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Stability Analysis of Fractional Duffing Oscillator 1

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Abstract

We solve fractional Duffing equation with two fractional derivatives namely

$$D^{\alpha}q(t)+\lambda D^{\beta}q(t)-\mu q(t)+\nu q^{3}(t)=f(t)$$
$$0<\beta<1,\quad 1<\alpha<2$$

With initial condition ***q (0) =1,*** $D^{\beta}q(0)=0$
Here λ is the damping coefficient, μ the stiffness coefficient, and v the coefficient of nonlinearity with f(t) as the forcing function. Using complex analysis method we find that a Hankel contour ensue and we are unable to calculate residues because the location of the poles and branch points cannot be explicitly found except for special cases relating α and β in a simple way. Using the Laplace transform method in this first part we solve the linear homogenous Duffing equation in terms of Mittag-Leffler function.

We then consider in the second part of the paper solving the nonlinear case using homotopy analysis method. We found that there are more interesting cases that were not evident in the integer calculus case. We found that for the case α = 2β with 0<β<1 there are 2n poles and branch points, where $n\in\mathbb{N}$. ***We found that there are infinitely many solutions and as*** $n\to\infty\ \ \beta\to 0$ ***implicating that the solution goes to a constant as β decreases acting as an asymptote just as was observed in the linear homogeneous case (observation (iii) above).***
For the linear case (v = 0) with f(t) =0 we observe that for μ < 0 and α = 2β

- ***The solution is non-periodic***
- ***For β ϵ (0,1) the solutions exhibit the characteristics of decaying exponential function***
- ***For fixed λ solutions with lower fractional derivatives are more damped than those with higher derivatives. The solution for β = 1/3 have some distinguishing property with faster damping.***

We discuss the case for which α ≠ 2β. We found that for the case α = nβ we have a case and results similar to those discussed in [1]. For the case for which $\alpha-\beta\neq n\in\mathbb{N}$ ***we have a very non-trivial analysis for various f(t). Numerical consideration and simulations showed some chaotic nature as in [2]***
Using some existing results in the literature we show that the fractional Duffing oscillator with two fractional derivatives can be seen as an appropriate model for earthquake prediction. We then have a comparative analysis of our results and the results known for the integer calculus case. We finally engage in the stability analysis of our problem.

1.0 Introduction

The Duffing oscillator has been a long standing model for many physical phenomena.

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Most physical (real-life) problems can be efficiently modeled by second-order nonlinear differential equations. As matter is made up of smaller particles which are in continuous motion (vibration), one cannot over emphasize the importance of the study of nonlinear vibrations. The study of nonlinear vibrations is impossible without employing nonlinear differential equations (mostly second-order). One of the most famous of these nonlinear differential equations is the Duffing equations which have been presented considering various types of nonlinearity [3]. They describe many kinds of nonlinear oscillatory systems in physics, mechanics and engineering. Duffing equations with cubic and quintic nonlinearities have recently been extensively studied as extensions of the one with cubic nonlinearity in spite of its complexity in [4, 5]. Interesting results have been obtained in the geometric and analytic studies of the un-damped and unforced cubic-quintic Duffing equation using different methods [6, 2]. This un-damped and unforced cubic-quintic Duffing equation is an ideal one since in practice it is impossible to eliminate damping in dynamical systems, though it can be reduced to an extent of negligence. The solution of the nth-order Duffing equation with periodic forcing has been attempted using homotopy analysis method (HAM). Duffing oscillators of various nonlinearity and degrees of damping have been shown to exhibit chaotic characteristics in [8]

In recent years fractional calculus has found its way into many application areas in physical and engineering sciences. It's utility has found ways into many areas of applications like speech signals [9], image processing [10], and many models [11-13] and control of engineering systems [14]

Advancement of calculus of variations and optimal control to fractional dynamic systems using analytic and numerical tools and techniques has been undertaken [17], [18].

Fundamental explorations of the mechanical, electrical, and thermal constitutive relations and other properties of various engineering materials such as viscoelastic polymers, foams, gels, and animal tissues, and their engineering and scientific applications has been explored [17-21].

Fundamental understanding of wave and diffusion phenomenon, their measurements and verifications, including applications to plasma physics [17, 18] has become possible..

Bioengineering and biomedical applications, thermal modeling of engineering systems such as brakes and machine tools [14], image and signal processing [10] are areas already opened up by studies of fractional calculus.

Soltciro et al [22] showed that fractional order models capture phenomena and properties that classical integer order simply neglects.

An enthusiastic reader can gain insight into theory and applications of fractional calculus in [23-28], [17-29]. Researches that are based on the theory of fractional calculus are ongoing and it is among the expanding, booming and promising frontiers of mathematics. It is becoming obvious that greater applications of fractional calculus to human problems and particularly with problems that has some memory aspects will rule the world in no distant future. The history of fractional calculus can be seen in [30-31] and references therein.

Recently Kemfile and co-workers demonstrated the power of fractional calculus definition of linear fractional differential operators via generalized Fourier transform. The Duffing oscillator has been used as a model for prediction of earthquake [32-34]. Recently Oyesenya and Collins [33] espounced new methods of solving linear fractional differential equations relying on the basic theories in [26, 36]. In the method the generalized Mittag-Leffler function $E_\alpha(z)$ is imposed using the infinite series of components. Drozdov [35] considered fractional oscillator driven by a Gaussian noise and in [36] and [37] the dynamics and damping characteristics of a fractional oscillator were treated. The intrinsic damping of the fractional oscillator was considered in [37] while the chaotic dynamics of fractional oscillators has had its own share of treatment [38, 39]

2.0 Homotopy Analysis Method

The question the reader may be asking will possibly be on the use of HAM in solving the problem at hand. Some methods like perturbation, homotopy perturbation method (HPM), Adomian decomposition method (ADM), and variational iteration method have been used for solving nonlinear problems. HAM has the following advantage overthese other methods:

1. It is independent of any parameter, large or small
2. It provides great freedom in the choice of equation and solution type
3. It provides a convenient way of ensuring convergence of solution.

