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Extended Nonlinear Model of Stochastic Processes

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Abstract

A unidirectional model of the nonlinear stochastic process is investigated. As a result, this study seems to have extended the model equation beyond the present state.

An interesting result appears to emerge when the model is applied to the study of water waves in an undisturbed field. In this case, as the solution was extended to third order, singular behaviour appears in the previously smooth parameters

1.0 Introduction

Nonlinearity which normally develop in an oscillating process has been intensively investigated; theoretically and experimentally. Its impact in almost the entire family of geophysical phenomena is established [1, 2, 4].

Nonlinearity is confirmed to, often, tend to destroy phase coherence among specific components in an involving process. Thus, instead of abnormal growth, randomness is indeed introduce in the system. In this consideration, Longuet Higgins [4] proposed the Euleriant and Lagrangan aspects of surface wave evolution. Consequently, a number of investigators were involved. Notable among them are: Mori and Yasuda [6], Philip et al. [5], Arena and Fedele [1, 2].

In particular Arena and Fedele proposed a model involving the family of narrow banded nonlinear stochastic processes. This model was introduced to study a variety of processes in water wave phenomenon.

Consequently, the present investigation intends to tackle the identical problem associated with nonlinear effects in wave phenomena. It will follow the same approach adopted by Arena and Fedele [1] but will represent a significant extension in the entire development.

2.0 A Model Describing a Narrow Banded Oscillatory Process.

In cartesian coordinate system, an oscillatory process that is structurally uniform in y-coordinate is moving in (x, ƶ) plane. In this description, x ishorizontal and ƶ is vertical coordinate in the system; t represents time.

Thus, the analytical representation for the process is expressed in the form:

$$\emptyset(x,ƶ,ƚ) = [f(x,ƶ)acosx] + a^2[g(x,ƶ)cos^2 X + h(x,ƶ)\sin^2 X]$$
$$+a^3[g_1(x,ƶ)\cos^3 X + g_2 sin^3 X] + a^4[h_1 \cos^4 X + h_2 \sin^4 X] \dots\dots\dots\dots\dots (1)$$

$\times = \omega t + \varepsilon, \omega$ =the modulation frequency with a as the typical height . ε is the phase angle, $0 < \varepsilon < \pi$. The first three terms in (1) are as suggested by Arena and Fedele (2001). This is in relation to their study concerning the narrow banded nonlinear processes. The remaining terms are as introduced in this study.

We now define the variables

$$Y_1 = \frac{a}{\sigma} cos \times, Y_2 = \frac{a}{\sigma} sin \times \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

σ =variance and is calculated from given time series. In term of the variables (Y_1, Y_2), (1) is now of the form

$$\emptyset(x,ƶ,t) = \emptyset(Y_1,Y_2) = \sigma [F(x,ƶ)Y_1 + G(x,ƶ)Y_1^2 + H(x,ƶ)Y_2^2 +$$
$$G_1(x,ƶ)Y_1^3 + G_2(x,ƶ)Y_2^3 + H_1(x,ƶ)Y_1^4 + H_2(x,ƶ)Y_2^4 \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

Where

$F(x,ƶ) = f(x\ ƶ), G(x,ƶ) = ag(x,ƶ)$.

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$H(x,z) = ah(x,z), G_1(x,z) = ag(x,z), G_2(x,z) = ag_2(x,z)$ ……………. (4)

In terms of statistical mean and variances,

$\overline{Y_1} = \overline{Y_2} = 0 , \sigma_{y_1}^2 = \sigma_{y_2}^2 = 1$ ……………………………………………………. (5)

The joint probability $N(Y_1, Y_2) of Y_1 and\ Y_2$ simplifies to

$$N(Y_1, Y_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(Y_1^2+Y_2^2)} = \frac{1}{2\pi} e^{-\frac{a^2}{2\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Using symbol bar ($\bar{}$) to indicate the mean value we have $\overline{(cosx)}^{2n+1} = (\overline{sinx})^{2n+1} = 0$ for $0 < x < \pi$ where $\cos x = \frac{1}{2}[e^{ix} + e^{ix}]$ and $\sin x = \frac{1}{2i}[e^{ix} - e^{-ix}], n = 0,1,2,3, \ldots, i = \sqrt{-1}$

