) - h^2 \beta_v^{(2)} y''(x_n + vh) - h^3 \beta_k^{(3)} y'''(x_n + kh)$$
$$= C_{p+1} h^{k+2} y^{(k+2)}(x_n) + 0(h^{k+3}).$$

The paper is organized as follows. In section 2 is a systematic approach to constructing multi-derivative LMM. The way to do this follows for example the approach in [6, 15]. Section 3 gives the graphs of the stability region of the hybrid scheme in (5) with a varying step number and section 4 presents the results of our numerical experiments.

2.0 The Derivation of the Multi-Derivative Hybrid LMM (5)

Consider derivation of the hybrid predictor (5a) and the output scheme (5b). When for example $k = 1$ $(p = q = 3)$ in (5a), expanding the resulting expression in Taylor series about x_n and equating the coefficients of the powers of h yields,

$$\begin{cases} h^0: \quad -\Omega_0 - \Omega_1 + 1 = 0, \qquad h: \quad v - \phi_1 - \Omega_1 = 0, \\ h^2: \quad \frac{1}{2}(v^2 - 2\phi_1 - 2\Omega_1) = 0, \quad h^3: \quad \frac{1}{6}(v^3 - 3\phi_1 - 6\varphi_1 - \Omega_1) = 0. \end{cases} \qquad (7)$$

By this and with a choice of v ,

$$\Omega_0 = 1 - 3v + 3v^2 - v^3, \quad \Omega_1 = 3v - 3v^2 + v^3, \quad \phi_1 = -2v + 3v^2 - v^3, \quad \varphi_1 = \frac{v}{2} - v^2 + \frac{v^3}{2}. \qquad (8)$$

Substituting (8) into (5a) and setting $v = k - \frac{1}{2}$ gives

$$y_{n+\frac{1}{2}} = \frac{1}{8}\left(y_n + 7y_{n+1}\right) - \frac{3}{8}hf_{n+1} + \frac{1}{16}h^2 g_{n+1}, \qquad q = 3, \quad C_4^{(1)} = \frac{-1}{384}. \tag{9}$$

In a similar manner, the output scheme for $k = 1\ (p = 3)$ in (5b) is

$$y_{n+1} = y_n + hf_{n+\frac{1}{2}} + \frac{h^3}{24} l_{n+1}, \qquad p = 3, \qquad C_4^{(2)} = \frac{-1}{48}. \tag{10}$$

In general, the discrete coefficients of the methods in (5) for step number $k \le 14$, are in Tables 1 and 2. The composite scheme (9) and (10) produces a stable method as can be verified in Fig. 3.