HAM is the brain child of Shijun Liao [40-45]. It has been used advantageously by many others in the solution of problems in science and engineering. It involves choosing a convenient initial guess.

Consider a differential equation

$$A\left[u(t)\right]=0 \qquad (1)$$

Where A is a nonlinear operator, u(t) an unknown function with t being time. Let $u_0(t)$ denote an initial approximation of u(t) and L denote an auxiliary linear operator with the property

$$Lf = 0 \text{ when } f = 0 \qquad (2)$$

A homotopy is then constructed as

$$H\left[\phi(t;q);q\right]=(1-q)L\left[\phi(t;q)-u_0(t)\right]+qA\left[\phi(t;q)\right] \quad (3)$$

Where q ∈ [0,1] is an embedding parameter and $\phi(t;q)$ is a function of t and q. we have

$$H\left[\phi(t;q);q\right]\Big|_{q=0}=L\left[\phi(t;0)-u_0(t)\right] \qquad q=0$$
$$H\left[\phi(t;q);q\right]\Big|_{q=1}=A\left[\phi(t;1)\right] \qquad q=1 \quad (4)$$

Thus using (2) it is clear that

$$\phi(t;0)=u_0(t)$$

is the solution of

$$H\left[\phi(t;q);q\right]\Big|_{q=0}=0$$

and

$$\phi(t;1)=u(t)$$

is the solution of the equation

$$H\left[\phi(t;q);q\right]\Big|_{q=1}=0$$

We see then that as the embedding parameter q increases from 0 to 1 the initial approximation u_0 (t) moves to u (t).
This method has been used successfully to solve many problems in engineering and science [46, 47]. It is found that HAM is more general than Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM)and the δ-expansion method [48].

3.0 Some Mathematical Preliminaries

In dealing with problems of application of fractional calculus the following functions feature prominently and we define them in this section for a smooth reading of this article.

1. Gamma function, the generalized factorial function denoted as $\Gamma(z)$
2. Mittag-Leffler function, the generalized exponential function denoted $E_{\alpha,\beta}$
3. H-function
4. Fox function

$D^{\alpha}u$ **- fractional derivative operator of non-integer order α**

Fractional calculus has its beginning in a letter dated September 30, 1695 in which L' Hospital wrote to Leibnitz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function $f(x)=\dfrac{D^n x}{Dx^n}$

- L' Hospital posed the question to Leibnitz, what would the result be if n = ½. Leibnitz response: "An apparent paradox, from which one day useful consequences will be drawn."
- In these words fractional calculus was born.
- Using their own notation and methodology many found definitions that fit the concepts of non-integer order integral or derivative. The most famous in the world of fractional calculus are the Riemann-Liouville and Grunwald-Letnikov definitions.
- Caputo reformulated the more classic definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations.
- In 1996 Kolwankar [48] reformulated again the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions.
- Some mathematical functions that occur regularly in the definitions of fractional calculus include Gamma function, Beta function, Error function, Mittag-Leffler function, Mellin-Ross function, Mijer G-function, and the Fox H-function. Also included in the understanding and applications of fractional calculus are the integral transforms – Laplace transform, Fourier transform, Mellin transform and others.

Definitions:
Let us recall some relationships from our study of integral equations.
Define

$$I_n(x)=\int_a^x (x-\xi)^{n-1} f(\xi)d\xi \quad n\in R_+, a \ (cons\tan t) \tag{5}$$

Then by differentiating

$$\frac{dI_n}{dx}=(n-1)\int_a^x (x-\xi)^{n-2} f(\xi)d\xi+\left[(x-\xi)^{n-1} f(\xi)\right]\Big|_{\xi=x}$$
$$=(n-1)I_{n-1} \quad (n>1) \tag{6}$$

If n = 1 we have $\frac{dI_1}{dx}=f(x)$.

A repeated use of the above procedure leads to the relation

$$\frac{d^k I_n}{dx^k}=(n-1)(n-2)...(n-k)I_{n-k} \ \ and$$
$$\frac{d^{n-1} I_n}{dx^{n-1}}=(n-1)!I_1(x)=(n-1)!f(x))$$

$$\int ... \int_0^t f(\tau)d\tau=\frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

Thus we have
and we now write

$$J^n f(t)=f_n(t)=\frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau \quad (*)$$

Hence by replacing the factorial expression by its gamma function equivalent we can generalize (*) for all $\alpha \epsilon R_+$ and we have

$$J^n f(t)=\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \tag{7}$$

This is Riemann-Liouville fractional integral.
As seen above the Riemann-Liouville approach derives its definition from repeated integral. Let us now approach the same problem of definition from the derivative side.
The fundamental definition of the derivative is given by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Applying this formula again, we can find the second derivative

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}$$
$$=\lim_{h_1\to 0}\frac{\lim_{h_2\to 0}\frac{f(x+h_1+h_2)-f(x+h_1)}{h_2}-\lim_{h_2\to 0}\frac{f(x+h_2)-f(x)}{h_2}}{h_1}$$

Which with $h = h_1 = h_2$ gives

$$f''(x)=\lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$

With n-repetitions of the derivative we have the expression for nth derivative

$$d^{a} f(x) = \lim_{h \to 0} \frac{1}{h^{n}} \sum_{m=0}^{\frac{t-a}{h}} (-1)^{m} \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x-mh) \quad (8)$$

This is Grunwald-Letnikov (GL) fractional derivative.

We note from this definition by GL that we can deduce some generalization. For example we have

$$\binom{-n}{m} = \frac{-n(-n-1)(-n-2)(-n-3)...(-n-m+1)}{m!}$$

This can be written as

$$\binom{-n}{m} = (-1)^{m} \frac{(n+m-1)!}{(n-1)!m!}$$

Generalizing for negative real using Gamma function we have

$$\binom{-n}{m} = (-1)^{m} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)m!} \quad (9)$$

And we have the Grunwald-Letnikov fractional integral defined by

$$d^{-\alpha} f(x) = \lim_{h \to 0} h^{\alpha} \sum_{m=0}^{\frac{t-a}{h}} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(x-mh) \quad (10)$$

It has been rigorously shown [25] that the two formulations are equivalent. But it is found that Grunwald-Letnikov definition is very easily utilized for numerical evaluation.

