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A New Family of Multivariate Semi-Pareto Distributions and its Characterizations Properties

[1]S.U Gulumbe and [2]S.M. Umar

[1]Department of Mathematics, Usmanu Danfodio University, P.M.B 2346, Sokoto.
[2]Department of Mathematical Sciences, Bayero University, P.M.B. 3011, Kano.

Abstract

This paper proposed new family of multivariate semi-Pareto distribution. It then gives the general solutions of Euler's functional equation for homogeneous functions in the form of two lemmas the proofs of which are presented. Particular solution of the same equation, when $\ell = 1$, is also stated in the form of corollary that is also proved in this paper. Two theorems, one for each characterization property of this new family of distributions, are developed. These theorems are then proved through the method of geometric minimization procedure using general and particular solutions of Euler's functional equation for homogeneous functions. Simulated data is used by means of Mat lab programming language to fit this new family of distributions. The result obtained is then compared with the previous one given in the literature.

Keywords: Multivariate semi-Pareto distribution; Joint survival function; Geometric minimization procedure; Euler`s functional equation.

1.0 Introduction

Multivariate semi-Pareto distribution denoted by $MSP^{(k)}\left(\underline{\sigma}, \underline{\alpha}, p\right)$, is one of the most important distributions that can be applied in daily socio-economics activities. It fits to the upper tails of some multivariate continuous income data as well as some other socio-economic multivariate variables.

Apostol (1985) discussed the characterization of Marshall-Olkin multivariate exponential distribution using an integrated lack of memory equation. Euler (1925) stated the Euler`s functional equations and presented its general solutions in the form of Lemmas which are also available in Aczel (1966). Castillo and Ruiz-Cobo (1992) studied the general and particular solutions of Euler`s functional equations of homogeneous functions.

The univariate semi-Pareto distribution was first introduced by Pillai (1991). Some special bivariate semi-Pareto distribution with homogeneous scale parameters were studied by Balakrishna and Jayakumar (1997). Characterization property of the bivariate semi-Pareto distribution for homogeneous scales parameters $\sigma_i \equiv 1,\ i = 1, 2, ..., k,$ can also be found in Balakrishna and Jayakumar (1997).Yeh (2004) had studied the characterization of multivariate Pareto (III), $MP^{(n)}$ (III) already discussed by Arnold (1983), through the geometric minimization procedures. All these results were extended to the multivariate semi-Pareto ($MSP^{(k)}$) distribution by Yeh (2007). Characterizations properties of multivariate semi-Pareto distribution, using the method of geometric minimization procedures, were also studied by Yeh (2007). The proofs of these characterizations properties (given in Yeh 2007) were based on the general and the particular solutions of the Euler's functional equations of $k \geq 1$ variables.

Thomas and Jose (2002) introduced and studied the univariate Marshall-Olkin Pareto processes. Recently, Thomas and Jose (2004) developed a new family of distributions that were earlier studied by Marshall-Olkin (1997) which is similar to those of Pillai et al (1995). In that research work, also, they developed the Marshall-Olkin bivariate semi-Pareto distribution as a generalization of the bivariate semi-Pareto distribution of Balakrishna and Jayakumar (1997). Its characterization property is

Corresponding author: S.U Gulumbe, E-mail: surajoumar@yahoo.com, Tel.: +2347030608737

also studied in that paper. Miroslav et al (2008) studied a Marshall-Olkin form of bivariate exponential distribution. In that paper, they went ahead to estimate the unknown parameters and also studied the asymptotic properties of the estimated parameters. Umar (2009a) gives modified multivariate semi logistics distribution and its minification processes. Similarly, Umar (2009b) presented minification processes models of multivariate semi-Pareto. Characterizations properties of this family of distribution are also studies in that paper. Umar (2010a) developed characterizations properties of generalized multivariate Pareto (III). Minifications processes of this distribution are also presented. Umar (2010b) proposed a new method of parameter addition to a family of bivariate exponential and Weibull distributions.

In this paper, we propose new family of multivariate semi-Pareto distribution. Tools used in getting the characterizations properties of this family of distributions are presented. Characterizations properties of this new family of multivariate semi-Pareto distribution are then presented and proved using general and particular solutions of Euler`s functional equations of homogeneous functions through the method of geometric minimization procedures. Mat lab programming language is employed to simulate the data and then use it to fit the proposed new family of distributions.

2.0 New Family of Multivariate Semi-Pareto Distribution

According to Yeh (2007), a more general class of k-variate $(k \geq 2)$ semi-Pareto distributions than those proposed by Balakrishna and Jayakumar (1997) as well as Thomas and Jose (2002, 2004) is that of multivariate semi-Pareto distributions. This class of distributions is denoted by $MSP^{(k)}(\underline{\sigma}, \underline{\alpha}, p)$ .Yeh (2007) defined this family of distributions as follows:

Definition 2.1

A random vector $\underline{X} = (X_1, X_2, ..., X_k)$ is said to have a k-variate semi-Pareto distribution, denoted by $\underline{X} \sim MSP^{(k)}(\underline{\alpha}, \underline{\sigma}, p)$, with parameters $p \in (0,1)$, $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k) > \underline{0}$ and the scale parameter $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_k) > \underline{0}$, if its survival function is of the form

$$\overline{G}_{\underline{X}}(\underline{X}) = \frac{1}{1 + \Psi(x_1, x_2, ..., x_k)} \qquad x_i > 0, \quad for \;\; i = 1, 2, ..., k \;\; and \;\; k \geq 2,$$

such that

$$\Psi(x_1, x_2, ..., x_k) = \frac{1}{p} . \Psi\left(p^{1/\alpha_1} x_1, p^{1/\alpha_2} x_2, ..., p^{1/\alpha_k} x_k\right). \qquad \ldots(2.1)$$

Now, based on the above definition we have the following definitions.