Table 1: The coefficients of the hybrid predictor (5a), p=k+2

K	1	2	3	4	5	6	7	8	9	10	11
$\beta_k^{(4)}$	$\frac{1}{16}$	$\frac{3}{64}$	$\frac{5}{128}$	$\frac{35}{1024}$	$\frac{63}{2048}$	$\frac{231}{8192}$	$\frac{429}{16384}$	$\frac{6435}{262144}$	$\frac{12155}{524288}$	$\frac{46189}{2097152}$	$\frac{88179}{4194304}$
$\beta_k^{(3)}$	$\frac{-3}{8}$	$\frac{-21}{64}$	$\frac{-115}{384}$	$\frac{-1715}{6144}$	$\frac{-5397}{20480}$	$\frac{-20559}{81920}$	$\frac{-275847}{1146880}$	$\frac{-1700127}{7340032}$	$\frac{-29582839}{132120576}$	$\frac{-573713569}{2642411520}$	$\frac{-584295049}{2768240640}$
α_0	$\frac{1}{8}$	$\frac{-1}{128}$	$\frac{1}{576}$	$\frac{-5}{8192}$	$\frac{7}{25600}$	$\frac{-7}{49152}$	$\frac{33}{401408}$	$\frac{-429}{8388608}$	$\frac{715}{21233664}$	$\frac{-2431}{104857600}$	$\frac{4199}{253755392}$
α_1	$\frac{7}{8}$	$\frac{3}{16}$	$\frac{-5}{256}$	$\frac{7}{1152}$	$\frac{-45}{16384}$	$\frac{77}{51200}$	$\frac{-91}{98304}$	$\frac{495}{802816}$	$\frac{-7293}{16777216}$	$\frac{13585}{42467328}$	$\frac{-51051}{209715200}$
α_2		$\frac{105}{128}$	$\frac{15}{64}$	$\frac{-35}{1024}$	$\frac{7}{512}$	$\frac{-495}{65536}$	$\frac{1001}{204800}$	$\frac{-455}{131072}$	$\frac{8415}{3211264}$	$\frac{-138567}{67108864}$	$\frac{95095}{56623104}$
α_3			$\frac{1805}{2304}$	$\frac{35}{128}$	$\frac{-105}{2048}$	$\frac{77}{3072}$	$\frac{-2145}{131072}$	$\frac{1001}{81920}$	$\frac{-7735}{786432}$	$\frac{53295}{6422528}$	$\frac{-969969}{134217728}$
α_4				$\frac{55685}{73728}$	$\frac{315}{1024}$	$\frac{-1155}{16384}$	$\frac{1001}{24576}$	$\frac{-32175}{1048576}$	$\frac{17017}{655360}$	$\frac{-146965}{6291456}$	$\frac{159885}{7340032}$
α_5					$\frac{300013}{409600}$	$\frac{693}{2048}$	$\frac{-3003}{32768}$	$\frac{1001}{16384}$	$\frac{-109395}{2097152}$	$\frac{323323}{6553600}$	$\frac{-205751}{4194304}$
α_6						$\frac{3505733}{4915200}$	$\frac{3003}{8192}$	$\frac{-15015}{131072}$	$\frac{17017}{196608}$	$\frac{-692835}{8388608}$	$\frac{2263261}{26214400}$
α_7							$\frac{335572523}{481689600}$	$\frac{6435}{16384}$	$\frac{-36465}{262144}$	$\frac{46189}{393216}$	$\frac{-2078505}{16777216}$
α_8								$\frac{1401794537}{2055208960}$	$\frac{109395}{262144}$	$\frac{-692835}{4194304}$	$\frac{323323}{2097152}$
α_9									$\frac{222757759081}{332943851520}$	$\frac{230945}{524288}$	$\frac{-1616615}{8388608}$
α_{10}										$\frac{4376973241927}{6658877030400}$	$\frac{969969}{2097152}$
α_{11}											$\frac{49619129184677}{76735630540800}$

Continuation of Table 1

K	12	13	14
$\beta_k^{(4)}$	$\frac{676039}{33554432}$	$\frac{1300075}{67108864}$	$\frac{5014575}{268435456}$
$\beta_k^{(3)}$	$\frac{-13661878997}{66437775360}$	$\frac{-69339054385}{345476431872}$	$\frac{-210885500055}{1074815565824}$
α_0	$\frac{-29393}{2415919104}$	$\frac{52003}{5670699008}$	$\frac{-185725}{26306674688}$
α_1	$\frac{96577}{507510784}$	$\frac{-734825}{4831838208}$	$\frac{1404081}{11341398016}$
α_2	$\frac{-1174173}{838860800}$	$\frac{2414425}{2030043136}$	$\frac{-2204475}{2147483648}$
α_3	$\frac{2187185}{339738624}$	$\frac{-391391}{67108864}$	$\frac{21729825}{4060086272}$
α_4	$\frac{-22309287}{1073741824}$	$\frac{54679625}{2717908992}$	$\frac{-10567557}{536870912}$
α_5	$\frac{735471}{14680064}$	$\frac{-111546435}{2147483648}$	$\frac{10935925}{201326592}$
α_6	$\frac{-4732273}{50331648}$	$\frac{6128925}{58720256}$	$\frac{-1003917915}{8589934592}$
α_7	$\frac{7436429}{52428800}$	$\frac{-16900975}{100663296}$	$\frac{165480975}{822083584}$
α_8	$\frac{-47805615}{268435456}$	$\frac{7436429}{33554432}$	$\frac{-152108775}{536870912}$
α_9	$\frac{7436429}{37748736}$	$\frac{-132793375}{536870912}$	$\frac{22309287}{67108864}$
α_{10}	$\frac{-7436429}{33554432}$	$\frac{37182145}{150994944}$	$\frac{-717084225}{2147483648}$
α_{11}	$\frac{2028117}{4194304}$	$\frac{-16900975}{67108864}$	$\frac{10140585}{33554432}$
α_{12}	$\frac{1172798911730641}{1841655132979200}$	$\frac{16900975}{33554432}$	$\frac{-152108775}{536870912}$
α_{13}		$\frac{15630801570008773}{24899177397878784}$	$\frac{35102025}{67108864}$
α_{14}			$\frac{2284668726871879}{3688767021907968}$