4.0 Caputo Derivative:

While the RL fractional integral is found using the procedure of (a) integrating the function for some m-integer and (b) differentiating for m which we refer to as LHM the procedure in the reverse order i.e. (b) differentiating the function and then (a) integrating lead us to another definition called Caputo Fractional Derivative i.e.

$$D^{\alpha} f(t) = \begin{cases} \dfrac{d^{m}}{dt^{m}}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{0}^{t} \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\right], m-1<\alpha<m \\ \dfrac{d^{m}}{dt^{m}} f(t) \qquad \alpha = m \end{cases}$$

$$D_{C}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_{0}^{t} \dfrac{f^{m}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \quad m-1<\alpha<m \\ \dfrac{d^{m}}{dt^{m}} f(t) \qquad \alpha = m \end{cases} \quad (11)$$

5.0 Some Observation:

- In Caputo definition we find that a link between what is possible and what is practical. The Caputo definition allows us to use the integer order initial conditions in the solution of fractional differential equations. In addition, the Caputo fractional derivative of a constant is zero unlike the RLD. It is interesting to note that the RLD of a constant is not zero. We shall see why this is so below.

Riemann-Liouville Derivative of the function x^{p}. By definition we have

$$\frac{d^{q}x^{p}}{dx^{q}} = \frac{1}{\Gamma(-q)}\int_{0}^{x} \frac{y^{p}}{(x-y)^{q+1}} dy \qquad for\ q<0$$

With a change of variable we have

$$\frac{d^{q}x^{p}}{dx^{q}} = \frac{x^{p-q}}{\Gamma(-q)}\int_{0}^{1} u^{p}(1-u)^{-q-1} du \qquad q<0$$

$$=\frac{x^{p-q}}{\Gamma(-q)}B(p+1,-q), \quad q<0 \quad p>-1$$

Therefore the fractional integral of x^p is, for p > 1.

$$\frac{d^q x^p}{dx^q}=\frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q} \qquad q<0, \quad p>-1 \tag{12}$$

where B(p.q) is the beta function. Therefore to find the fractional derivative we use the definition given by

$$\frac{d^q f(x)}{[d(x-a)]^q}=\frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_0^x \frac{f(y)}{(x-y)^{q-n+1}}$$

$$=\sum_{k=0}^{n-1}\frac{(x-a)^{-q+k} f^k(a)}{\Gamma(-q+k+1)}+\frac{d^{q-n} f^n(x)}{[d(x-a)]^{q-n}} \qquad for \;\; n-1<q<n$$

$$\frac{d^q x^p}{dx^q}=\frac{d^n}{dx^n}\frac{d^{q-n}x^p}{dx^{q-n}}$$

$$=\frac{d^n}{dx^n}\left[\frac{x^{p-q+n}}{\Gamma(n-q)}\int_0^1 u^p(1-u)^{n-q-1}\,du\right], \quad for \;\; 0\le q<n$$

$$=\frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q} \qquad q\ge 0, \quad p>-1$$

To obtain

$$\frac{d^q x^0}{dx^q}=\frac{\Gamma(1)}{\Gamma(\pi 1-q)}x^{-q}\neq 0$$

For p=0 (where x^p is a constant) we have
for any constant C.

$$D^\alpha C=\frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}\neq 0 \tag{13}$$

Therefore we have

$$\lim_{x\to 0^+}\frac{d^q x^p}{dx^q}=\begin{cases}0 \;\; if \;\; p<q \;\; or \;\; q=p+b\\ \Gamma(p+1) \;\; if \;\; q=p\\ \infty \qquad otherwise\end{cases} \tag{14}$$

We note also the limiting behavior of the above formula
We now look at the nature of these definitions and their usefulness in solution of physical problems
Let us see some transform:

6.0 Laplace Transform:

This is by far the most widely used and very effective procedure for solving fractional differential equations.
By definition the Laplace transform of a function f (t) of exponential order is given by

$$L\{f(t)\}:=\int_0^\infty e^{-st}f(t)dt \;=\tilde{f}(s) \tag{15}$$

Also the convolution gives

$$f(t)*g(t):=\int_0^t f(t-\tau)g(\tau)d\tau \;=g(t)*f(t)$$

Thus

$$l\{f(t)*g(t)\}=\tilde{f}(s)\tilde{g}(s) \tag{16}$$

We note also that

$$l\left\{f^{n}\left(t\right)\right\}=s^{n}\tilde{f}\left(s\right)-\sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}\left(0\right)=s^{n}\tilde{f}\left(s\right)-\sum_{k=0}^{n-1}s^{k}f^{(n-k-1)}\left(0\right)$$

For the RL fractional derivative the Laplace transform can be taken as

$$L\left\{D^{\alpha}f\left(t\right)\right\}=s^{m}\tilde{g}\left(s\right)-\sum_{k=0}^{m-1}s^{k}g^{(m-k-1)}\left(0\right)=s^{\alpha}\tilde{f}\left(s\right)-\sum_{k=0}^{m-1}s^{k}D^{(\alpha-k-1)}f\left(0\right) \tag{17}$$

We thus see that the required initial conditions are, for all k to n-1 terms, fractional order derivatives of f (t).

For the Caputo derivative we have

$$L\left\{D_{C}^{\alpha}f\left(t\right)\right\}=s^{-(m-\alpha)}\tilde{g}\left(s\right)=s^{\alpha}\tilde{f}\left(s\right)-\sum_{k=0}^{m-1}s^{\alpha-k-1}f^{(k)}\left(0\right) \tag{18}$$

In this formulation the order α does not appear in the derivatives of f(t), but rather in the preceding multiplier $s^{\alpha-k-1}$. So quite conveniently integer order derivatives of f(t) (f^{1}, f^{2}, etc) are used as the initial conditions and therefore easily interpreted from physical data and observations.