Definition 2.2

Let $\underline{X} = (X_1, X_2, ..., X_k)$ be a k-variate random vector with joint survival function as given in definition 2.1. The new family of multivariate distributions is given in the form:

$$\overline{F}_{\underline{X}}(\underline{X}) = \frac{\beta \overline{G}_{\underline{X}}(\underline{X})}{1 - (1-\beta)\overline{G}_{\underline{X}}(\underline{X})}, \qquad \underline{X} = (x_1, x_2, ... x_k) \geq 0, \qquad 0 < \beta < 1. \qquad \ldots(2.2)$$

Definition 2.3

Consider definitions (2.1) and (2.2), the proposed new family of multivariate semi-Pareto distribution has the survival function given as:

$$\overline{F}_{\underline{X}}(\underline{X}) = \frac{\beta \overline{G}_{\underline{X}}(\underline{X})}{1 - (1-\beta)\overline{G}_{\underline{X}}(\underline{X})}, \qquad \underline{X} \geq 0, \qquad 0 < \beta < 1.$$

$$= \frac{\beta \left\{1 + \Psi(x_1, x_2, ..., x_k)\right\}^{-1}}{1 - (1-\beta)\left\{1 + \Psi(x_1, x_2, ..., x_k)\right\}^{-1}},$$

$$= \frac{\beta}{\beta + \Psi(x_1, x_2, ..., x_k)}, \qquad x_i > 0, \;\; for \quad i = 1, 2, ..., k; \; (k \geq 2) \;\; and \quad 0 < \beta < 1, \qquad \ldots(2.3)$$

such that

$$\Psi(x_1, x_2, ..., x_k) = \frac{1}{p} . \Psi\left(p^{1/\alpha_1} x_1, p^{1/\alpha_2} x_2, ..., p^{1/\alpha_k} x_k\right), \qquad \ldots(2.4)$$

for shape parameters $p \in (0,1)$ *and* $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k) > \underline{0}$ as well as the scale parameter $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_k) > \underline{0}$.

3.0 Characterizations Properties

Before we give the characterizations properties of this new family of distributions, two lemmas concerning the general solutions of Euler`s functional equations for homogeneous functions as well as a corollary concerning a particular of Euler`s functional equations for homogeneous functions will be presented and proved. These lemmas and corollary are available in Yeh (2007).

Lemma 3.1: The Euler`s functional equation of $k(\geq 2)$ variables

$$F(px_1, px_2, ..., px_k) = p^{\alpha} F(x_1, x_2, ..., x_k), \quad \alpha \neq 0,\ p \neq 0 \quad and \quad x_1 \neq 0, \qquad \ldots(3.1)$$

has its general solution in the form

$$F(x_1, x_2, ..., x_k) = x_1^{\alpha} \cdot f\left(\tfrac{x_2}{x_1}, \tfrac{x_3}{x_1}, ..., \tfrac{x_k}{x_1}\right), \qquad \ldots(3.2)$$

where $f(\bullet)$ is an arbitrary function of $(k-1)$ variables in terms of $\left\{\tfrac{x_2}{x_1}, \tfrac{x_3}{x_1}, ..., \tfrac{x_k}{x_1}\right\}$.

Proof:

The $k-$variate $(k \geq 2)$ Euler`s functional equation is solved as

$$F(x_1, x_2, ..., x_k) = F\left(x_1 \cdot, x_1 \cdot \tfrac{x_2}{x_1}, ..., x_1 \cdot \tfrac{x_k}{x_1}\right) = x_1^{\alpha} \cdot F\left(1, \tfrac{x_2}{x_1}, ..., \tfrac{x_k}{x_1}\right) \triangleq x_1^{\alpha} \cdot f\left(\tfrac{x_2}{x_1}, ..., \tfrac{x_k}{x_1}\right), \qquad ...(3.3)$$

where $f(\bullet)$ is an arbitrary function of $(k-1)$ variables in terms of $\left\{\frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_k}{x_1}\right\}$ that satisfy equation (3.1) for $\alpha \neq 0$ *and* $x_1 \neq 0$. Therefore, equation (3.2) is the general solution of equation (3.1) and thus this lemma follows.

Lemma (3.1) above, can be extended to the almost homogeneous functions and stated as follows:

Lemma 3.2: The Euler`s functional equation of $k(\geq 2)$ variables for almost homogeneous functions given by:

$$F\left(p^{\ell_1} x_1, p^{\ell_2} x_2, ..., p^{\ell_k} x_k\right) = p^{\ell} F(x_1, x_2, ..., x_k), \ \ell \neq 0 \ \ and \ \ \ell_i \neq 0,\ i = 1, 2, ...k, \qquad \ldots(3.4)$$

has its general solution in the form:

$$F(x_1, x_2, ..., x_k) = x_1^{\ell/\ell_1} \cdot f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, ..., \frac{x_k}{x_1^{\ell_k/\ell_1}}\right), \qquad \ldots(3.5)$$

where $f(\bullet)$ is an arbitrary function of $(k-1)$ positive variables in terms of $\left\{\frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_k}{x_1}\right\}$, and $\ell \neq 0,\ \ell_i \neq 0$ *and* $x_i > 0,\ i = 1, 2, ...k.$

In this paper we consider the case of only positive real values of $p > 0$ *and* $x_i > 0,\ i = 1, 2, ..., k$.

