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Another Characterization of Local Completeness

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Abstract

A separated locally convex space is locally complete if and only if every lower semicontinuous seminorm is bounded.

Incidentally, we also establish

(i) a seminorm analogue of [9, Definition 13-1-4 of the inductive limit topology], THEOREM 2.3, and

(ii) the counterpart of [4, IMPORTANT CONSEQUENCE 5.7] for $\beta^*(E, E')$, THEOREM 3.9.

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1.0 Introduction

All topologies are assumed separated, and by a *lcs* (E, τ) shall be meant a separa- ted locally convex space, with continuous dual E'. The field of scalars of our spaces shall be $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$, the reals or the complex numbers. All notation and terminologies shall be standard as found, for example, in [9], [3], [2], [5] and [4]. /// signifies the end or absence of a proof; and if τ_1, τ_2 are topologies on $X \neq \varnothing$, by $\tau_1 \geq \tau_2$ is meant that τ_1 is finer than τ_2.

2.0 $1\tau^{\Re}$

An absolutely convex bounded subset B of the lcs (E, τ) is called a *disc* of (E, τ), and the seminormed space (E_B, q_B), where E_B is the linear span of B in E and q_B the Minkowski functional of B in E_B, is a normed space[5, third paragraph]. We denote by σp, following [4, Section 1], the pseudometric topology of the semi- norm $p : E \to \mathbb{R}$; and so for disc B of (E, τ), $(E_B, \sigma q_B)$ is a lcs.

Let $\Re$ be a collection of discs of lcs (E, τ) satisfying the condition that $\bigcup_{B\in\Re} E_B$ spans E. Denote by $\tau^{\Re}$ the *inductive limit*[9, Definition 13-1-4, p.210] *topology* on E by the inclusions $i_{E_B} : (E_B, \sigma q_B) \to E$, $B \in \Re$.

We note the following two lemmas.

LEMMA 1 [3, Paragraph following Definition 2.6.1, p.108] Let A and B be non-empty subsets of a vector space X such that A is balanced. Then, A absorbs B if and only if there exists $\mu \in \mathbf{K}$ such that $B \subseteq \mu A$. ///

LEMMA 2 [3, Fifth and sixth lines p.208] For disc B of an lcs (E, τ), the lcs $(E_B, \sigma q_B)$ has $\{\varepsilon B : \varepsilon > 0\}$as a base of neighbourhoods of zero. ///

With notation and language as in the paragraph preceding LEMMA 1, immediate from LEMMAS 1 and 2 and [9, Theorem 13-1-11, p.211] is

THEOREM 3 A base of neighbourhood of zero of $\tau^{\Re}$ is the collection of all absolutely convex subsets V of E such that V absorbs every $B \in \Re$. ///

COROLLARY 4 Members of $\Re$ are $\tau^{\Re}$ -bounded

Proof [9, Problem 4-4-3, p.49]. ///

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LEMMA 5 [7, Proposition 3.2.2, p.82] For any disc B of lcs (E, τ), $\tau|E_B \le \sigma q_B$. i.e., the restriction of τ to E_B is coarser than σq_B. ///

FACT 6 $\tau \le \tau^{\mathfrak{R}}$. i.e., τ is coarser than $\tau^{\mathfrak{R}}$.

Proof From LEMMA 5 follows that τ is a *test topology*[9, Definition 13-1-1, p.209]. Therefore, $\tau \le \tau^{\mathfrak{R}}$. ///

FACT 7 $\tau^{\mathfrak{R}}$ is the finest locally convex topology on E with respect to which the members of $\mathfrak{R}$ are bounded.

Proof By COROLLARY 4, members of $\mathfrak{R}$ are $\tau^{\mathfrak{R}}$-bounded. Suppose τ' is another locally convex topology on E w.r.t which the members of $\mathfrak{R}$ are bounded. Then, clearly, immediate from definition, $\tau'^{\mathfrak{R}} = \tau^{\mathfrak{R}}$. But $\tau' \le \tau'^{\mathfrak{R}}$ by BACT 6, and so $\tau' \le \tau'^{\mathfrak{R}} = \tau^{\mathfrak{R}}$. ///

Clearly, from the proof of FACT 7 above is

THEOREM 8 If $\mathfrak{R}$ is a collection of discs of a dual pair $\langle E, E'\rangle$[4, Section 2][9, Theorem 8-4-1, p.114] such that $\bigcup_{B\in\mathfrak{R}} E_B$ spans E, the topology $\tau^{\mathfrak{R}}$ is duality invariant. That is, if τ and τ' are topologies of a dual pair $\langle E, E'\rangle$, then $\tau^{\mathfrak{R}} = \tau'^{\mathfrak{R}}$. ///