7.0 Special functions of the Fractional Calculus

Some of the special functions that occur in fractional calculus include Gamma function, Beta function and Mittag-Leffler functions. The Mittag-Leffler functions ($E_{\alpha}\left(z\right)$) plays a very significant role in the solution of fractional differential equations. We are therefore concentrating on it in this presentation.

The function given by

$$E_{\alpha}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma\left(\alpha k+1\right)} \tag{19}$$

Is the one parameter generalization of the exponential function.

A two parameter function of the Mittag-Leffler type is defined by

$$E_{\alpha,\beta}\left(z\right)=\theta\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma\left(\alpha k+\beta\right)}\quad \alpha>0,\ \beta>0 \tag{20}$$

The hyperbolic sine and cosine are particular cases of Mittag-Leffler function.

$$E_{2,1}\left(z^{2}\right)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma\left(2k+1\right)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{\left(2k\right)!}=\cosh z \tag{21}$$

The very beautiful thing about Mittag-Leffler function is that solutions of fractional differential equations can be expressed in terms of Mittag-Leffler function as we shall illustrate below.

8.0 Derivatives of Mittag-Leffler function:

The RL fractional order differentiation of the Mittag-Leffler (ML) function is [25]:

$$D_{t}^{\gamma}\left(t^{\alpha k+\beta+1}E_{\alpha,\beta}^{(k)}\left(\lambda t^{\alpha}\right)\right)=t^{\alpha k+\beta-\gamma-1}E_{\alpha,\beta-\gamma}^{(k)}\left(\lambda t^{\alpha}\right) \tag{22}$$

The Laplace transform of ML function is given as

$$L\left\{t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}\left(at^{\alpha}\right)\right\}=\frac{m!s^{\alpha-\beta}}{\left(s^{\alpha}-a\right)^{m+1}} \tag{23}$$

It thus follow that the inverse transform can be deduced from here and we have

$$L^{-1}\left[\frac{m!s^{\alpha-\beta}}{\left(s^{\alpha}-a\right)^{m+1}}\right]=t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)} \tag{24}$$

Thus for the case for which the transform is

$$Q((s) = \frac{x_1}{s^\alpha + \delta s^\beta + \mu} = \frac{x_1}{\left(s^\beta + \omega\right)^2} \quad \alpha > \beta$$

We deduce that

$$L^{-1}\left[\frac{x_1}{\left(s^\beta + \omega\right)^2}\right] = t^{\alpha-\beta} E_{\alpha,\beta}\left(-at^{=\alpha}\right) \tag{25}$$

9.0 The Linear Fractional Duffing Oscillator

Let us now consider the linear fractional Duffing oscillator with equation

$$\begin{gathered} D^\alpha q(t) + \lambda D^\beta q(t) - \mu q(t) = 0 \\ 0 < \beta < 1, \quad 1 < \alpha < 2 \end{gathered} \tag{26}$$

With initial condition q(0) =1, $D^\beta q(0) = 0$

We take Laplace transform of (26) to have

$$Q(s) = \frac{x_1 + s x_0 + \delta s^{\beta-1} x_0}{s^\alpha + \delta s^\beta - \mu} \tag{27}$$

Where Q(s) =L(q(t)) and s is the transform parameter L(u(t)) being the Laplace transform of u(t).
In studying this problem let

$$s = re^{i\theta} \tag{28}$$

We analyze the denominator of (27) with a view to getting the poles and branch points. Using (28) leads us to the equations

$$\begin{aligned} r^\alpha \cos\alpha\theta + \delta r^\beta \cos\beta\theta - \mu = 0 \\ r^\alpha \sin\alpha\theta + \delta r^\beta \sin\beta\theta = 0 \end{aligned} \qquad (29))$$

Equation (29ii) gives

$$r^\beta \left(r^{\alpha-\beta} \sin\alpha\theta + \delta \sin\beta\theta\right) = 0$$

So that for $r^\beta \neq 0$ we have

$$\begin{aligned} r^{\alpha-\beta} &= -\frac{\delta \sin\beta\theta}{\sin\alpha\theta} \\ \Rightarrow r &= \left(-\frac{\delta \sin\beta\theta}{\sin\alpha\theta}\right)^{\frac{1}{\alpha-\beta}} \end{aligned} \tag{30}$$

Substituting (30) into (29i) gives

$$\left(-\frac{\delta \sin\beta\theta}{\sin\alpha\theta}\right)^{\frac{\alpha}{\alpha-\beta}} \cos\alpha\theta + \delta\left(-\frac{\delta \sin\beta\theta}{\sin\alpha\theta}\right)^{\frac{\beta}{\alpha-\beta}} \cos\beta\theta = \mu$$

This can be written in the form

$$R \sin\left(\alpha - \beta\right)\theta = \mu \tag{31}$$

Where

$$R^2 = \left(\frac{\delta^{2\beta} \sin^{2\beta} \beta\theta}{\sin^{2\beta} \alpha\theta}\right)^{\frac{1}{\alpha-\beta}} + \delta^2 \left(\frac{\delta^{2\alpha} \sin^{2\alpha} \beta\theta}{\sin^{2\alpha} \alpha\theta}\right)^{\frac{1}{\alpha-\beta}} \tag{32}$$

So that

$$s = \left(-\frac{\delta \sin\beta\theta}{\sin\alpha\theta}\right)^{\frac{1}{\alpha-\beta}} \exp\left(\frac{i}{\alpha-\beta} \sin^{-1}\left(\frac{\mu}{R}\right)\right) \tag{33}$$

Considering (27) we take two cases viz: (i) $\mu > 0$ (ii) $\mu < 0$.
With R > 0 and considering the case $\mu > 0$ we have

$$\sin\left(\alpha - \beta\right)\theta > 0$$

Which implies $0 < \theta < \dfrac{\pi}{\alpha-\beta} < \pi$ showing that

$$0 < \theta < \pi \tag{34}$$

For the case μ< 0 we have

$$\sin(\alpha-\beta)\theta < 0$$

This implies $-\dfrac{\pi}{\alpha-\beta} < \theta < 0$ and

$$-\pi < \theta < 0 \tag{35}$$

Equations (34) and (35) show that the negative real axis is a branch cut with a branch point at 0
This gives a Hankel contour.
We note here that for α – β ϵ N particularly for α – β =1 the Hankel contour still persists. For fractional α and β we cannot calculate explicitly the branch points or poles. For only one fractional derivative this can be done as was the case in [6] and [2].