Proof:

Let us take $p = x_1^{-1/\ell_1}$ and substitute in equation (3.4). Then, this equation becomes

$$F\left(\left(x_1^{-1/\ell_1}\right)^{\ell_1} x_1, \left(x_1^{-1/\ell_1}\right)^{\ell_2} x_2, ..., \left(x_1^{-1/\ell_1}\right)^{\ell_k} x_k\right) = \left(x_1^{-1/\ell_1}\right)^{\ell} F(x_1, x_2, ..., x_k).$$

This implies that $F\left(1, \frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, ..., \frac{x_k}{x_1^{\ell_k/\ell_1}}\right) = x_1^{-\ell/\ell_1} \cdot F(x_1, x_2, ..., x_k).$

Therefore, a direct and general solution of Euler`s functional equation given by equation (3.4) is

$$F(x_1, x_2, ..., x_k) = x_1^{\ell/\ell_1} F\left(1, \frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, ..., \frac{x_k}{x_1^{\ell_k/\ell_1}}\right)$$

$$= x_1^{\ell/\ell_1} \cdot f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, ..., \frac{x_k}{x_1^{\ell_k/\ell_1}}\right),$$

where $f(\bullet)$ is an arbitrary function of $(k-1)$ positive variables in terms of $\left\{\frac{x_2}{x_1}, \frac{x_3}{x_1}, ..., \frac{x_k}{x_1}\right\}$, and $\ell \neq 0,\ \ell_i \neq 0\ \ and\ \ x_i > 0,\ \ i = 1, 2, ...k.$ This establish the proof of this lemma.

The corollary below gives the particular solution of Euler's functional equation of almost homogeneous functions given in equation (3.4) when $\ell = 1.$

Corollary 3.1: The particular solution of Euler's functional equation of the form

$$F\left(p^{\ell_1}x_1, p^{\ell_2}x_2, ..., p^{\ell_k}x_k\right) = pF\left(x_1, x_2, ..., x_k\right) \qquad \ldots (3.6)$$

is either

(i) $$F\left(x_1, x_2, ..., x_k\right) = \sum_{i=1}^{k} x_i^{1/\ell_i} \qquad \ldots (3.7)$$

or

(ii) $$F\left(x_1, x_2, ..., x_k\right) = \sum_{i=1}^{k} \left(\frac{x_i}{\sigma_i}\right)^{1/\ell_i} \qquad \ldots (3.8)$$

for some scale parameters $\underline{\sigma} = \left(\sigma_1, \sigma_2, ..., \sigma_k\right) > \underline{0}.$

Proof: Consider equation (3.5) above and choose arbitrary function $f(\cdot)$ as

$$f\left(\frac{x_2}{x_1^{\ell_2/\ell_1}}, \frac{x_3}{x_1^{\ell_3/\ell_1}}, ..., \frac{x_k}{x_1^{\ell_k/\ell_1}}\right) = 1 + \sum_{i=2}^{k}\left(\frac{x_i}{x_1^{\ell_i/\ell_1}}\right)^{1/\ell_i} = 1 + \frac{\sum_{i=2}^{k} x_i^{1/\ell_i}}{x_1^{1/\ell_1}}.$$

Hence, the particular solution of Euler's functional equation for almost homogeneous functions given by equation (3.4) when $\ell = 1,$ is

$$F\left(x_1, x_2, ..., x_k\right) = x_1^{1/\ell_1} \cdot \left\{1 + \frac{\sum_{i=2}^{k} x_i^{1/\ell_i}}{x_1^{1/\ell_1}}\right\} = \sum_{i=1}^{k} x_i^{1/\ell_i}.$$ Hence, corollary (3.1), (i) follows.

Similarly, for corollary (3.1), (ii), consider the scale transformation on each variable as $\frac{x_i}{\sigma_i},\ \ for\ \ \sigma_i > 0,\ \ i = 1, 2, ..., k.$ It is straightforward to check that equation (3.8) does satisfy equation (3.6) as

$$F\left(p^{\ell_1}x_1, p^{\ell_2}x_2, ..., p^{\ell_k}x_k\right) = \sum_{i=1}^{k}\left(\frac{p^{\ell_i}x_i}{\sigma_i}\right)^{1/\ell_i} = p\sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{1/\ell_i}$$
$$= p \cdot F\left(x_1, x_2, ..., x_k\right).$$

Therefore, $F\left(x_1, x_2, ..., x_k\right) = \sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{1/\ell_i}$ is a particular solution of Euler's functional equation for almost homogeneous functions given by equation (3.6). Thus, corollary (3.1), (ii) holds.

From equations (2.3) and (2.4) above, it worth noting that $\Psi\left(x_1, x_2, ..., x_k\right) = \sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{\alpha_i}$ is the particular solution of equations (2.4) and (3.1), and thus, the new family of multivariate semi-Pareto, reduces to the family of multivariate Pareto (III) distribution with joint survival function

$$\overline{F}_{\underline{X}}\left(x_1,x_2,...,x_k\right)=\frac{1}{1+\frac{1}{\beta}\sum_{i=1}^{k}\left(\frac{x_i}{\sigma_i}\right)^{\alpha_i}},\quad for\quad \underline{X}=\left(x_1,x_2,...,x_k\right)>\underline{0}. \qquad \ldots(3.9)$$

Now, we give the characterizations properties of the new family of multivariate semi-Pareto distributions derived in this paper. These characterizations properties are developed through the method of geometric minimization procedure using general and particular solutions of Euler's functional equation for homogeneous functions.