Thus, the mean value of $\emptyset(Y_1, Y_2 t)$is of the form

$$\overline{\phi}(x, z, t) = \frac{\sigma}{24}\left[(G + H) + \frac{3}{8}(H_1 + H_2)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The variance of $\emptyset(x, z, t) = \sigma_\emptyset^2$where

$$\sigma_\emptyset^2 = \frac{\sigma^2 F}{\beta^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$and\ \beta^2 = [1 + 2R_6]^{1/2} , R_6 = \sum_{i=1}^{6} \alpha_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$\alpha_i(i = 1,2,3,4,5,6)$ are the nonlinearity parameters and they determine the nonlinear effects related to the involving processes in this consideration. They are calculated from the following relations:

$$\alpha_1 = \frac{G}{|F|}, \alpha_2 = \frac{H}{|F|}, \alpha_3 = \frac{G_i}{|F|}, \alpha_4 = \frac{G_2}{|F|}, \alpha_5 = \frac{H_1}{|F|}, \alpha_6 = \frac{H_2}{|F|}$$

Previous attempt ended nicely with second order approximation ($\alpha, and\ \alpha_2$). Thus the present study seems to represent a significant extension in this aspect of the nonlinear physics.

Define a parameter $\zeta = \frac{\emptyset - \bar{\emptyset}}{\sigma_\emptyset}; \sigma_\emptyset = variance\ of\ \emptyset,$ thus,

$$\zeta = \beta[Y_1 + (Y_1 + Y_2)R_6] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

3.0 Calculation of Special Density

The characteristic function associated with $\zeta$is taken as $e^{i\zeta w}$ where

$$e^{\overline{i\zeta w}} = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iw\zeta} N(Y_1, Y_2) dY_1 dY_2 = \frac{1}{\pi}\exp\left[\frac{-a^2}{2\delta_2}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp\ [iw\zeta] dY_1 dY_2$$

$$= \frac{1}{\pi}\exp\left[\frac{-a^2}{2\sigma^2}\right] e^{i\omega R_1} I_1 I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$R_1 = -\beta\left[(\alpha_1 + \alpha_2) + \frac{2}{3}(\alpha_3 + \alpha_4) + \frac{3}{8}(\alpha_5 + \alpha_6)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$I_1 = \int_{-\infty}^{\infty} \exp[-[Y_1 + \alpha_1 Y_1^2 + \alpha_3 Y_1^3 + \alpha_5 Y_1^5]] dY_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$I_2 = \int_{-\infty}^{\infty} \exp[-[\alpha_2 Y_2^2 + \alpha_4 Y_2^3 + \alpha_6 Y_2^4]] dY_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Numerical estimate of the terms involving $Y_1^3, Y_1^4\ and\ Y_2^3, Y_2^4$ term suggest that they contribute only 6% to the integral (13) and (14). Thus,

$$f_\zeta = \frac{L}{\pi}\exp\left[\frac{-a^2}{2\sigma^2} + i\omega R_1\right]\int_{-\infty}^{\infty} e^{i\zeta\omega}$$
$$\left\{\exp\left[-\frac{1}{2}\frac{\omega^2\beta^2}{1 - 4(\alpha_1\beta\omega)} - i\omega\beta(\alpha_1 + \alpha_2) - 2i\omega\beta(\alpha_3 + \alpha_4)\right.\right.$$
$$\left.\left. + \frac{(\beta\omega)^2}{1 + 4(\omega\alpha_1\beta)^2}\right][1 - 4\omega^2\beta^2\alpha_1\alpha_2 - 2i\omega\beta R_6]^{-\frac{1}{2}}\right\} d\omega \ldots\ldots (15)$$

(15) can be evaluated using contour integral method in ω- plane, if note is taken of the branch points related the values of ω which satisfies the quadratic equation;

$$4\omega^2\beta^2\alpha_1\alpha_2 + 2i\omega\beta R_6 - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

However, one can realistically make the following assumption which simplify (6)

$$G = -H, G_1 = -H_1, \alpha_5 + \alpha_6 = \frac{-3}{8}[\alpha_1 + \alpha_2] \text{ thus,}$$

$$\beta = \left[1 + \frac{5}{4}(\alpha_5^2 + \alpha_6^2)\right]^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

$$\zeta = \beta[Y_1 + \alpha_1(Y_1^2 - Y_2^2) + \alpha_3(Y_1^3 - Y_2^3) + \alpha_5(Y_1^4 - Y_2^4)] \ldots\ldots\ldots\ldots (18)$$

$$f_\zeta = \frac{1}{\pi}\exp\left(\frac{-a^2}{2\sigma^2}\right)\int_{-\infty}^{\infty}\exp -\frac{1}{2}\left[\frac{\omega^2\beta^2}{1+4\,\omega^2\beta^2}\right]cos\left[\omega\left\{\zeta + \left(\frac{(\omega\beta)^2 a}{1+4(\omega\beta\sigma)^2}\right)\right\}\right]d\omega \ldots\ldots\ldots (19)$$