Table 2: The Coefficients of the Multi-derivative LMM (5b), p=k+2

k	1	2	3	4	5
γ_k	$\frac{1}{24}$	$\frac{5}{276}$	$\frac{3520}{3821}$	$\frac{28982}{3156475}$	$\frac{2781980}{365587289}$
$\beta_v^{(2)}$	-	$\frac{1}{23}$	$\frac{268}{3821}$	$\frac{56736}{631295}$	$\frac{38665280}{365587289}$
$\beta_v^{(1)}$	1	$\frac{22}{23}$	$\frac{137}{11463}$	$\frac{561984}{631295}$	$\frac{945468160}{1096761867}$
α_0	1	$\frac{-1}{23}$	$\frac{33}{3821}$	$\frac{9133}{3156475}$	$\frac{4175971}{3290285601}$
α_1		$\frac{24}{23}$	$\frac{-367}{3821}$	$\frac{90274}{3156475}$	$\frac{-4638825}{365587289}$
α_2			$\frac{4155}{3821}$	$\frac{-499704}{3156475}$	$\frac{22898450}{365587289}$
α_3				$\frac{3575038}{3156475}$	$\frac{-757508830}{3290285601}$
α_4					$\frac{431031315}{365587289}$
α_5					
α_6					
α_7					
α_8					

k	6	7	8	9
γ_k	$\frac{533685720}{80952008767}$	$\frac{662656620360}{112742462210171}$	$\frac{5533461018960}{1035223276257953}$	$\frac{1068294918974040}{216616761138439597}$
$\beta_v^{(2)}$	$\frac{1378337280}{11564572681}$	$\frac{14756253012480}{112742462210171}$	$\frac{146279430406144}{1035223276257953}$	$\frac{32650312419680256}{216616761138439597}$
$\beta_v^{(1)}$	$\frac{9668208640}{11564572681}$	$\frac{91507051274240}{112742462210171}$	$\frac{4082025914925056}{5176116381289765}$	$\frac{830443336390508544}{1083083805692197985}$
α_0	$\frac{-53114041}{80952008767}$	$\frac{42705586150}{112742462210171}$	$\frac{-55135339493671}{232925237158039425}$	$\frac{476104089662525}{303263465593815435 8}$
α_1	$\frac{1677137536}{242856026301}$	$\frac{-479711835533}{112742462210171}$	$\frac{2948978382664}{1035223276257953}$	$\frac{-76716876404856042}{37907933199226929475}$
α_2	$\frac{-2793258285}{80952008767}$	$\frac{7572137187818}{338227386630513}$	$\frac{-16561165117994}{1035223276257953}$	$\frac{2637677288170386}{216616761138439597}$
α_3	$\frac{9182827520}{80952008767}$	$\frac{-8404344884705}{112742462210171}$	$\frac{522737862410792}{9317009486321577}$	$\frac{-9874428158400681}{216616761138439597}$
α_4	$\frac{-75693124945}{242856026301}$	$\frac{20704452765010}{112742462210171}$	$\frac{-145001616226040}{1035223276257953}$	$\frac{25969438719590337}{216616761138439597}$
α_5	$\frac{99287549376}{80952008767}$	$\frac{136155417204485}{338227386630513}$	$\frac{7139363593261144}{25880581906448825}$	$\frac{-51852751583872185}{216616761138439597}$
α_6		$\frac{143740453918138}{112742462210171}$	$\frac{-4684166945440486}{9317009486321577}$	$\frac{2126308578136021506}{5415419028460989925}$
α_7			$\frac{1370888590667704}{1035223276257953}$	$\frac{-928267113190173597}{1516317327969077179}$
α_8				$\frac{4167778216868225469}{3032634655938154358}$