Theorem 3.1: Let $\left\{\underline{X}^i=\left(X_1^i,X_2^i,...,X_k^i\right)\right\}_{i=1}^{n}$ be a sequence of independently and identically distributed nonnegative random vectors with common joint survival function $\overline{F}_{\underline{X}}\left(\cdot\right)$. For a fixed $p\in\left(0,1\right)$, let N_p be a random vector which is independent of all $\underline{X}^i$ and is distributed geometrically with parameter p . Let $\underline{m}_p=\left(X_{(1)},X_{(2)},...,X_{(k)}\right)$ be the k-dimensional geometric minima with $X_{(j)}=\min\left\{X_j^1,X_j^2,...,X_j^n\right\}$ for each $j=1,2,...,k.$ Then,

$$p^{-\underline{\alpha}^{-1}}\underline{m}_p\triangleq\left(p^{-\frac{1}{\alpha_1}}X_{(1)},p^{-\frac{1}{\alpha_2}}X_{(2)},...,p^{-\frac{1}{\alpha_k}}X_{(k)}\right)\underset{=}{d}\ \underline{X}^1$$

if and only if $\underline{X}^1$ follows a new family of multivariate semi-Pareto distribution with $0<\beta<1$ and the parameter vector $-\underline{\alpha}^{-1}=\left(-\alpha_1^{-1},-\alpha_2^{-1},...,-\alpha_k^{-1}\right).$

Proof:

Let us assume that $p^{-\underline{\alpha}^{-1}}\underline{m}_p$ has the same distribution as $\underline{X}^1$ and try to show that $\underline{X}^1$ is distributed as new family of multivariate semi-Pareto distribution with $0<\beta<1$.

Let $\overline{H}_{\underline{m}_p}\left(\cdot\right)$ be the joint survival function of $p^{-\underline{\alpha}^{-1}}\cdot\underline{m}_p$, that is,

$$\begin{aligned}\overline{H}_{\underline{m}_p}\left(\underline{x}\right)&=P\left(p^{-\underline{\alpha}^{-1}}\cdot\underline{m}_p\geq\underline{x}\right)=P\left(\underline{m}_p\geq p^{\underline{\alpha}^{-1}}\underline{x}\right)=P\left(X_{(1)}>p^{\frac{1}{\alpha_1}}x_1,X_{(2)}>p^{\frac{1}{\alpha_2}}x_2,...,X_{(k)}>p^{\frac{1}{\alpha_k}}x_k\right)\\&=\sum_{n=1}^{\infty}P\left(\underline{X}^1\geq p^{\underline{\alpha}^{-1}}\underline{x}\right)^n\cdot p\left(1-p\right)^{n-1}\\&=\frac{p\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}}\underline{x}\right)}{1-\left(1-p\right)\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}}\underline{x}\right)}=\overline{F}_{\underline{X}}\left(\underline{X}\right), \qquad \ldots(3.12)\end{aligned}$$

where $\underline{x}=\left(x_1,x_2,...,x_k\right)>\underline{0}$ and $p^{\underline{\alpha}^{-1}}\cdot\underline{x}=\left(p^{\frac{1}{\alpha_1}}x_1,...,p^{\frac{1}{\alpha_k}}x_k\right)$. Hence, the common joint survival function of each $\underline{X}^i$, $\overline{F}_{\underline{X}}\left(\cdot\right)$ satisfies the functional equation.

Let $\Psi\left(\underline{x}\right)=\frac{\beta\left(1-\overline{F}_{\underline{X}}\left(\underline{x}\right)\right)}{\overline{F}_{\underline{X}}\left(\underline{x}\right)}$, therefore $\overline{F}_{\underline{X}}\left(\underline{x}\right)=\frac{1}{1+\frac{1}{\beta}\Psi\left(\underline{x}\right)}$ and hence, $\overline{F}_{\underline{X}}\left(p^{-\underline{\alpha}^{-1}}\underline{x}\right)=\frac{1}{1+\frac{1}{\beta}\Psi\left(p^{-\underline{\alpha}^{-1}}\underline{x}\right)}$. Substituting $\overline{F}_{\underline{X}}\left(\cdot\right)$ in equation (3.12), we have $\overline{F}_{\underline{X}}\left(\underline{x}\right)=\frac{1}{1+\frac{1}{\beta}\Psi\left(\underline{x}\right)}=\frac{p}{p+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}}\underline{x}\right)}$ $for\quad 0<\beta<1.$ Hence the functional equation $\Psi\left(\underline{x}\right)=\frac{1}{p}\Psi\left(p^{\underline{\alpha}^{-1}}\underline{x}\right)$ is satisfied. But this is the functional equation (2.4) given in definition (2.3) satisfied by new family of distributions given in equation (2.3).Therefore $\underline{X}^1$ is distributed as the new family of multivariate semi-Pareto with parameter $0<\beta<1$ and parameter vector $-\underline{\alpha}^{-1}=\left(-\alpha_1^{-1},-\alpha_2^{-1},...,-\alpha_k^{-1}\right).$

Conversely, let $\underline{X}^1$ be distributed as this proposed new family of multivariate semi-Pareto and show that $p^{-\underline{\alpha}^{-1}} \underline{m}_p$ has the same distribution as $\underline{X}^1$.

