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On the Core and Some Isotopic Characterisations of Generalised Bol Loops

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Abstract

In this work, a study of properties of the core of a generalised Bol loop is carried out. Isotopic characterisation of generalised Bol loops is examined. It is shown that the set of semi-automorphisms of the core of the generalised Bol loop (G, ·) are the automorphisms of the core (G, +) which fixed the identity element of (G, ·). It is also proved that isotopic generalised Bol loops have isomorphic cores. The necessary conditions for the core to be an isotopic invariant for a generalised Bol loops are also obtained.

1.0 Introduction

Loops $(G,\cdot)$ satisfying the identity

$$(xy \cdot z)y = x(yz \cdot y) \quad (1)$$

$for\ all\ x, y, x\ \in G$ are called right Bol loop. These loops are algebraic duals of the loops $(G,\cdot)$ satisfying the identity

$$y(z \cdot yx) = (y \cdot zy)\, x \quad (2)$$

$for\ all\ x, y, x\ \in G$ called left Bol loops.

Moufang loops are found to satisfy the identities (1) and (2) simultaneously.

The notion of the core in loops was first introduced by R.H. Bruck [9] in connection with invariants of classes of isotopic Moufang loops. If $(G,\cdot)$ is a Moufang loop, Bruck defined the core$(G,\cdot)$ as the groupoid $(G,+)$ consisting of the elements of G under the binary operation '+' defined by $x + y = xy^{-1}$ for all $x, y \in G$ where y^{-1} is the inverse of y in G. Robinson[11] carried out a study on the core of Bol loops. If the Bol loop is Moufang then core of the Bol loop is also Moufang.

The object of this present study is to examine the core of generalised Bol loops using procedures analogous to those employed by Robinson[11] in his work on the core of Bol loops. Isotopic characterisation of generalised Bol loops is also investigated. It is shown that the set of semi-automorphisms of the generalised Bol loop $(G,\cdot)$ are automorphisms of the core $(G,+)$ which fixes the identity of $(G,\cdot)$. Isotopic generalised Bol loops are also shown to isomorphic cores. The necessary conditions for the core to be an isotopic invariant for generalised Bol loops are also formulated.

2.0 Preliminaries

Theorem 2.1: If$(G,\cdot)$is a generalised Bol loop with identity 1, then the following hold:

1. $(G,\cdot)$satisfies the right inverse property, that is $(xy)y^{\rho} = x$, where y^{ρ} is defined by $y \cdot y^{\rho} = 1$ for all $y \epsilon G$.
2. The left and right inverse elements y^{γ} and y^{ρ} coincide, that is $y^{\gamma} = y^{\rho} = y^{-1}$
3. $(G,\cdot)$ satisfies the generalised right alternative law, that is $(xy)y^{\alpha} = x(yy^{\alpha})$

Theorem 2.2: A loop $(G,\cdot)$ is generalised Bol loop if and only if for $x \epsilon G$

$$(R(x)^{-1}, L(x)R(x^{\alpha}), R(x^{\alpha}))$$

Is an autotopism of $(G,\cdot)$.

Theorem 2.3: If $(G,\cdot)$ is a generalised Bol loop, then

$$(xy \cdot x^{\alpha})^{-1} = (x^{\alpha})^{-1}y^{-1} \cdot x^{-1}$$

For all $x, y \epsilon G$.

Lemma 2.4: Let $(G,\cdot)$ be a quasigroup and let $f, g \epsilon\ G$ then for all $x, y \epsilon\ G$

$$x^{\circ}y = xR(g)^{-1} \cdot yL(f)^{-1} \quad (3)$$

then$(G, ^{\circ})$is a loop with identity $f \cdot g$ and$(G,\cdot)$and$(G, ^{\circ})$are isotopic.

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Remark 2.5: A loop $(G,\circ)$obtained from a quasigroup$(G,\cdot)$ in the manner described in Lemma 2.4 is called a principal loop isotope of $(G,\cdot)$.
Lemma 2.6.: If a quaisgroup$(G,\circ)$ and a loop $(H,\circ)$ are isotopic then $(H,\circ)$ is isomorphic to a principal isotope of $(G,\cdot)$.