Continuation of Table2

K	10	11	12	13	14
γ_k	$\frac{2795035290258138480 0}{6075834388554584988829}$	$\frac{634706031083800737 60}{1466592090368740588077 1}$	$\frac{691348269213490753972800}{168653982416502671589645833}$	$\frac{46162169127620846907647040}{1182335514125570564119305 1009}$	$\frac{601157281265711669811 31200}{1609168792058097965545512 3091}$
$\beta_v^{(2)}$	$\frac{8802345494599630848 0}{5523485807776895444 39}$	$\frac{2454289600152858918912}{1466592090368740588077 1}$	$\frac{2267589728260052075151360}{12973383262807897814588141}$	$\frac{10745532414493501296042049536}{59116775706278528205965255045}$	$\frac{3030877143325097210847166464}{16091687920580979655455123091}$
$\beta_v^{(1)}$	$\frac{41195206241080993382 4}{55234858077768954443 9}$	$\frac{3130472408772561076224}{4313506148143354670815}$	$\frac{64150186228792302193606656}{90813682839655284702116987}$	$\frac{1422922070417683189417163882496}{2069087149719748487208783926575}$	$\frac{161626194542763955618469380096}{241375318808714694831826846365}$
α_0	$\frac{-662638463001283149}{6075834388554584988829}$	$\frac{5765452993123809276}{7332960451843702940385 5}$	$\frac{-9853584754820756860887}{168653982416502671589645833}$	$\frac{658034728430088003592323 36}{1477919392656963205149131376125}$	$\frac{-885882220696145457549644}{25557386697393320629252254321}$
α_1	$\frac{9132570530844687460}{6075834388554584988829}$	$\frac{-16947081576000039915}{14665920903687405880771}$	$\frac{7545953448668123102425088}{8264045138408630907892645817}$	$\frac{-436213498861578724846989 39}{59116775706278528205965255045}$	$\frac{244385439556174238340329304}{4022921980145244913863780772 75}$
α_2	$\frac{-294305041208669194161}{303791719427729249441 45}$	$\frac{1167812166721216413 76}{14665920903687405880771}$	$\frac{-1131656622086543439608568}{168653982416502671589645833}$	$\frac{16702680153842737136561216608}{2896722009607647882092297497205}$	$\frac{-81001688305407690900463293}{16091687920580979655455123091}$
α_3	$\frac{236093993110975533840}{6075834388554584988829}$	$\frac{-1254426033958321785986 7}{3666480225921851470192 75}$	$\frac{5198703248319165439099520}{168653982416502671589645833}$	$\frac{-333981022337837401089045912}{1182335514125570564119305 1009}$	$\frac{886157846751264145212361105 6}{3379254463322005727645575849 11}$
α_4	$\frac{-662835831721633212510}{6075834388554584988829}$	$\frac{1509410463751992566952}{14665920903687405880771}$	$\frac{-83762467278700025975533671}{843269912082513357948229165}$	$\frac{1150690415229684899776718512}{1182335514125570564119305 1009}$	$\frac{-1550429798521884737248387596}{16091687920580979655455123091}$
α_5	$\frac{1394430012011705083128}{6075834388554584988829}$	$\frac{-16949770682863540021962}{73329604518437029403855}$	$\frac{40314282445626059977818624}{168653982416502671589645833}$	$\frac{-370796878872202279165094116119}{1477919392656963205149131376125}$	$\frac{427341590082200920637439728 8}{16091687920580979655455123091}$
α_6	$\frac{-231970432788695236671 0}{6075834388554584988829}$	$\frac{5942412809451214777296}{14665920903687405880771}$	$\frac{-75447439381239354998623824}{168653982416502671589645833}$	$\frac{29742965826861549504543732288}{59116775706278528205965255045}$	$\frac{-229507201627873962246618365201}{402292198014524491386378077275}$
α_7	$\frac{1629904606326423801153 6}{3037917194277292494414 5}$	$\frac{-8471812126556714642586}{14665920903687405880771}$	$\frac{113353233176559513139090176}{168653982416502671589645833}$	$\frac{-2806457974569476155031003472}{3477457394486972247409720885}$	$\frac{15779383465886159384025405216}{16091687920580979655455123091}$
α_8	$\frac{-44400417521726303737 65}{6075834388554584988829}$	$\frac{26032198810865675758849 2}{36664802259218514701927 5}$	$\frac{-6927706427723598031127899605}{8264045138408630907892645817}$	$\frac{12543204219497030359143857136}{1182335514125570564119305 1009}$	$\frac{-1302747170664097278260456316}{946569877681234097379713123}$
α_9	$\frac{8658474158734163157060}{6075834388554584988829}$	$\frac{-12589844730432288779427}{14665920903687405880771}$	$\frac{772078026486603543208567296}{843269912082513357948229165}$	$\frac{-29623259830930518177452724411}{25188887040066503322541717367}$	$\frac{698666954923529978721420709624}{434475573855686450697288323457}$
α_{10}		$\frac{10826806143794111005233 6}{73329604518437029403855}$	$\frac{-16783413511749406637286 5912}{168653982416502671589645833}$	$\frac{17080265325718548135762508294 08}{1477919392656963205149131376125}$	$\frac{-180701426871105294584618089473}{112641815444066857588185861637}$
α_{11}			$\frac{257775501191113033686122112}{168653982416502671589645833}$	$\frac{-1105587010456922765003488984}{969127470594729970589594345}$	$\frac{576408641000229729423520060272}{402292198014524491386378077275}$
α_{12}				$\frac{93459744216689183413568681232}{59116775706278528205965255045}$	$\frac{-625353447071479613194321508 44}{4827506376174293896636536927 3}$
α_{13}					$\frac{26292817369524410435272538136}{16091687920580979655455123091}$