Since $\underline{X}^1$ is distributed as this new family of distributions, then its survival function is of the form given in definition (2.3). Let $\overline{H}_{\underline{m}_p}(\cdot)$ be the joint survival function of $p^{-\underline{\alpha}^{-1}} \cdot \underline{m}_p$. It is derived as in equation (3.10) by conditioning on N_p. Hence,

$$\overline{H}_{\underline{m}_p}(\underline{x}) = P\left(p^{-\underline{\alpha}^{-1}} \cdot \underline{m}_p \geq \underline{x}\right) = P\left(\underline{m}_p \geq p^{\underline{\alpha}^{-1}} \underline{x}\right) = P\left(X_{(1)} > p^{\frac{1}{\alpha_1}} x_1, X_{(2)} > p^{\frac{1}{\alpha_2}} x_2, ..., X_{(k)} > p^{\frac{1}{\alpha_k}} x_k\right)$$

$$= \sum_{n=1}^{\infty} P\left(\underline{X}^1 \geq p^{\underline{\alpha}^{-1}} \underline{x}\right)^n \cdot p(1-p)^{n-1}$$

$$= \frac{p\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)}{1-(1-p)\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)} \qquad ...(3.11)$$

where $\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)$ satisfies

$$\overline{F}_{\underline{X}}\left(p^{\underline{\alpha}^{-1}} \underline{x}\right) = \frac{1}{1+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)}. \qquad \text{… (3.12)}$$

Putting equation (3.12) in equation (3.11), we have

$$\overline{H}_{\underline{m}_p}(\underline{x}) = \frac{\dfrac{p}{1+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)}}{1-\dfrac{(1-p)}{1+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)}} = \frac{p}{p+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)}. \qquad \text{… (3.13)}$$

Considering equation (3.13) together with the definition (2.3) above, we can see that the functional equation $\Psi(\cdot)$ satisfies

$$\Psi(x_1, x_2, ..., x_k) = \tfrac{1}{p}\Psi\left(p^{\frac{1}{\alpha_1}} x_1, p^{\frac{1}{\alpha_2}} x_2, ..., p^{\frac{1}{\alpha_k}} x_k\right) \triangleq \tfrac{1}{p}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right). \qquad \text{… (3.14)}$$

Hence, equation (3.13) is the same as $\overline{H}_{\underline{m}_p}(\underline{x}) = \dfrac{p}{p+\frac{1}{\beta}\Psi\left(p^{\underline{\alpha}^{-1}} \underline{x}\right)} = \dfrac{1}{1+\frac{1}{\beta}\Psi(x_1, x_2, ..., x_k)} = \overline{F}_{\underline{X}}(\underline{X})$ for all $\underline{x} > \underline{0}$ and $0 < \beta < 1$. Thus, $p^{-\underline{\alpha}^{-1}} \underline{m}_p \underline{\underline{d}} \underline{X}^1$ follows. Therefore theorem (3.1) is completely proved.

Theorem (3.1) is extended to any finite steps of repeated geometric minimization procedures as described above, to give another characterization property of this proposed new family of multivariate semi-Pareto distribution. This extension is presented in the following theorem.

Theorem 3.2: Let $\left\{\underline{X}_1^{(1)}, \underline{X}_2^{(1)}, ..., \underline{X}_n^{(1)}, ...\right\}$ be a sequence of independently and identically distributed nonnegative k-variate random vectors with common joint survival function $\overline{F}_1(\cdot)$. For each $\ell = 2, 3, ...,$ define $\overline{F}_\ell(\cdot)$ sequentially in such a manner that $\overline{F}_\ell(\cdot)$ is the joint survival function of a geometric $(p_{\ell-1})$ minima $(0 < p_{\ell-1} < 1)$, $\underline{X}_{(N_{\ell-1})}^{(\ell-1)}$ of a random sample of $\left\{\underline{X}_i^{(\ell-1)}\right\}$ which is independently and identically distributed as $\overline{F}_{\ell-1}(\cdot)$. If there exist two parameter vectors $\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_k) > \underline{0}$ as the scale parameter of $\overline{F}_\ell(\cdot)$ and $\underline{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_k) > \underline{0}$ as the inequality parameter of $\overline{F}_\ell(\cdot)$, then the following two statements are equivalent:

(1). For each finite $\ell = 2, 3, ...,$

$P\left(\underline{X}_{(N_{\ell-1})}^{(\ell-1)} \geq \left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = P\left(\underline{X}_i^{(1)} \geq \underline{x}\right)$. Therefore,

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \overline{F}_1(\underline{x}) \quad \ldots (3.15)$$

for any $\underline{x} = (x_1, x_2, ..., x_k) > \underline{0},$ where $\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x} \triangleq \left[\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma_1} x_1, ..., \left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma_k} x_k\right)\right].$

or equivalently,

$$\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{(N_{\ell-1})}^{(\ell-1)} \underline{\underline{d}}\ \underline{X}_i^{(1)} \sim \overline{F}_1(\cdot). \quad \ldots (3.16)$$

(2). The common joint survival function of $\left\{\underline{X}_i^{(1)},\ i \geq 1\right\},\ \overline{F}_1(\cdot)$ is the new family of multivariate semi-Pareto distribution with $p = \prod_{j=1}^{\ell-1} p_j$.

Proof:
Let us assume statement **(1)** is true and try to show that statement **(2)** holds.