Let $(G,\cdot)$ be a loop which satisfies the right inverse property and let $f, g\epsilon\ G$, define $x^\circ y$ as in (3). Since $f\cdot g$ is the identity for $(G,\circ)$, for each $x\epsilon\ G$ let $x^{\rho\sigma}$ denote that unique element of G such that $x^\circ x^{\rho\sigma}=f.g$. Define mappings L,R,J and R° by

$xL=x^\gamma, xR=x^\rho, xJ=x^{-1}, xR^\circ=x^{\rho\sigma}$ (4)

For all$x\epsilon G$. Then for all $x\epsilon G$,

$$xR(g)^{-1}\cdot xR^\circ L(f)^{-1}=f\cdot g$$

and by right inverse property of $(G,\cdot)$

$$xR(g)^{-1}=(f\cdot g)\cdot(xR^\circ L(f)^{-1}R)$$

implying that

$$xR(g)^{-1}=(xR^\circ L(f)^{-1}R)L(f\cdot g)$$

hence we have

$$R^\circ=\ R(g)^{-1}L(f\cdot g)^{-1}L(f)$$

if$g=f^{-1}$ and the left inverse mapping L is replaced with inverse mapping J we have

$R^\circ=R(f)JL(f)$ (5)

Theorem 2.7: Let $(G,\cdot)$ be a generalised Bol loop. A loop isotope $(H,\circ)$ of $(G,\cdot)$ is a generalised Bol loop if there exists a self-map of $(H,\circ)$ that fixes its identity element.
Theorem 2.8: Let $(G,\cdot)$ be a generalised Bol loop. Any loop isotope $(H,\circ)$ of $(G,\cdot)$ with identity element $e$is an α −generalised Bol loop if and only if e^α is in the right nucleus of H.
Definition 2.1: A mapping θ of a generalised Bol loop $(G,\cdot)$ into itself is called a semi-endomorphism of$(G,\cdot)$ if

$(yx.y^\alpha)\theta=(y\theta.x\theta)(y^\alpha)\theta\ and\ 1\theta=1$ (6)

For further definition and properties of concepts in Theory of Loops employed in this paper readers are to consult Bruck[9] and Pflugfelder [10].

3.0 Main Results

Theorem 3.1: If (G,·) a generalised Bol loop with core $(G,+)$, then the following hold:

i. $y+y=y^\alpha$
ii. $y+y^\alpha=y$

Proof:

i. For every $y\epsilon G$, we have $y+y=yy^{-1}\cdot y^\alpha=y^\alpha$
ii. For every $y\epsilon G$, we have that $y+y^\alpha=y(y^\alpha)^{-1}\cdot y^\alpha$
If $(y^\alpha)^{-1}=z$ then$y^\alpha=z^{-1}$. By right inverse property it follows that $yz.z^{-1}=y$ hence we have $y+y^\alpha=y$■

Theorem 3.2: If (G,·)is a commutative generalised Bol loop with core $(G,+)$, then $(G,+)$ satisfies

$$x+(x+y)=y,\qquad for\ all\ x,y\epsilon G$$

Proof:$x+(x+y)=x(xy^{-1}\cdot x^\alpha)=x(xy^{-1}\cdot x^\alpha)^{-1}\cdot x^\alpha$which by commutativity is

$$x((x^\alpha)^{-1}y\cdot x^{-1})x\cdot x^\alpha=(x^\alpha)^{-1}y\cdot x^\alpha=y(x^\alpha)^{-1}\cdot x^\alpha=y$$■