If we take $\alpha = 2\beta$ equation (24) becomes $s^{2\beta} + \delta s^{\beta} - \mu = 0$. Putting $s^{\beta} = \tau$ we have

$$\tau^2 + \delta\tau - \mu = 0 \tag{36}$$

This is a quadratic with roots

$$\tau = \frac{-\delta \pm \sqrt{\delta^2 + 4\mu}}{2} = s^{\beta} \tag{37)}$$

giving

$$s = \left(\frac{-\delta \pm \left(\delta^2 + 4\mu\right)^{1/2}}{2}\right)^{1/\beta} \tag{38}$$

This case was treated in [55].

With 0 < β <1, β will take values $\beta = \dfrac{1}{p},\ \ p > 0,\ \ p \neq 1$. We expect p ϵ N.

With $(\delta^2 + 4\mu)^{1/2} = \nu$ and denoting the values of s in (38) by $s_i^{\beta}\quad i = 1, 2$ we have

$$s_1^{\beta} = \tfrac{1}{2}\left(-\delta + \nu\right),\ s_2^{\beta} = \tfrac{1}{2}\left(-\delta - \nu\right),\quad \nu > 0\,.$$

These are complex conjugates of each other. For these cases we have

$$\left|s_i^{\beta}\right| = \tfrac{1}{2}\left(\delta^2 + \nu^2\right)^{1/2} \quad \arg s_i^{\beta} = \tan^{-1}\tfrac{d}{2}\left(\tfrac{\nu}{\delta}\right),$$
$$d = \begin{cases} -1\, for\ \ i = 1 \\ 1\, for\ \ i = 2 \end{cases} \tag{39}$$

And we have

$$s^{\beta}{}_{1,2} = \left[\tfrac{1}{2}\left(\delta^2 + \nu^2\right)^{1/2}\right]^{1/2} \exp\left[\frac{\tan^{-1}\frac{d}{2}\left(\frac{\nu}{\delta}\right) + 2k\pi}{p}\right],\ k = 0,1,\ldots \tag{40}$$

We note the following

- For μ < 0 $(\delta^2 + 4\mu)^{1/2} < \delta$
- For μ >0 $(\delta^2 + 4\mu)^{1/2} > \delta$
- Thus for μ < 0 (13) gives distinct solutions

$$q = \left[\tfrac{1}{2}\left(-\delta \pm \left(\delta^2 + 4\mu\right)^{1/2}\right)\right] \tag{41}$$

- For μ< 0

$$s_1^{\beta}=\tfrac{1}{2}\left(-\delta+\nu\right)\Rightarrow s=\sqrt[\beta]{q},q>0$$
$$s_2^{\beta}=\tfrac{1}{2}\left(-\delta-\nu\right)\Rightarrow s=\sqrt[\beta]{-r},r>0\,i.e.\left(ir\right)^{1/\beta}\quad \nu>0 \tag{42}$$

- For μ > 0

$$s_1^{\beta}=\tfrac{1}{2}\left(-\delta+\nu\right)\Rightarrow s=\sqrt[\beta]{z},z>0,0<\delta<z$$
$$s_2^{\beta}=\tfrac{1}{2}\left(-\delta-\nu\right)\Rightarrow s=\sqrt[\beta]{-p},p>0\,i.e.\left(ip\right)^{1/\beta}\quad \nu>0 \tag{43}$$

- If β = 1 we then have for μ < 0 $q^{1/2},\ q>0$ *and* $\sqrt[\beta]{-z}=ir^{1/2},r>0$, (19) and for μ > 0 we have $\sqrt[\beta]{-p}=\left(-p\right)^{1/2}=ip^{1/2},\ 0<\delta<p$ (20)

This gives same result as the normal derivative order equation.
For various values of p we have from (15) and (16)
p = 2 i.e. β = ½ gives 4 roots; p =3 gives 6 roots; p = 4 gives 8 roots and for p = n, that is β = 1/n we have 2n roots. In contrast with the purely integer values p =1 which gives only 2 roots. Thus the influence of the fractional derivative becomes evident. And since $\frac{1}{n}<\frac{1}{n-1}<\frac{1}{n-2}<......<\frac{1}{2}$ it follows that as n → ∞,β → 0 implicating that for the case α = 2β with 0 < β < 1 there are 2n poles and or branch points where n ϵ N and the solution goes to a constant as β decreases acting as an asymptote as was observed in [55].