From $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{(N_{\ell-1})}^{(\ell-1)} \underline{\underline{d}}\ \underline{X}_i^{(1)}$ for any finite $\ell = 2, 3, ...,$ the recursive relation among the joint survival functions $\left\{\overline{F}_\ell(\cdot)\right\}_{\ell=1}^{\infty}$ is derived by conditioning on the geometric $N_{\ell-1}$ random variable. The joint survival function of the geometric $(p_{\ell-1})$ minima $\underline{X}_{(N_{\ell-1})}^{(\ell-1)}, \overline{F}_\ell(\cdot)$ is derived as

$$\begin{aligned}\overline{F}_\ell(\underline{x}) = P\left(\underline{X}_{(N_{\ell-1})}^{(\ell-1)} \geq \underline{x}\right) &= \sum_{n=1}^{\infty} P\left(\min_{1\leq i\leq n} \underline{X}_i^{(\ell-1)} \geq \underline{x}\right) \cdot P(N_{\ell-1} = n) \\ &= \sum_{n=1}^{\infty} \left(\overline{F}_{\ell-1}(\underline{x})\right)^n \cdot p_{\ell-1}(1-p_{\ell-1})^{n-1} \\ &= \frac{p_{\ell-1}\overline{F}_{\ell-1}(\underline{x})}{1-(1-p_{\ell-1})\overline{F}_{\ell-1}(\underline{x})}. \quad ...(3.17)\end{aligned}$$

From $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{(N_{\ell-1})}^{(\ell-1)} \underline{\underline{d}}\ \underline{X}_i^{(1)} \sim \overline{F}_1(\cdot),$ it can be seen that for any $\underline{x} > \underline{0},$

$P\left[\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{X}_{(N_{\ell-1})}^{(\ell-1)} \geq \underline{x}\right] = P\left(\underline{X}_i^{(1)} \geq \underline{x}\right)$ and hence $P\left[\underline{X}_{(N_{\ell-1})}^{(\ell-1)} \geq \left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right] = P\left(\underline{X}_i^{(1)} \geq \underline{x}\right),$ that is, $\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\underline{\gamma}} \underline{x}\right) = \overline{F}_1(\underline{x})$ or equivalently,

$$\overline{F}_\ell(\underline{x}) = \overline{F}_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\underline{\gamma}} \underline{x}\right). \quad \ldots (3.18)$$

Now, let $\Psi_\ell(\underline{x}) \triangleq \dfrac{\beta\left(1-\overline{F}_\ell(\underline{x})\right)}{\overline{F}_\ell(\underline{x})}$, then $\overline{F}_\ell(\underline{x}) \triangleq \dfrac{1}{1+\frac{1}{\beta}\Psi_\ell(\underline{x})}$ for each $\ell \geq 1$ and $0 < \beta < 1$ and substitute this in equation (3.17) above to get

$$\overline{F}_{\ell}\left(\underline{x}\right)=\frac{1}{1+\frac{1}{\beta}\Psi_{\ell}\left(\underline{x}\right)}=\frac{\dfrac{p_{\ell-1}}{1+\frac{1}{\beta}\Psi_{\ell-1}\left(\underline{x}\right)}}{1-\dfrac{\left(1-p_{\ell-1}\right)}{1+\frac{1}{\beta}\Psi_{\ell-1}\left(\underline{x}\right)}}=\frac{p_{\ell-1}}{p_{\ell-1}+\frac{1}{\beta}\Psi_{\ell-1}\left(\underline{x}\right)}=\frac{1}{1+\frac{1}{\beta}\frac{1}{p_{\ell-1}}\Psi_{\ell-1}\left(\underline{x}\right)}.$$

Therefore, $\Psi_{\ell}\left(\underline{x}\right)=\dfrac{1}{p_{\ell-1}}\Psi_{\ell-1}\left(\underline{x}\right).$ It follows by iteration that

$$\Psi_{\ell}\left(\underline{x}\right)=\left(\prod_{j=1}^{\ell-1}p_j\right)^{-1}\Psi_1\left(\underline{x}\right), \qquad \ldots (3.19)$$

for all $\ell=2,3,\ldots$.

Similarly, from equation (3.18), we have $\dfrac{1}{1+\dfrac{1}{\beta}\Psi_{\ell}\left(\underline{x}\right)}=\dfrac{1}{1+\dfrac{1}{\beta}\Psi_1\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\underline{\gamma}}\underline{x}\right)}$ and hence

$$\Psi_{\ell}\left(\underline{x}\right)=\Psi_1\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\underline{\gamma}}\underline{x}\right). \qquad \ldots (3.20)$$

Considering equations (3.19) and (3.20), it follows that the functional equation $\Psi_1\left(\cdot\right)$ satisfies $\Psi_1\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\underline{\gamma}}\underline{x}\right)=\left(\prod_{j=1}^{\ell-1}p_j\right)^{-1}\Psi_1\left(\underline{x}\right)$ or equivalently,

$$\Psi_1\left(\underline{x}\right)=\left(\prod_{j=1}^{\ell-1}p_j\right)\Psi_1\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\underline{\gamma}}\underline{x}\right). \qquad \ldots (3.21)$$

That is, for any $\underline{x}=\left(x_1,x_2,\ldots,x_k\right)>\underline{0},\ \Psi_1\left(\cdot\right)$ satisfies $\Psi_1\left(x_1,\ldots,x_k\right)=\left(\prod_{j=1}^{\ell-1}p_j\right)\cdot\Psi_1\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\gamma_1}x_1,\ldots,\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\gamma_k}x_k\right)$. Therefore for $\Psi_1\left(\cdot\right)$, there exists a positive real number $p=\prod_{j=1}^{\ell-1}p_j,\ \left(0<p<1\right),$ such that

$$\Psi_1\left(\underline{x}\right)=p\cdot\Psi_1\left(p^{-\underline{\gamma}}\underline{x}\right). \qquad \ldots (3.22)$$