3.0 The Stability of the Multi-Derivative Hybrid LMM (5)

The stability of the hybrid algorithms in (5) are of consideration here. Consider for example, a case when $k=1,\ (p=q=3)$ in (5) and substituting the coefficients of the methods in (5a) and (5b) from Tables 1 and 2 respectively into (6) gives,

$$\pi_1(w,z)=w-1-z\left(\frac{1}{8}+\frac{7w}{8}-\frac{3wz}{8}+\frac{wz^2}{8}\right)-\frac{wz^3}{24}. \qquad (11)$$

Now, for $k=2,\ (p=q=4)$, we have

$$\pi_2(w,z)=w^2-\frac{24w}{23}+\frac{1}{23}-\frac{22}{23}z\left(-\frac{1}{128}+\frac{3w}{16}+\frac{105w^2}{128}-\frac{21w^2z}{64}+\frac{3w^2z^2}{64}\right)$$
$$-\frac{1}{23}z^2\left(-\frac{1}{128}+\frac{3w}{16}+\frac{105w^2}{128}-\frac{21w^2z}{64}+\frac{3w^2z^2}{64}\right)-\frac{5w^2z^3}{276}. \qquad (12)$$

In a similar manner we obtain the stability polynomial of (5) for step number $k=3(1)14.$ The computing power available have been constrained by machine limitation to stop investigation of the methods at $k=14$, However, it is conjectured the existence of stable methods with step number $k\geq 15$. The Fig. 3presents the computed stability region (exterior of the closed curve) of the methods in (5). The boundary locus of $\pi_k(w,z)=0$ shows that the method in (5) is $A(86^0)-$stable for $k=1$, $A(90^0)-$stable for $k=2(1)5$ and $A(\alpha)-$stable for $k=6(1)14$. Table 3 gives the stability characteristics of the TDLMM (2), TDBDF (4) and the multi-derivative hybrid LMM in (5).

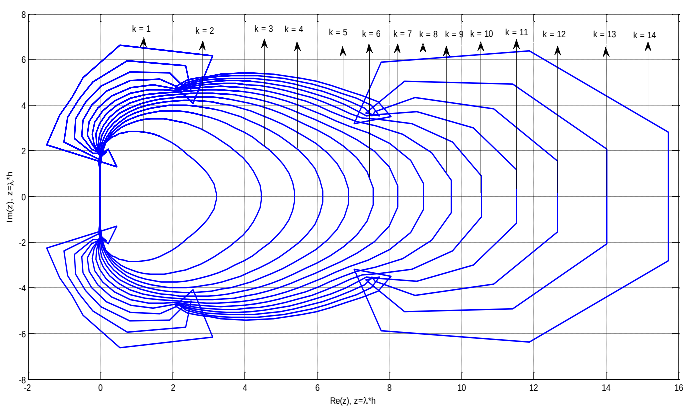

Fig. 3: The stability region (exterior of the closed curve) of the multi-derivative hybrid LMM (5).