$$\varphi = \frac{\zeta F}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

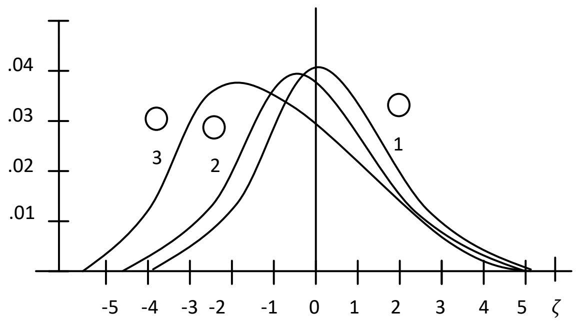

Fig1: The spectrum of f_ζ as a function of ζ

1. Models for first order
2. Models for $\alpha_1 + \alpha_2$
3. Models for R_6

Numerical calculation from Eqn (18a) are shown in Fig:1. There are significant shift to the left as the number of parameters $((\alpha_1, \alpha_2, \alpha_3, \alpha_4\alpha_5\alpha_5)$ increases. This implies that the increasing strength of nonlinearity given rise to distribution that is basically asymmetric and approximately Raleigh in shape.

4.0 Absolute Maximum and Minimum

Absolute maximum and minimum for wave crest and trough will now be established in this consideration. Thus, we have the expansion which was deduced from Eqa (1) as follows:

$$\emptyset(x, z, t) = \sum_{n=1}^{3}\beta_n \cos nX \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

$$\beta_1 = f(x,z)a + 3a^3 g(x,z), \beta_2 = \frac{a^2}{2}[g(x,z) - \hbar(x,z) + h_1(x,z)]$$

$\beta_3 = \frac{a^3}{4}g_1(x,z)$,if required, $\beta_4 = \frac{a^4}{8}[h,(x,z + h_2(x,z)]$

For the extreme values of $\emptyset(x,z,t)$,

$$\frac{d\emptyset}{d\chi} = 0 = sinX\,[B_1 - (4B_2 + 6B_3)\cos X - 3B_3]$$

thus, the point of interest in this consideration is given by sinX = 0,

i.e $\chi = 0\ or \pi$

$$cosX = \left[\frac{(B_1 - 3B_2)}{3B_2 + 6B_3}\right] is\ not\ relevant\ in\ this\ study\,.$$

Thus, the extreme points describing the extreme values are $X = 0\ or\ \pi$.

To determine the nature of the points, we have:

$\frac{d^2\emptyset}{d\chi^2} < 0\ when \chi = 0$ provides the crest

$\frac{d^2\emptyset}{d\chi^2} > 0\ when\ \chi = \pi$ provides the trough

The corresponding crest elevation height $\emptyset_h$ and trough depth $\emptyset_L$ are as follows: (eqn(1))

$\emptyset_h = f(x,z)a + a^2 g(x,z) + a^3 g_1(x,z) + a^4 h_1(\eta,z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$
$\phi_\angle = -af(x,z) + a^2 g(x,z) - a^3 g_1(x,z) + a^4 h_1(x,z) \ldots\ldots\ldots\ldots\ldots\ldots (22)$
From (21) and (22), it is deduced that crest height is larger in magnitude than trough depth. This seems realistic deduction because, all the quantities in the expression are positive from physical realities. Calculations involving rogue wave dataegive $\emptyset_L = .7854\varphi_h$. This compares reasonally well with observed result of $\emptyset_L = .82\varphi_h$[3].

5.0 Crests and Troughs Distribution.

It is often necessary during the operations involving risk design (off and on shore structures) to provide an adequate knowledge of wave crest height and trough depth in the locality. More realistic approach is to calculate the probability of exceeding absolute crest and trough respectively in the locality.
Consider the equation for ξ_h where:
$\xi_h = \frac{\phi_h}{\sigma_\phi} = \beta(u + \alpha_1 u^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$
u is a rapidly varying quantity. Thus, we obtain
$\beta(u + \alpha_1 u^2) - \xi_h = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$
(24) is solved as quadratic equation to give two real roots that are different i.e, $u_1 and\ u_2$ given by
$u_1 = \frac{1}{2\propto}[-1 + Q_1] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$
$u_2 = \frac{1}{2\propto_1}[-1 - Q_1] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$
$Q_1 = \left(-1 + \frac{4\xi_h\alpha_1}{\beta}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$
Thus, the probability of high crest $\xi_h$exceeding ξ is given by