In the remaining part of this section, consideration would be given to the case in which the mapping $\alpha\colon y\to y^\alpha$ commutes with the set of bijections defined on the generalised Bol loop $(G,\cdot)$, that is $\alpha T=T\alpha$ for every bijection$T$ on $(G,\cdot)$.
Theorem 3.3: If θ is a semi-endomorphism of a generalised Bol loop $(G,\cdot)$ then

$$(y^{-1})\theta=(y\theta)^{-1}\qquad\qquad for\ all\ y\epsilon\ G$$

Proof: Setting$x=y^{-1}$ in (6) above, we have, $(y^\alpha)\theta=(yy^{-1}.y^\alpha)\theta=(y\theta y^{-1}\theta).(y^\alpha)\theta$
Implying that $y\theta y^{-1}\theta=1$ which again imply that$y^{-1}\theta=(y\theta)^{-1}$■
Theorem 3.4: If the mapping α fixes the identity element of $(G,\cdot)$ then the semi-automorphisms of $(G,\cdot)$ are the automorphisms of $(G,+)$ which fixed the identity element of $(G,\cdot)$.
Proof: Let 1 be the identity element of $(G,\cdot)$ and suppose $\alpha\colon 1\to 1$. If θ is a semi-automorphism of $(G,\cdot)$, then $(x+y)\theta=(xy^{-1}\cdot x^\alpha)\theta=(x\theta\cdot(y^{-1})\theta)\cdot(x^\alpha)\theta\ =(x\theta\cdot(y\theta)^{-1})(x\theta)^\alpha=(x\theta+y\theta)$ hence θ is an automorphism of $(G,+)$.
Suppose θ is an automorphism of $(G,+)$ such that $1\theta=1$, then $y^{-1}\theta=(1+y)\theta=1\theta+y\theta=1+y\theta=(y\theta)^{-1}$.
Also, $(xy\cdot x^\alpha)\theta=(x+y^{-1})\theta=x\theta+(y\theta)^{-1}=(x\theta)(y\theta)\cdot(x\theta)^\alpha=(x\theta)(y\theta)\cdot(x^\alpha)\theta$
Hence θ is a semi-automorphismof $(G,\cdot)$.■
Lemma 3.1: Let $(G,\cdot)$ be a generalised Bol loops and let $f\in G$.

$$x^\circ y=xR(f)\cdot yL(f)^{-1}$$

for all $x,y\in G$ that is $(G,\circ)$ is a principal isotope of $(G,\cdot)$. Let T be a permutation of G and let $(G,+)$and $(G,\mp)$ be the cores of $(G,\cdot)$and $(G,\circ)$ respectively. Then

$$xT \breve{\mp} yT = (x+y)T \quad (7)$$

for all $x, y \in G$ if and only if

$$[(fx \cdot y^{-1}) \cdot x^{\alpha}]T^{-1} = [(fx)T^{-1} \cdot (fy)T^{-1}J] \cdot (fx^{\alpha})T^{-1} \quad (8)$$

for all $x, y \in G$ where $J: x \to x^{-1}$.

Proof: Define R as in (4) above and using the fact that

$$R^{\circ} = R(f)JL(f)$$

then

$$x\breve{\mp}y = (x^{\circ}y^{-1})^{\circ}x^{\alpha} = (x^{\circ}yR^{\circ})^{\circ}x^{\alpha} = [xR(f) \cdot yR^{\circ}L(f)^{-1}]R(f) \cdot x^{\alpha}L(f)^{-1}$$
$$= [xR(f) \cdot yR(f)JL(f)L(f)^{-1}]R(f) \cdot x^{\alpha}L(f)^{-1} = [xR(f) \cdot yR(f) \cdot x^{\alpha}L(f)^{-1}]$$

then

$$xT\breve{\mp}yT = (x+y)T$$

holds for all $x, y \in G$ if and only if

$$[xTR(f) \cdot yTR(f)J]R(f) \cdot (xT)^{\alpha}L(f)^{-1} = (xy^{-1} \cdot y^{\alpha})T$$

for all $x, y \in G$. Replacing x by $xL(f)T^{-1}$ and y by $yR(f)^{-1}T^{-1}$, then the last equation hold if and only if

$$(xL(f)R(f) \cdot y^{-1}R(f)) \cdot \left(xL(f)\right)^{\alpha}L(f)^{-1} = \{[xL(f)T^{-1} \cdot yR(f)^{-1}T^{-1}J] \cdot (xL(f)T^{-1})^{\alpha}\}T$$