From equation (2.4), it is observed that the joint survival function $\overline{F}_1\left(\underline{x}\right)$ can be written as $\overline{F}_1\left(\underline{x}\right)=\dfrac{1}{1+\frac{1}{\beta}\Psi_1\left(\underline{x}\right)}$ with $\Psi_1\left(\underline{x}\right)=p\cdot\Psi_1\left(p^{-\underline{\gamma}}\underline{x}\right).$ Thus, for all $i\geq 1,\ \underline{X}_i^{(1)}$ is distributed as this new family of distributions with $0<\beta<1.$ Hence, statement **(2)** follows.

Conversely, assume that statement **(2)** holds and show that $\left(\prod_{j=1}^{\ell-1}p_j\right)^{-\underline{\gamma}}\underline{X}_{(N_{\ell-1})}^{(\ell-1)}\ \underline{\underline{d}}\ \underline{X}_i^{(1)}\sim\overline{F}_1\left(\cdot\right)$ or equivalently $\overline{F}_{\ell}\left(\left(\prod_{j=1}^{\ell-1}p_j\right)^{\underline{\gamma}}\underline{x}\right)=\overline{F}_1\left(\underline{x}\right).$

Now, let $\underline{X}_i^{(1)}$ be distributed as this new family of distributions with $0<\beta<1$, and $p=\prod_{j=1}^{\ell-1} p_j$ then by definition (2.3),

$\overline{F}_1(\cdot)$ can be written as $\overline{F}_1(\underline{x})=\dfrac{1}{1+\frac{1}{\beta}\Psi_1(\underline{x})}$ with

$$\Psi_1(\underline{x})=p\cdot\Psi_1\left(p^{-\gamma}\underline{x}\right), \qquad \ldots (3.23)$$

and also, the joint survival function of the scaled geometric minima $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma}\cdot\underline{X}_{(N_{\ell-1})}^{(\ell-1)}$ is $P\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma}\underline{X}_{(N_{\ell-1})}^{(\ell-1)}\geq\underline{x}\right)=P\left(\underline{X}_{(N_{\ell-1})}^{(\ell-1)}\geq\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)=\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)$. Considering equation (3.17), we have

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)=\frac{p_{\ell-1}\overline{F}_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)}{1-(1-p_{\ell-1})\overline{F}_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)}. \qquad \ldots (3.24)$$

Using the same expression for any $\ell\geq 1$, $\Psi_\ell(\underline{x})\triangleq\dfrac{\beta\left(1-\overline{F}_\ell(\underline{x})\right)}{\overline{F}_\ell(\underline{x})}$ gives $\overline{F}_\ell(\underline{x})=\dfrac{1}{1+\frac{1}{\beta}\Psi_\ell(\underline{x})}$. Putting this in equation (3.24), yields $\Psi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)=\dfrac{1}{p_{\ell-1}}\Psi_{\ell-1}\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)$. It follows by iteration that $\Psi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)=\dfrac{1}{\prod_{j=1}^{\ell-1} p_j}\Psi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)$. From equation (3.23) with $p=\prod_{j=1}^{\ell-1} p_j$, equation (3.24) becomes

$$\overline{F}_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)=\frac{1}{1+\frac{1}{\beta}\Psi_\ell\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)}=\frac{1}{1+\frac{1}{\beta\left(\prod_{j=1}^{\ell-1} p_j\right)}\Psi_1\left(\left(\prod_{j=1}^{\ell-1} p_j\right)^{\gamma}\underline{x}\right)}=\frac{1}{1+\frac{1}{\beta}\Psi_1(\underline{x})}=\overline{F}_1(\underline{x}).$$

Thus, $\left(\prod_{j=1}^{\ell-1} p_j\right)^{-\gamma}\cdot\underline{X}_{(N_{\ell-1})}^{(\ell-1)}\ \underline{\underline{d}}\ \underline{X}_i^{(1)}$, and hence statement **(1)** is true. Hence, Theorem 3.2 follow.

4.0 Fitting Data into New Family of Distributions

In this section, a Mat lab package is employed to fit this family of distributions to the data using particular solution of Euler's functional equation.

Figure 4.1 below, shows the various values of $\overline{F}_i(x_i)=\left\{1+\frac{1}{\beta}\left(\frac{x_i}{\sigma_i}\right)^{\frac{1}{\gamma_i}}\right\}^{-1}$, that is, new family of multivariate Pareto (III) with univariate Pareto (III) marginal distributions, for different values of $x_i>0,\delta_i>0,\gamma_i>0$ as well as for $\beta=0.02,\beta=0.95$ and $\beta=1$. The first two values of β are two extreme ends values of new family of multivariate Pareto (III) with univariate Pareto (III) marginal distributions while last value of $\beta(\beta=1)$, is for multivariate Pareto (III) with univariate Pareto (III) marginal distributions given in the literature.