$$p(\xi_h > \xi) = \begin{cases} p(u > u_1) = e^{-\frac{1}{2}u_1^2} \\ p(u_2 < u < u_1) = e^{-\frac{1}{2}u_2^2} - e^{-\frac{1}{2}u_1^2} \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Numerically, $u_1 > u_2$.
For the trough, (24) is re-written as
$\propto_1 u^2 - u + \frac{\xi_L}{\beta} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$
Numerical values of quantities in (29) gives$\beta > 4 \propto_1 \xi_h$ hence, (29) has real roots provided by
$u_1 = \frac{1}{2\propto}[1 + Q_2], u_2 = \frac{1}{2\propto_1}[1 - Q_2] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$

$$Q_2 = \left[1 - \frac{4 \propto_1 \xi_L}{\beta}\right]^{\frac{1}{2}}$$

For ξ deeper than trough ξ_L, the probability of this being the case is provided by

$$p(\xi_L > \xi) = \begin{cases} p(u > u_1) = e^{-\frac{1}{2}u_1^2} \\ p(u_\angle < u < u_1) = e^{-\frac{1}{2}u_2^2} - e^{-\frac{1}{2}u_1^2}, u_1 > u_2 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

An extended form of (24) is a cubic equation, thus,

$$\propto_2 u_h^3 + \propto_1 u^2 + u - \frac{\xi_h}{\beta} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

(32) is a cubic equation with three roots involving complex numbers, hence, are not appropriate in this consideration.
Nonlinearity parameters $\propto_i$ (i=1,2.....6) are displayed in table 1
We have taken$|F|$ to be proportional to the significant wave heightH_s (1F1=1.9H_s). The numerical values of $H_s$are in (m) and are obtained from global data.

Table I: Nonlinearity parameters as functions of significant wave heightsH_s.

$\lvert F\rvert(m)$	$\propto_1$	$\propto_2$	$\propto_3$	$\propto_4$	$\propto_5$	$\propto_6$
9.5	°412	°482	°351	°332	°303	°275
10.0	°371	°362	°348	°311	°285	°251
10.5	°330	°310	°301	°284	°251	°231
11.0	°287	°251	°221	°210	°231	°211

The dependence of these parameters on $|F|$ are clearly demonstrated. Solution obtained from Euler's equations of an irrotational fluid using Stokes expansion are involved $\sigma_\phi = 0.32\sigma_C^2, and\ \sigma_C$ is obtain from global datae for extreme water waves [3].

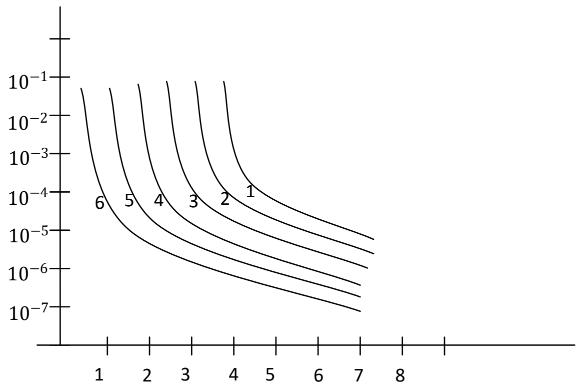

Fig II: Exceedence probability as a function of non-dimensional variable

$\zeta$and $\sum_{n=1}^{k} \alpha_n * (1)k = 1, (2)k = 2, (3)k = 3, (4)k = 4, (5)k = 5, (6)k = 6$

In Fig. II, $\xi = \frac{\emptyset - \emptyset_0}{\sigma_\emptyset^2}$, where ϕ_0 refers to linear wave form, ϕ refers to randomly distributed but with narrows banded wave form, σ_ϕ^2 in the variance calculated from ϕ.

The Figure II suggests that higher order nonlinearity implies lower exceedence probability. Obviously, linear processes are easier to predict than nonlinear random processes. This difficulty increases with increasing intensity of nonlinearity and this table is as one would expect.

The Fig II represents the crest elevation, and that for the corresponding trough is similar.

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6.0 Application to the Evolution of Sea Surface Waves in an Undisturbed Field.

This study will be applied to provide more realistic function to model the evolution of sea-surface wave profile in an undisturbed wave field. The consideration will be an extension to the related study in [2]. In [1] the foregoing theory of stochastic family with narrow banded spectrum was successfully applied to explain to second order the observed nonlinear effects for the wave profile in Stoke's expansion.