for all $x, y \in G$. On the assumption that $\alpha T = T\alpha$ for every bijection T on $(G,\cdot)$ then the last equation holds if and only if

$$(xL(f)R(f) \cdot y^{-1}R(f)) \cdot x^{\alpha} = \{[xL(f)T^{-1} \cdot yR(f)^{-1}T^{-1}J] \cdot (x^{\alpha}L(f)T^{-1})\}T$$

for all $x, y \in G$. Using the Bol identity (1), this can be rewrittenas

$$[(fx \cdot f)y^{-1} \cdot f)] \cdot x^{\alpha} = \{[(fx)T^{-1} \cdot [f(f^{-1}y \cdot f^{-1})]T^{-1}J] \cdot (fx^{\alpha})T^{-1}\}T$$

for all $x, y \in G$. By applying T^{-1} to both sides, we have

$$\{[(fx)(fy^{-1} \cdot f)]x^{\alpha}\}T^{-1} = \{(fx)T^{-1} \cdot [f(f^{-1}y \cdot f^{-1})]T^{-1}J\} \cdot (fx^{\alpha})T^{-1}$$

for all $x, y \in G$. If we set $f^{-1}y \cdot f^{-1} = z$, implying that$fy^{-1} \cdot f = z^{-1}$ then we have

$$[(fx \cdot z^{-1})x^{\alpha}]T^{-1} = [(fx)T^{-1} \cdot (fz)T^{-1}J] \cdot (fx^{\alpha})T^{-1}$$

which is (8) aside from change of notation hence (7) holds for all $x, y \in G$ if and only if (8) holds for all $x, y \in G$. ■

Theorem 3.5: Isotopic generalised Bol loops have isomorphic cores.

Proof: Let $(G,\cdot)$ be a generalised Bol loop. In view of Lemma 2.6 only the principal isotopes of $(G,\cdot)$ would be considered, i.e. isotopes $(G, \circ)$ of the form

$$x^{\circ}y = xR(f) \cdot yL(f)^{-1}$$

for all $x, y \in G$. Let $(G, +)$ and $(G, \breve{\mp})$ be the cores of $(G,\cdot)$ and $(G, \circ)$ respectively and let $J: y \to y^{-1}$. Since $(G,\cdot)$is a generalised Bol loop, then $(fx \cdot y^{-1})x^{\alpha} = f(xy^{-1} \cdot x^{\alpha})$for all $x, y \in G$. This implies that $(fx \cdot y^{-1})x^{\alpha} = (xy^{-1} \cdot x^{\alpha})L(f)$for all $x, y \in G$. Applying $L(f)^{-1}$ on both sides we obtain$[(fx \cdot y^{-1})x^{\alpha}]L(f)^{-1} = xy^{-1} \cdot x^{\alpha} = xyJ \cdot x^{\alpha}$ for all $x, y \in G$. This implies that $[(fx \cdot y^{-1})x^{\alpha}]L(f)^{-1} = [(fx)L(f)^{-1} \cdot (fy)L(f)^{-1}J] \cdot fx^{\alpha}L(f)^{-1}$ for all $x, y \in G$. If we set $T = L(f)$ we have $[(fx \cdot y^{-1})x^{\alpha}]T^{\alpha} = [(fx)T^{-1}J] \cdot (fx^{\alpha})T^{-1}$ for all $x, y \in G$ hence by Lemma 3.1

$$xT\breve{\mp}yT = (x+y)T$$

for all $x, y \in G$. i.e. $(G, \breve{\mp})$ and$(G, +)$ are isomorphic.■

Corollary 3.1: The core is an isotopic invariant for generalised Bol loops if for each loop isotope $(H, \circ)$ of a generalised Bol loop $(G,\cdot)$ there exist a mapping β in the class of mappings α defined such that β fixes the identity element of $(H, \circ)$.