Table 4.1:

A table showing the various values of new family of multivariate Pareto (III) with univariate Pareto (III) marginal distributions for $\beta = 0.02, \quad \beta = 0.95$ *and* $\beta = 1.$

				$\overline{F}_i(x_i)$		
i	x_i	σ_i	γ_i	**β = 0.02**	**β = 0.95**	**β = 1**
1	0.1000	0.1000	0.1000	0.0196	0.4872	0.5000
2	10.0000	0.1655	0.1483	0.0108	0.3405	0.3522
3	20.0000	0.2310	0.1966	0.0082	0.2831	0.2936
4	30.0000	0.2966	0.2448	0.0064	0.2346	0.2440
5	40.0000	0.3621	0.2931	0.0050	0.1929	0.2011
6	50.0000	0.4276	0.3414	0.0039	0.1574	0.1644
7	60.0000	0.4931	0.3897	0.0031	0.1275	0.1334
8	70.0000	0.5586	0.4379	0.0024	0.1027	0.1075
9	80.0000	0.6241	0.4862	0.0019	0.0823	0.0862
10	90.0000	0.6897	0.5345	0.0015	0.0656	0.0689
11	100.0000	0.7552	0.5828	0.0012	0.0522	0.0548
12	110.0000	0.8207	0.6310	0.0009	0.0414	0.0435
13	120.0000	0.8862	0.6793	0.0007	0.0327	0.0344
14	130.0000	0.9517	0.7276	0.0006	0.0258	0.0272
15	140.0000	1.0172	0.7759	0.0004	0.0204	0.0214
16	150.0000	1.0828	0.8241	0.0003	0.0161	0.0169
17	160.0000	1.1483	0.8724	0.0003	0.0126	0.0133
18	170.0000	1.2138	0.9207	0.0002	0.0099	0.0105
19	180.0000	1.2793	0.9690	0.0002	0.0078	0.0082
20	190.0000	1.3448	1.0172	0.0001	0.0061	0.0065
21	200.0000	1.4103	1.0655	0.0001	0.0048	0.0051
22	210.0000	1.4759	1.1138	0.0001	0.0038	0.0040
23	220.0000	1.5414	1.1621	0.0001	0.0030	0.0031
24	230.0000	1.6069	1.2103	0.0000	0.0023	0.0025
25	240.0000	1.6724	1.2586	0.0000	0.0018	0.0019
26	250.0000	1.7379	1.3069	0.0000	0.0014	0.0015
27	260.0000	1.8034	1.3552	0.0000	0.0011	0.0012
28	270.0000	1.8690	1.4034	0.0000	0.0009	0.0009
29	280.0000	1.9345	1.4517	0.0000	0.0007	0.0007
30	290.0000	2.0000	1.5000	0.0000	0.0005	0.0006

Also figure 4.1 below, is the graph that represents the information given in table 4.1 above for various values of β.

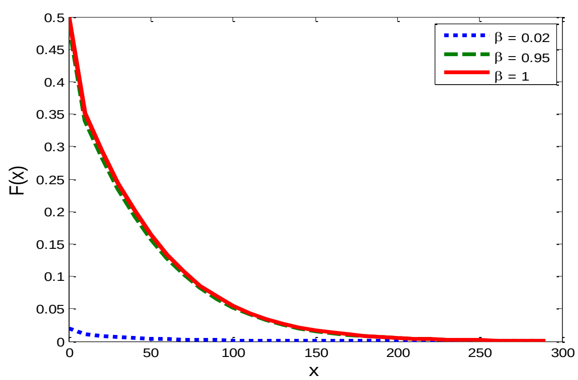

Figure 4.1: A graph showing the various values of the new family of multivariate Pareto (III) with univariate Pareto (III) marginal distributions for $\beta = 0.02, \beta = 0.95$ *and* $\beta = 1.$

5.0 Discussions of the Results

It is well known that Pareto distributions are generally used to describe the allocations of wealth among individuals. Since it clearly show the way in which larger portion of the wealth of any society is owned by a smaller percentage of the people in that society.

Now, considering the Table 4.1 and Figure 4.1 above, it can be seen that the percentage, probability or fraction of the population that owns a small amount of wealth per person is higher when parameter $\beta = 1,$ that is in the distributions given in the literature. In the new family of distributions obtained in this research work, in which parameter β is within the interval $0 < \beta < 1,$ it is observed that the percentage, probability or fraction of population that owns a small amount of wealth per person is decreasing exponentially as newly introduced parameter, β approaches zero. Indeed, this percentage, fraction or probability of population approaches zero as the value of newly introduced parameter β turns to zero.

6.0 Conclusions

From the results obtained, we can conclude that multivariate semi-Pareto distribution given in the literature can be extended to a new family of multivariate semi-Pareto distribution. Similarly, generalized multivariate Pareto (III) with univariate Pareto (III) distributions as marginal can be modified to its new family, that is, new family of generalized multivariate Pareto (III) with univariate Pareto (III) distribution as marginal. Characterizations properties of multivariate semi-Pareto distribution can be extended to that of the new family of multivariate semi-Pareto distribution using the method of geometric minimization procedures.

It can also be concluded here that simulated data can be fitted into this new family of distributions. Mat lab package is employed in fitting data into this new family of distributions. Based on the results obtained, we can conclude that the new family of multivariate semi-Pareto distributions obtain in this paper, has advantage over the distributions stated in the literature. These are so, since the percentage, probability or fraction of population that owns a small amount of wealth per person is rather lower when new family of distributions are used and it continues to reduce as the value of parameter β approaches zero. On the other hand, the percentage, probability or fraction of population that owns a small amount of wealth per person attains its peak when parameter $\beta = 1,$ that is, when the distributions given in the literature (Yeh, 2007) are used.

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