9.0 Further Analysis:

We now give the analysis for the case α ≠ nβ. For this case we note that

$$g_{\beta}(p)=\frac{1}{ap^{\beta}+bp^{\alpha}+c}\text{ for }\beta>\alpha$$
$$=\frac{1}{c}\frac{cp^{-\alpha}}{ap^{\beta-\alpha}+b}.\frac{1}{1+\frac{cp^{-\alpha}}{ap^{\beta-\alpha}+b}}$$

can be rewritten as (44)

$$g_{\beta}(p)=\frac{1}{c}\sum_{k=0}^{\infty}\left(-1\right)^{k}\left(\frac{c}{a}\right)^{k+1}\frac{p^{-\alpha k-\alpha}}{\left(p^{\beta-\alpha}+\frac{b}{a}\right)^{k+1}}$$

This shows that for our problem, making appropriate correspondences, we have

$$Q(s)=\frac{1}{\mu}\sum_{k=0}^{\infty}\left(-1\right)^{k}\mu^{k+1}\frac{s^{-\beta k-\beta}}{\left(s^{\alpha-\beta}+\delta\right)^{k+1}} \tag{45}$$

We note that, as shown above, the equation gives the result of taking Laplace transform of (26). To get the solution to we need to take inverse Laplace transform. Doing this by contour integration process is a near-impossible task. We therefore take recourse to the results given in [27] that

$$L^{-1}\left[\frac{m!s^{\alpha-\beta}}{\left(s^{\alpha}-a\right)^{m+1}}\right]=t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}$$

to arrive at the result

$$q(t)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{k!}\mu^{k}t^{\alpha(k+1)-1}E_{\alpha-\beta,\alpha+\beta k}^{(k)}\left(-\delta t^{\alpha-\beta}\right) \tag{46}$$

We note that for α = 2 our result coincides with the result in [39].
We note that as was proved in [26] (Theorem 5.2**)**

$$y_j\left(x\right)=\sum_{n=0}^{\infty}\frac{\mu^{n}}{n!}x^{\alpha n+\alpha-j}\,{}_1\Psi_1\left[\begin{matrix}\left(n+1,1\right)\\ \left(\alpha n+\alpha+1-j,\alpha-\beta\right)\end{matrix}\middle|\lambda x^{\alpha-\beta}\right] \tag{47}$$
$$\left(j=1,...,l\right)$$

are solutions to the equation

$$\left(D_{0+}^{\alpha}y\right)(x)-\lambda\left(D_{0+}^{\beta}y\right)(x)-\mu y(x)=0$$

$$\left(x>0;l-1<\alpha\leq l;l\in\square\;;\alpha>\beta>0\right)$$

provided that the series converges.
Where

$${}_1\Psi_1\left[\begin{matrix}n+1,1\\(\alpha n+\beta,\alpha)\end{matrix}\Big|z\right]=\sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma(\alpha n+\beta+\alpha j)}\frac{z^j}{j!}=\left(\frac{\partial}{\partial z}\right)^n E_{\alpha,\beta}(z)$$

is the generalized Wright function
This shows that the solution to our equation is given by

$$y_j(x)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}x^{\alpha n+\alpha-j}\,{}_1\Psi_1\left[\begin{matrix}(n+1,1)\\(\alpha n+\alpha+1-j,\alpha-\beta)\end{matrix}\Big|(-\delta)x^{\alpha-\beta}\right]\quad(j=1,\ldots,l) \tag{48}$$

If we now consider the equation

1. $D^{\alpha}q+\delta D^{\beta}q-\mu q=f(t)$
2. $D^{\alpha}q+\delta D^{\beta}q-\mu q+\gamma q^3=f(t)$

and take $f(t)=\lambda\sin\omega t$ we have, taking Laplace transform,

$$Q(s)=\frac{x_1+sx_0+\delta s^{\beta-1}x_0}{s^{\alpha}+\delta s^{\beta}-\mu}-\lambda\frac{\omega}{\left(s^{\alpha}+\delta s^{\beta}-\mu\right)\left(s^2+\omega^2\right)} \tag{49}$$

Taking $x_0=0,\ x_1=1$ we have $Q(s)=\dfrac{1}{\mu}\displaystyle\sum_{k=0}^{\infty}(-1)^k\mu^{k+1}\frac{(1-\lambda P_0)s^{-\beta k-\beta}}{\left(s^{\alpha-\beta}+\delta\right)^{k+1}}$

Where $P_0=\omega\left(s^2+\omega^2\right)^{-1}$ and hence

$$q(t)=\frac{1}{\mu}\sum_{k=0}^{\infty}(-1)^k\mu^{k+1}\left[\frac{1}{k!}t^{\alpha(k+1)+\beta(k+1)-1}E^{(k)}_{\alpha-\beta,\alpha+\beta(2k+1)}\left(-\delta t^{\alpha-\beta}\right)\right]$$

$$-\quad-\frac{1}{\mu}\sum_{k=0}^{\infty}(-1)^k\mu^{k+1}\sum_{j=0}^{\infty}\left[\frac{(-1)^j\lambda\omega^3}{k!}t^{\alpha(k+1)+\beta(j+1)}E^{(k)}_{\alpha-\beta,\alpha+\beta k+2(j+1)}\left(-\delta t^{\alpha-\beta}\right)\right] \tag{50}$$

10.0 Stability Analysis

In this section we give some relevant preliminary stability results available in the literature which we use in our study of the stability of the linear Duffing oscillator.
We note that some stability analysis have been done for some class of fractional differential systems covering fractional delay differential equations, fractional differential equations for multi-orders, fractional differential systems with Riemann-Liouville derivative, using methods of steps and inverse Laplace transform [50-56]. The analysis in [55] and in [15] are quite illuminating.
We note that we can write equation (48) as

$$D^{\alpha}q=f\left(t,q,D^{\beta}q\right),\quad \alpha>\beta$$

$0<\beta<1,\ 1<\alpha<2$

If we consider the case $\alpha = n\beta$ with $n\in\Re_+$ then the following results become pertinent to our analysis

(a) Theorem 1.1: [17] page 169 Theorem 8.1, Theorem 8.2
 Which we now state:

Stability Theorem 1: consider the equation

$$D^{n_k}_{*0}y(x)=f\left(x,y(x),D^{n_1}_{*0}y(x),D^{n_2}_{*0}y(x),\ldots..,D^{n_{k-1}}_{*0}y(x)\right)$$

Subject to the initial conditions;

$$y^{(j)}(0)=y_0^{(j)}, \qquad j=0,1,.....,\lfloor n_k \rfloor -1$$

Where $n_k > n_{k-1} > ... > n_1 > 0, n_j - n_{j-1} \le 1 \ \forall \ j=2,3,...,k \ and \ 0 < n_1 \le 1$