Proof: The core is an isotopic invariant for a loop if its core is isomorphic to the core of every of its loop isotope. Let $(H, \circ)$be a loop isotope of a generalised Bol loop $(G,\cdot)$ and let $(H, \breve{\mp})$and$(G, +)$ be the cores of $(H, \circ)$ and $(G,\cdot)$ respectively. Suppose there exists a mapping β in the class of mappings α defined such that

$$\beta: e^{\circ} \to e^{\circ}$$

wheree° is the identity element of $(H, \circ)$. In view of Theorem 2.7 $(H, \circ)$ is also a generalised Bol loop. Therefore, by Theorem 3.5 we have that

$$xT\breve{\mp}yT = (x+y)T$$

holds for every loop isotope $(H, \circ)$ of $(G,\cdot)$. ■

Corollary 3.2: The core is an isotopic invariant for generalised Bol loops if every loop isotope $(H, \circ)$ of a generalised Bol loop $(G,\cdot)$ with identity element e, there exists a mapping γ in the class of mappings α defined such that e^{γ} is an element of the right nucleus of H.

Proof: Suppose $(H, \circ)$ is a loop isotope of a generalised Bolloop $(G,\cdot)$, let $(H, \breve{\mp})$and$(G, +)$ be cores of $(H, \circ)$ and $(G,\cdot)$ respectively. Also, suppose there exists a mapping γ in the class of mappings α defined such that e^{γ} is an element of the right nucleus of H where e^{γ} is the image of the identity element e of H under the mapγ. It follows from Theorem 2.8 that $(H, \circ)$ is also a generalised Bol loop. Therefore the result follows from Theorem 3.5.■

Theorem 3.6: If$(G,\cdot)$ is a generalisedBol loop such that for each principal isotope $(G, \circ)$of $(G,\cdot)$ the condition $x + y = x\breve{\mp}y$ holds for all for all $x, y \in G$ where $(G, +)$and$(G, \breve{\mp})$ are the core of $(G,\cdot)$and $(G, \circ)$ respectively then $(G,\cdot)$ is a group.

Proof: Suppose $(G,\cdot)$ is generalised Bol loop such that $x+y=x\mp y$ for all $x,y\in G$. Then by Lemma 3.1 with the bijection$T$taken to be the identity map$I$, we have $(fx\cdot z^{-1})x^{\alpha}=[(fx)(fz)^{-1}](fx^{\alpha})$ for all $f,x,z\in G$. If we set $z=f^{-1}$, we have $(fx\cdot f)x^{\alpha}=(fx)(fx^{\alpha})$which implies that $x^{\alpha}L(fx.f)=(fx^{\alpha})L(fx)$ which implies that $x^{\alpha}L(fx.f)=x^{\alpha}L(f)L(fx)$ which implies that $L(fx.f)=L(f)L(fx)$ for all $f,x,\in G$. This shows that $(G,\cdot)$ is associative hence a group. ■

Isotopy Characterisation of Generalised Bol loops

An Isotopism of $(G,\cdot)$ unto $(G,\cdot)$ is called an autotopism. The set of all autotopism a loop $(G,\cdot)$ forms a group with "componentwise multiplication"

$$(U_1,U_2,U_3)(V_1,V_2,V_3)=(U_1V_1,U_2V_2,U_3V_3)$$

The identity element of this group is (I,I,I) where $xI=x$ for all $x\in G$ and $(U,V,W)^{-1}=(U^{-1},V^{-1},V^{-1})$

Theorem 3.7: A loop $(G,\cdot)$ is a generalised Bol loop if and only if $A=(R(y),L(x)R(y^{\alpha})^{-1}L(xy)^{-1},R(y^{\alpha})^{-1})$ is an autotopism of$(G,\cdot)$.

Proof: By definition $(G,\cdot)$ is a generalised Bol loop if and only if

$$x(yz\cdot y^{\alpha})=(x\cdot yz)y^{\alpha})$$

If and only if

$$L(yz)L(y^{\alpha})=L(y)R(y^{\alpha})L(x)$$

Pre-multiplying and post multiplying both sides of this equation by $R(y^{\alpha})^{-1}L(y)^{-1}$ and $R(y^{\alpha})^{-1}L(xy)^{-1}$ then the latter hold if and only if