Thus, the surface profile $\eta(x,t)$ is expanded in the form:

$\eta(x,t) = acos\chi + kaf_m cos2\chi$. … … … … … … … … … … … … … … … … …. (33)

$f_m(kd) = \frac{[2 + cosh^2(2kh)]\, cosh\, 2kd}{sinh^2 kd}$ ………………………… (34)

$\chi = kx - \omega t + \theta, 0 < \theta < \pi, k = wave\ number, \omega = coresponding$
$frquency\ of\ the\ wave\ envelope, d = still\ water\ depth.$

As in previous sections (eqn2), (33) takes the form:

$\eta(x,t) = \sigma[Z_1 + \epsilon f_m, (z_1^2 - z_2^2)]$ … … … … … … … … … … … … … … … … (35)

Where $F = 1, G = \in f_m, H = -G, \in = ka =$ steepness parameter,

$kd =$ nonlinearity parameter;$\sigma = \propto_1 = -\propto_2 = \frac{\in}{2}$

Extended to third order in Stoke's expansion, then, we obtained

$$\eta(x,t) = a\ cos\chi + ka^2 f_m\ cos2\chi + 3a^3k^2 f_n(kd) cos3\chi \ldots\ldots (36)$$

$$\mathrm{f_n(kd)} = \frac{[3 + 8\cosh^3 kd]}{64\sinh^6(kd)} cosh3kd \ldots\ldots (37)$$

(34) and (37) were derived from the Stoke's expansion applied to Euler's equations of inviscid and irrotational fluid flow.

Equ(36) may be put in the form

$\eta(x,t) = a\cos X + kdf_m(1 - 2\sin^2 X) + 3k^2a^3 f_n(kd)[4\cos^3 X - 3cosX$

Introducing (2) as before

$\eta(x,t) = (1 - 9k^2a^2)\sigma Y_1 - 2\sigma^2 k\ f_m(kd) Y_1^2 + 12k^2\sigma^3 f_n(kd) Y_2^2$
$+ ka^2 f_n(kd)$ … (38)

Since, $1 - 2\sin^2\chi = 2\cos^2\chi - 1$

Equation (2) is introduced to obtain from (38) an improved form as:

$\eta(x,t) = \sigma[F\gamma_1 - G\gamma_2^2 + H\gamma_1^3] + ka^2 f_n\,(kd)$……………… … … … … …. (39)

Where $F = 1 - 9K^2a^2, G = 2k\sigma f_m, H = 12k^2\sigma\ f_m\,(kd)$

$\propto_1 = \frac{G}{|F|}, \propto_2 = \frac{H}{|F|}$

By introducing the third other term in (36), the corrected coefficient $F = F(ka)\ gives\ F = 0$ when $ka = \frac{1}{3}; Thus, \propto_1\ and \propto_2$ are now singular, and depend on wave steepness parameter. (see Fig III). In terms of geophysical representations, ka ranges from 0.08 to 0.29 and is the range [7] globally accepted by marine observers..

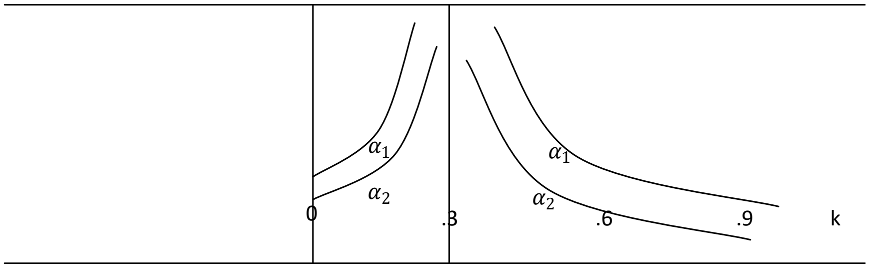

Fig III: The explosive characteristics of $\alpha_1 and\ \alpha_2$ being the effect of third order term thus introduced.

7.0 Conclusions

In this study, the evolutional changes associated with nonlinear strochastic phenomena is analysed. The approach is both deterministic and statistical. The build-up of nonlinearity expressed in the form of Stoke's expansion is significantly elongated. Consequently, the shift in otherwise,, symmetric spectral profile which usually characterises Gaussian sea is now more pronounced than previously established in previous investigations. Further, the study has successfully introduced six nonlinear parameters rather than two used in previous investigations in [1, 2]. It thus appears that the shift in spectral shape is related to the number of these parameters introduced in the calculations.

Finally, the theory was applied to the Stoke's expansium for wave in an undisturbed field. Extended to third order in Stoke's expansion, surprisingly, the parameters develop singular behavior when the wave skeepness tends to $\frac{1}{3}$ (non-dimensional).

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