Assume that $n_j \in \mathrm{Q} \ \forall \ j=1,2,....,k,$ define M to be the least common multiple of the denominators of $n_1, n_2,...., n_k$ and set $\gamma = 1/M \ and \ N = Mn_k$. Then this initial value problem is equivalent to the system of equations

$$D_{*0}^{\gamma} y_0(x) = y_1(x)$$
$$D_{*0}^{\gamma} y_1(x) = y_2(x)$$
$$.$$
$$.$$
$$.$$
$$D_{*0}^{\gamma} y_{N-2}(x) = y_{N-1}(x)$$
$$D_{*0}^{\gamma} y_{N-1}(x) = f\left(x, y_0(x), y_{n_1/\gamma} ... y_{n_{k-1/\gamma}}(x)\right)$$

Together with the initial conditions

$$y_j(0) = \begin{cases} y_0^{(j/M)} \ if \ j/M \in \mathbb{N}_0 \\ 0 \ elsewhere \end{cases}$$

In the following sense

1. Whenever $Y := (y_0,....,y_{N-1})^T \ with \ y_0 \in C^{\lceil n_k \rceil}[0,b]$ for some b > 0 is the solution of the system, the function $y := y_0$ solves the multi-term equation initial value problem
2. Whenever $y \in C^{\lceil n_k \rceil}[0,b]$ is a solution of the multi-term initial value problem the vector function

$$Y := (y_0,....,y_{N-1})^T := \left(y, D_{*0}^{\gamma} y, D_{*0}^{2\gamma} y,....., D_{*0}^{(N-1)\gamma} y\right)^T$$

Solves the multidimensional initial value problem.
Proof: See[17] page 169)

Stability Theorem 2.

Consider the linear system of fractional differential equations

$$D_*^{\alpha} x(t) = Ax(t), \quad x(0) = x_0 \qquad (+)$$

Where

$$x \in \mathbb{R}^n, \ A \in \mathbb{R}^n \times \mathbb{R}^n, \ \alpha = [\alpha_1, \alpha_2, ..., \alpha_n], \ indicating \ fractional \ order$$
$$D_*^{\alpha} = [D_*^{\alpha_1}, D_*^{\alpha_2}, ..., D_*^{\alpha_n}] \ \ 0 < \alpha_i \le 1, \ i = 1,2,...,n$$

If $\alpha = \alpha_1 = \alpha_2 = ... = \alpha_n$ then the system (+) is asymptotically stable iff $\left|\arg(spec(A))\right| < \frac{\alpha\pi}{2}$. In this case the components of the state decay towards 0 like $t^{-\alpha}$. For the case where the α_i's are rational numbers between 0 and 1, for i =1, 2, …,n, let $\gamma = 1/m$ where m is the least common multiple of the denominators $m_i \ of \ \alpha_i's, \ where \ \alpha_i = \frac{k_i}{m_i}, \ k_i, m_i \in \mathbb{N}, \ i = 1,2,...,n$ the system (+) is asymptotically stable if all the roots λ of the equation $\det\left(diag\left(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, ..., \lambda^{m\alpha_n}\right) - A\right) = 0$ satisfy $\left|\arg(\lambda)\right| > \gamma\pi/2$.

Proof: See [55] and [56].
We now analyse the stability of our problem (22).
First we note that this equation (22) is equivalent to the following system taking α = nβ

$$D^{\beta} q = q_1$$
$$D^{\beta} q_1 = q_2$$
$$D^{\beta} q_2 = q_3$$
.
.
.
$$D^{\beta} q_{n-1} = -\delta q_1 + \mu q$$

Thus

$$D^{\beta}\begin{pmatrix} q \\ q_1 \\ q_2 \\ . \\ q_{n-1}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \\ \mu & -\delta & 0 & . & 0 \end{pmatrix}\begin{pmatrix} q \\ q_1 \\ q_2 \\ . \\ q_{n-1}\end{pmatrix}$$

Using Laplace transform we have the characteristic matrix of the system given by

$$\lambda^{n} + \delta\lambda - \mu = 0$$

For α = 2β we have

$$D^{\beta}\begin{pmatrix} q \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mu & -\delta \end{pmatrix}\begin{pmatrix} q \\ q_1 \end{pmatrix}$$

P = [- $P = [-\lambda \quad 1 \quad 0; 0 \quad -\lambda \quad 1; \mu \quad -\delta \quad -\lambda]$

The characteristic equation is given by $m^2 + \delta m - \mu = 0$ yielding instability when subjected to the stability theorem test. This case was discussed in [53]

11.0 The Nonlinear Problem

We now consider the nonlinear problem

$$D^{\alpha} q + \delta D^{\beta} q - \mu q + \gamma q^3 = f(t) \tag{51}$$

Beginning with the homogenous case f (t) = 0. We define

$$\varepsilon = \frac{\gamma}{m}, \quad \varepsilon << 1$$

And our equation (29) becomes

$$D^{\alpha} q + \delta D^{\beta} q + \mu q + \varepsilon q^3 = f(t) \tag{52}$$

We define a perturbation series

$$q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + ... \tag{53}$$

And substitute into equation (29) with f(t) = 0 and equate powers of ϵ to have

$$\begin{aligned} L_{\beta}^{\alpha} q_0 &= 0 \\ L_{\beta}^{\alpha} q_1 &= -q_0^3 \\ L_{\beta}^{\alpha} q_2 &= -3q_0^2 q_1 \\ L_{\beta}^{\alpha} q_3 &= -3q_0 q_1^2 - 3q_0^2 q_2 - 3q_0 q_1^2 - 6q_0 q_1 q_2 \end{aligned} \tag{54}$$

Where

$$L_{\beta}^{\alpha} q_i \equiv D^{\alpha} q_i + \delta D^{\beta} q_i - \mu q_i \tag{55}$$

We note that the solution to (54a) is given by (23) namely,

$$q_0(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mu^k t^{\alpha(k+1)-1}E^{(k)}_{\alpha-\beta,\alpha+\beta k}\left(-\delta t^{\alpha-\beta}\right) \tag{56}$$

Substituting (56) into (54b) and solving we have

$$q_1(t)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mu^k t^{\alpha(k+1)-1}E^{(k)}_{\alpha-\beta,\alpha+\beta k}\left(-\delta t^{\alpha-\beta}\right)+q_{1p} \tag{57}$$

where q $_{1p}$ is the particular integral for (32b).