$$R(y^{\alpha})^{-1}L(y)^{-1}=L(x)R(y^{\alpha})^{-1}L(xy)^{-1} \quad (9)$$

Now if $(G,\cdot)$ is a generalised Bol loop

$$B=(R(y)^{-1},L(y)R(y^{\alpha}),R(y^{\alpha}))$$

is an autotopism of $(G,\cdot)$. Taking the inverse of B, we have

$$B^{-1}=(R(y),R(y^{\alpha})^{-1}L(y)^{-1},R(y^{\alpha})^{-1})$$

which is also an autotopism. Using (9) we have

$$B^{-1}=(R(y),L(x)R(y^{\alpha})^{-1}L(xy)^{-1},R(y^{\alpha})^{-1}) \quad (10)$$

Conversely, suppose

$$A=(R(y),L(x)R(y^{\alpha})^{-1}L(xy)^{-1},R(y^{\alpha})^{-1})$$

is an autotopism of $(G,\cdot)$ for all $x,y\in G$. Let $L(x)R(y^{\alpha})^{-1}L(xy)^{-1}=Y$ then by using (10) on any $a,b\in G$ we have

$$aR(y)\cdot bY=abR(y^{\alpha})^{-1}$$

This implies that $ay\cdot bY=abR(y^{\alpha})^{-1}$. For $a=1$, we have $y\cdot bY=bR(y^{\alpha})^{-1}$ which implies that $bYL(y)=bR(y^{\alpha})^{-1}$ and this shows that $bY=bR(y^{\alpha})^{-1}L(y)^{-1}$ hence

$$Y=R(y^{\alpha})^{-1}L(y)^{-1}$$

Hence we have

$$L(x)R(y^{\alpha})^{-1}L(xy)^{-1}=R(y^{\alpha})^{-1}L(y)^{-1}$$

implying that $(G,\cdot)$ is a generalised Bol loop.■

Corollary 3.3: If $(G,\cdot)$ is a generalised Bol loop, then

$$(R(y),L(x)R(y^{\alpha})^{-1}L(xy)^{-1},R(y^{\alpha})^{-1})$$

Is an autotopismof$(G,\cdot)$.

Proof: Since $(G,\cdot)$ is a right inverse property loop, then

$$R(y)^{-1}=R(y^{-1})$$

for all $y\in G$. The result follows from Theorem 3.7. ■

Theorem 3.8: Let $(G,\cdot)$ be a generalised Bol loop, let $f\in G$, and let $u\circ v=uR(f)\cdot vL(f)^{-1}$for all $u,v\in G$ i.e. $(G,\circ)$ is a principal isotope of $(G,\cdot)$. Then $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if there exists a pseudo-automorphism with companion f^{α}.

Proof:$(G,\cdot)$ and $(G,\circ)$ are isomomorphic if and only if there exists a permutation T of G such that $uT\circ vT=(u\cdot v)T$ for all $u,v\in G$. Equivalently, this holds if and only if $uTR(f)\cdot vTL(f)^{-1}=(u\cdot v)T$ for all $u,v\in G$. The last statement holds if and only if $\beta=(TR(f),TR(f)^{-1},T)$ is an autotopism of $(G,\cdot)$. By Theorem 2.2 $(R(f)^{-1},L(f)R(f^{\alpha}),R(f^{\alpha}))$ is an autotopism of $(G,\cdot)$ hence β is an autotopism of $(G,\cdot)$ if and only if $\gamma=\beta(R(f)^{-1},L(f)R(f^{\alpha}),R(f^{\alpha}))$ is an autotopism of $(G,\cdot)$. But $\gamma=(T,TR(f^{\alpha}),TR(f^{\alpha}))$. This completes the proof.■

Corollary 3.5 Let $(G,\cdot)$ be a generalised Bol loop, let $f\in G$ and let $u\circ v=uR(f)\cdot vL(f)^{-1}$ for all $u,v\in G$. If $f^{\alpha}\in N_{\rho}$, whereN_{ρ} is the right nucleus of $(G,\cdot)$, then $(G,\cdot)$ and $(G,\circ)$ are isomorphic.