Table 3: Stability characteristics and error constants of the TDMM (2), TDBDF (4) and MHLMM (5)

k	1	2	3	4	5	6	7	8	9
TDMM(2)[26]									
p	4	5	6	7	8	-	-	-	-
$C_{p+1}^{(k)}$	$\frac{-1}{480}$	$\frac{-1}{1800}$	$\frac{-11}{50400}$	$\frac{-89}{846720}$	$\frac{-5849}{101606400}$	-	-	-	-
α	90^0	90^0	90^0	89.86^0	89.1^0	-	-	-	-
TDBDF(4)[24]									
p	3	4	5	6	7	8	9	10	11
$C_{p+1}^{(k)}$	$\frac{-1}{24}$	$\frac{-2}{225}$	$\frac{-9}{2875}$	$\frac{-288}{204575}$	$\frac{-4500}{6123971}$	$\frac{-1000}{2356067}$	$\frac{-34300}{129973303}$	$\frac{-2195200}{12648444479}$	$\frac{-133358400}{1117849207079}$
α	87^0	90^0	90^0	90^0	87.5^0	88^0	86^0	83^0	79^0
MDLMM(5)									
q	3	4	5	6	7	8	9	10	11
p	3	4	5	6	7	8	9	10	11
$C_{q+1}^{(k)}$	$\frac{-1}{384}$	$\frac{-1}{1280}$	$\frac{-1}{3072}$	$\frac{-1}{6144}$	$\frac{-3}{32768}$	$\frac{-11}{196608}$	$\frac{-143}{3932160}$	$\frac{-13}{524288}$	$\frac{-221}{12582912}$
$C_{p+1}^{(k)}$	$\frac{-1}{48}$	$\frac{-33}{7360}$	$\frac{-9133}{5502240}$	$\frac{-4175971}{5302878000}$	$\frac{-53114041}{122837329104}$	$\frac{-21352793075}{81599624837136}$	$\frac{-55135339493671}{324698291165292480}$	$\frac{-95220817932505}{819896834796298776}$	$\frac{-220879487667094383}{2668718497225575835040}$
α	86^0	90^0	90^0	90^0	90^0	90^0	89^0	86^0	86^0

k	10	11	12	13	14
MDLMM(5)					
q	12	13	14	15	16
p	12	13	14	15	16
$C_{q+1}^{(k)}$	$\frac{-323}{25165824}$	$\frac{-323}{33554432}$	$\frac{-7429}{1006632960}$	$\frac{-37145}{6442450944}$	$\frac{-19665}{4294967296}$
$C_{p+1}^{(k)}$	$\frac{-211880397497299990893}{3475377270253222613610188}$	$\frac{-9853584754820756860887}{213535808357688629624025760}$	$\frac{-2741811368458700014968014}{76737561999508715573288854015}$	$\frac{-6865587210395127296009741}{243422017614088057318680461950}$	$\frac{-990795223332818617781741389}{43769391143980264662837934807520}$
α	84^0	77^0	74^0	70^0	54^0

The α in Table 3 implies the angle of absolute stability of the method.

- Although, the error constants of the MDLMM (5) are larger in size than that of the TDLMM (2), and the TDBDF (4), but the MDLMM presents more A(α)-stable methods of higher order than the TDLMM (2), and TDBDF (4),
- Again, the MDLMM (5) have more A-stable methods than the TDLMM and TDBDF methods. These serves as an advantage over the TDLMM [26] and TDBDF [24] methods in variable order implementation, see Table 3.

4.0 Implementation of the Methods (5): Numerical Experiment and Conclusion

Consider the application of the hybrid LMM (5) on some stiff problems. The results obtained are compared with methods of TDLMM (2) and TDBDF (4) with same order. Thus are the methods

(i). TDLMM (2)[26],

$$y_{n+1} = y_n + \frac{h}{4}(f_n + 3f_{n+1}) - \frac{h^2}{4} g_{n+1} + \frac{h^3}{24} l_{n+1}, \quad p = 4,$$

(ii). TDBDF (4)[24],

$$y_{n+2} = \frac{1}{15}(-y_n + 16y_{n+1}) + \frac{14}{15} hf_{n+2} - \frac{2}{5} h^2 g_{n+2} + \frac{4}{45} h^3 l_{n+2}, \quad p = 4,$$

(iii). MDLMM (5),

$$\begin{cases} y_{n+\frac{3}{2}} = \frac{1}{128}(-y_n + 24y_{n+1} + 105y_{n+2}) - \frac{21}{64} hf_{n+2} + \frac{3}{64} h^2 g_{n+2}, & q = 4, \\ y_{n+2} = \frac{1}{23}(-y_n + 24y_{n+1}) + \frac{22}{23} hf_{n+\frac{3}{2}} + \frac{1}{23} h^2 g_{n+\frac{3}{2}} + \frac{5}{276} h^3 l_{n+2}, & p = 4, \end{cases}$$

for a fixed step size implementation. The following stiff systems of initial value problems have been solved:

Problem 1: [21]

$$\begin{cases} y_1' = -8y_1 + 7y_2, \\ y_2' = 42y_1 - 43y_2, \end{cases} \quad y(0) = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \ x \in [0,1], \begin{cases} y_1(x) = 2e^{-x} - e^{-50x}, \\ y_2(x) = 2e^{-x} + 6e^{-50x}. \end{cases}$$

Problem 2: [11]

$$y' = \xi(y - \sin(x)) + \cos(x), \quad y(0) = 0, \quad y(x) = \sin(x), \ x \in [0, 1.56], \quad \xi = -10^4$$

Problem 3: [15]

$$\begin{cases} y_1' = -0.1y_1 + 199.9y_2, \\ y_2' = -200y_2, \end{cases} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x \in [0,1], \quad \begin{cases} y_1(x) = e^{-0.1x} + e^{-200x}, \\ y_2(x) = e^{-200x}, \end{cases}$$

If $\beta_k^{(3)} \neq 0$ in (5b) and the algorithm is applied on the initial value problem (IVPs (1)), the arising systems of nonlinear algebraic equations are resolved by the Newton-Raphson iterative scheme,

$$y_{n+k}^{[s+1]} - y_{n+k}^{[s]} = -\left(F'\left(y_{n+k}^{[s]}\right)\right)^{-1} F\left(y_{n+k}^{[s]}\right),$$

with $F'\left(y_{n+k}^{[s]}\right)^{-1}$ as the Jacobian matrix form,

$$\begin{cases} F\left(y_{n+k}^{[s]}\right) = y_{n+k}^{[s]} - \sum_{j=0}^{k-1} \alpha_j y_{n+j} - h\beta_v^{(1)} f\left(x_{n+v}^{[s]}, y_{n+v}^{[s]}\right) - h^2 \beta_v^{(2)} g\left(x_{n+v}^{[s]}, y_{n+v}^{[s]}\right) - h^3 \beta_k^{(3)} l\left(x_{n+k}^{[s]}, y_{n+k}^{[s]}\right) = 0, \ v = k - \frac{1}{2}, & (13a) \\ y_{n+v}^{[s]} = \Omega_k y_{n+k}^{[s]} + \sum_{j=0}^{k-1} \Omega_j y_{n+j} + h\Phi_k f\left(x_{n+k}^{[s]}, y_{n+k}^{[s]}\right) + h^2 \varphi_k g\left(x_{n+k}^{[s]}, y_{n+k}^{[s]}\right). & (13b) \end{cases}$$

The starter for the iterative scheme (13a) is the order $p = 3$ explicit third derivative Runge-Kutta methods in [5]. For stiff problems (1), h is constrained to be chosen small, for the sequence in (13a) will to converge. The step size $h = 0.0001$ has been adopted. The maximum absolute errors $E = \max\left\{\left|{}^1y(x_n) - {}^1y_n\right|, \left|{}^2y(x_n) - {}^2y_n\right|\right\}$ generated by the various methods at the end point of the interval are in Table 4. Find that the same order methods produced results of the same accuracy on problems 1-3.

Table 4: AbsoluteErrors: $E = \|y(x_n) - y_n\|_\infty, \quad p = 3$

Problem	x_n	TDLMM (2)	TDBDF (4)	MDLMM (5)
1	1.0	1.2643(- 05)	4.6528(- 05)	4.6528(- 05)
2	1.56	1.0887(- 06)	1.0815(- 06)	1.0815(- 06)
3	1.0	2.2787(-06)	8.6099(- 06)	8.6099(- 06)

The numerical results in Table 4 of theMDLMM (5) compares with the TDBDF methods in (4) and TDMM in (2) when applied to the stiff problems 1-3. Conclusively, this paper considered a hybrid TDBDFfor the direct solution of stiff IVPs in ODEs (1). The stability of the MDLMM (5) shows that the methods are stable for $k \leq 14$. The encouraging numerical results of Table 4 show that the MDLMM (5) is suitable for stiff ODEs (1). The methods on the problems 1-3 are of comparable performance with similar methods from [24] and [26] in (4) and (2) respectively.

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