12.0 Some Particular Cases

We deal with the linearly damped fractional oscillator taking α = 2β

$$\boldsymbol{D^{2\beta}q + \lambda D^{\beta}q - \mu q = 0 \quad 0 < \beta} < 1 \tag{58}$$

With the initial condition $\mathbf{q(0) = 1}$

Using Caputo's fractional derivative formulation and taking the Laplace transform of the equation we have

$$s^{2\beta}Q(s)-s^{2\beta-1}q(0)+\lambda s^{\beta}Q(s)-\lambda s^{\beta-1}q((0)-\mu Q(s)=0$$

giving

$$Q(s)=\frac{s^{2\beta-1}+\lambda s^{\beta-1}}{s^{2\beta}+\lambda s^{\beta-1}-\mu} \tag{59}$$

For us to be able to find the inverse Laplace transform, we resolve (56) into partial fractions and we have

$$Q(s)=\frac{k_1}{s^{\beta}-\alpha_1}+\frac{k_2}{s^{\beta}-\alpha_2} \tag{60}$$

Where

$$k_i=(-1)^{i+1}\frac{(\alpha_i)^{2-\frac{1}{\beta}}+\lambda(\alpha_i)^{1-\frac{1}{\beta}}}{\alpha_1-\alpha_2} \quad i=1,2 \tag{61}$$

Giving, on taking the inverse Laplace transform,

$$q(t)=\sum_{l=1}^{2}k_l t^{\beta-1}E_{\beta,\beta}\left(\alpha_l t^{\beta}\right) \tag{62}$$

Restricting ourselves to the case with real and distinct roots, the characteristic equation is

$$\tau^2+\lambda\tau-\mu=0$$

Where $\tau=s^{\beta}$ with the roots as

$$\tau_{1,2}=\frac{-\lambda\pm\sqrt{\lambda^2+4\mu}}{2} \tag{63}$$

The roots are always real and distinct since the radicand $\lambda^2 + 4\omega^2 > 0$, this is consistent with our assumption. The analysis and simulations are contained in Figures 1 – 2. From where we draw our conclusion.

13.0 The Stability Analysis

Writing equation (55) as

$$\begin{aligned} &D^{\alpha}q=y\\ &D^{\alpha}y=-\delta y+\mu q\\ &D^{\alpha}\begin{pmatrix}q\\ y\end{pmatrix}=\begin{pmatrix}0 & 1\\ \mu & -\delta\end{pmatrix}\begin{pmatrix}q\\ y\end{pmatrix}\end{aligned} \tag{64}$$

We can do a stability analysis of the linear Duffing oscillator

Equation (61) is in the form

$$D^{\alpha}x(t)=Ax(t),\quad x(t)=(\mathrm{q(t)},y(t))^T$$

With the eigenvalues of A given by

$$\lambda_{1,2} = -\tfrac{1}{2}\left(\delta \mp \sqrt{\delta^2 + 4\mu}\right)$$

A clear case of instability ensues. This is the same as for α = 1 when δ > 0 and δ < 0.
We note that our results in (34) and (35) above can be used to discuss the stability of our problem using the stability theorem 2 and we can conclude that the solutions are asymptotically stable.

14.0 Conclusion

We observe that

(i) For β ϵ (0,1) the solution exhibit the feature of a decaying exponential function.
(ii) For fixed λ solutions with lower fractional derivatives are more damped than those with higher derivatives.
(iii) The solution for β = 1/3 have a distinguishing property of faster damping crossing the solution for other values (β = 0.7, 0.8, 0.9) at t = 1.47.

In fact we observe that the solutions for the values of β ϵ [0.3,0.9) meet at t = 1.47 (Figure 1) showing there exists a critical value and just as it is observed in the analysis of fractional Langevin equation there may be resonance in this range. We note that the integer analysis does not reveal this aspect. In [2]such observation is given and there is a resonance range. Figure 2 shows the chaotic nature of the solution with time. Our simulations for the analysis of this problem under the homotopy analysis method are not included in this Part 1 of this article. There are very interesting results under this analysis that will be exhibited in Part II of our analysis.

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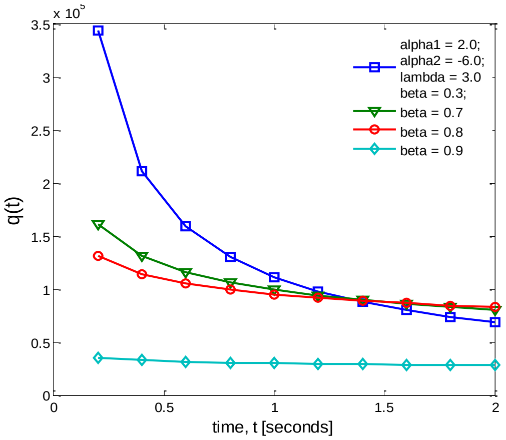

Figure 1:

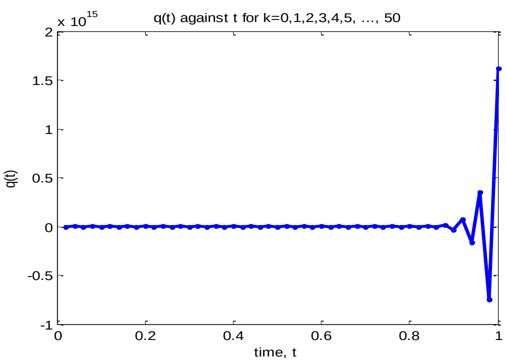

Figure 2: Chaotic nature of solution.