Proof: $f^{\alpha}\in N_{\rho}$ implies that $x\cdot yf^{\alpha}=xy\cdot f^{\alpha}$ for all $x,y\in(G,\cdot)$. This holds if and only if $xI\cdot yR(f^{\alpha})=xyR(f^{\alpha})$ for all $x,y\in(G,\cdot)$ where I is the identity map on G or equivalently if and only if $\big(I,R(f^{\alpha}),R(f^{\alpha})\big)$ is an autotopism of $(G,\cdot)$. Hence by Theorem 3.8 $(G,\cdot)$ and $(G,\circ)$ are isomomorphic.■

Theorem 3.9: If $(G,\cdot)$is a generalised Bol loop then $y^{\alpha} \in N_{\rho}$ if and only if $y \in N_{\mu}$.
Proof: Suppose $y^{\alpha} \in N_{\rho}$ then $(xy \cdot z)y^{\alpha} = x(yz \cdot y^{\alpha}) = (x.yz)y^{\alpha}$ for all $x, z \in (G,\cdot)$hence $xy \cdot z = x \cdot yz$ for all $x, z \in (G,\cdot)$ therefore $y \in N_{\mu}$.
Conversely, suppose $y \in N_{\mu}$ then $x(yz \cdot y^{\alpha}) = (xy \cdot z)y^{\alpha} = (x.yz)y^{\alpha}$for all $x, z \in G$. If $yz = q$ we have $x.qy^{\alpha} = xq.y^{\alpha}$ therefore $y^{\alpha} \in N_{\rho}$ ■
Corollary 3.5: If $(G,\cdot)$ is a generalised Bol loop with $f \in G$ and let $u^{\circ}v = uR(f) \cdot vL(f)^{-1}$ for all $u, v \in G$. If $f \in N_{\mu}$ whereN_{μ} is the middle nucleus of $(G,\cdot)$ then $(G,\cdot)$ and $(G,^{\circ})$ are isomorphic.
Proof: Suppose $f \in N_{\mu}$, then by Theorem 3.9 $f^{\alpha} \in N_{\rho}$ and the result follows from Corollary 3.4. ■

4.0 References

[1] J.O. Adeniran (1997), *On Generalised Bol loop identity and related loop identities*, M.Sc. Thesis, Obafemi Awolowo University, Ile-Ife, Nigeria.

[2] J.O. Adeniran& S.A. Akinleye (2001), *On some loops satisfying the Generalised Bol identity, Nig. Jour. Sc.*35, 101-107.

[3] J.O. Adeniran& T.G. Jaiyeola (2009), *On isotopic Characterisation of central loops,* Creative Math. & Inf., 18(1), 39-45.

[4] J.O. Adeniran, T.G. Jaiyeola& K.A. Idowu (2014), *Holomorph of generalized Bol loops*, Novi Sad J. Math., 44(1), 37-51.

[5] J.O. Adeniran & A.R.T. Solarin (1999), *A Note on Generalised Bol identity,* An. Stiint Univ. "Al.I.Cuza", Iasi, 45(2), 225-230

[6] N. Ajmal (1978), *A generalisation of Bol loops*, Ann. Soc. Sci. BruxellesSer 1, 92(4), 241-248

[7] V.D. Belousov (1938), *Foundations of theory of quasigroups and loops,* Nauka, Moscow

[8] G. Bol (1937), *Gewebe und Gruppen*, Math. Ann. 144, 414-431

[9] R.H. Bruck (1966), *Asurvey of Binary Systems,* Springer-Verlag, Berlin-Gottingen-Heidelberg

[10] H.O. Pflugfelder (1990), *Quasigroups and Loops: Introduction,* HeldermaanVerlag, Berlin

[11] D.A. Robinson (1964), *Bol loops*, Ph.D. Thesis, University of Wisconsin, Madison

[11] B.L. Sharma & L.V. Sabinin (1976), *On the existence of Half Bol loops,* An. Stiint Univ. "Al.I. Cuza", Iasi, 22(2), 147-148.

[12] B.L. Sharma & L.V. Sabinin (1979), On the algebraic properties of half Bol loops, Ann. Soc. Sci. BruxellesSer 1, 93(4), 227-240.