Let (E, τ) be a lcs and D the collection of **all** the discs of (E, τ). Clearly, for any $x \in E$, noting that a singleton is bounded[3, Second paragraph, p.109] and convex, and that [9, Problem 7-1-1, p.93 : the absolutely convex hull of a bounded set is bounded], it follows that there exists $B_x \in$ D such that $x \in B_x$; and so $\cup\{E_B : B \in \text{D}\}$ spans E. Then, by FACT 7, noting that a subset of a bounded set is bounded and that a subset bounded w.r.t a topology remains bounded w.r.t a coarser topology, τ^{D} is the finest locally convex topology on E having same bounded sets as (E, τ). τ^{D} is usually denoted τ^b. By [3, Exercises 3.7.8, p.226] τ^b is called the *bornological*[3, Definition 3.7.1, p.220] *topology associated with* τ. We note this in

FACT 9 For lcs (E, τ),

(i) τ^b is the finest locally convex topology on E with the same bounded sets as τ, and

(ii) τ^b is the finest locally convex topology with the same local null[5, third paragraph] sequences as τ.

Proof (ii) is immediate from (i) and the definition of a local null sequence. ///

3.0 2 $\tau^{\mathfrak{R}}$ and Generating Seminorms

Let (E_α, τ_α), $\alpha \in I \ne \emptyset$, be lcss and E a linear space. For each $\alpha \in I$, let $f_\alpha : E_\alpha \to E$ be a linear map. Suppose $\bigcup_{\alpha\in I} f_\alpha(E_\alpha)$ spans E. Recall that a locally convex topology τ on E is called a test topology if f_α is $(\tau_\alpha$-$\tau)$-continuous for each $\alpha \in I$. We have the following.

LEMMA 1 Let p be a seminorm on E. Then, the topology σp of the seminorm p on E, is a test topology $\Leftrightarrow$ the compositions $pof_\alpha : E_\alpha \to \mathbb{R}$, $\alpha \in I \ne \emptyset$, are continuous.

Proof We have $(E_\alpha, \tau_\alpha) \xrightarrow{f_\alpha} E \xrightarrow{p} \mathbb{R}$, $\alpha \in I$. By [9, Problem 4-5-1(a) $\Leftrightarrow$ (b), p.55], $(E, \sigma p) \xrightarrow{p} \mathbb{R}$ is continuous, and so if σp is a test topology on E, $(E_\alpha, \tau_\alpha) \xrightarrow{f_\alpha} (E, \sigma p)$ is continuous for each α, and therefore, pof_α is continuous for each α. For the implication $\Leftarrow$ let $B_p = \{x \in E : p(x) < 1\}$, and so for $\varepsilon > 0$, $\varepsilon B_p = \{z \in E : p(z) < \varepsilon\}$. A base of neighbourhoods of zero of the topology σp of p on E is $\{\varepsilon B_p : \varepsilon > 0\}$. If we now show that $f_\alpha^{-1}(\varepsilon B_p)$ is a neighbourhood of zero in (E_α, τ_α) for each $\alpha \in I$ and each $\varepsilon > 0$, then σp is a test topology [9, Problem 4-1-1, p.39]. Now

$$f_\alpha^{-1}(\varepsilon B_p) = f_\alpha^{-1}(p^{-1}((-\varepsilon, \varepsilon))) = (p \circ f_\alpha)^{-1}((-\varepsilon, \varepsilon))$$

which by the hypothesis of continuity of each composition pof_α, is a neighbourhood of zero of (E_α, τ_α). Hence, σp is a test topology. ///

LEMMA 2 Denote by $\text{ind·lim}(f_\alpha, \tau_\alpha)_{\alpha\in I}$ the inductive limit topology[9, Definition 13-1-4, p.210] by the linear maps f_α. Let

$p : (E, \text{ind·lim}(f_\alpha, \tau_\alpha)_{\alpha\in I}) \to \mathbb{R}$

be a seminorm. p is continuous $\Leftrightarrow$ the composition pof_α is continuous for each $\alpha \in I$, which by LEMMA 1, $\Leftrightarrow$ σp is a test topology on E.

Proof We have $E_\alpha \xrightarrow{f_\alpha} (E, \text{ind·lim}(f_\alpha, \tau_\alpha)_{\alpha\in I}) \xrightarrow{p} \mathbb{R}$. Clearly p is continuous implies pof_α is continuous for each $\alpha \in I$. Suppose pof_α is continuous for each $\alpha \in I$. By LEM- MA 1, σp is a test topology and so $\text{ind·lim}(f_\alpha, \tau_\alpha)_{\alpha\in I} \ge \sigma p$. And so by [9, Problem 4-5-1, p.55], $p : (E, \text{ind·lim}(f_\alpha, \tau_\alpha)_{\alpha\in I}) \to \mathbb{R}$ is continuous. ///

Since a locally convex topology is generated by its set of continuous seminorms [9, first line of first paragraph after Definition 7-2-3, p.94], we have a seminorm analogue of the Definition 13-1-4, p.210 of [9].

THEOREM 3 ind·lim(f_α, $\tau_\alpha)_{\alpha\in I}$ = $\vee\{\sigma p$: p is a seminorm on E, pof_α is continu-ous on (E_α, τ_α) for each $\alpha \in I$ } = $\vee\{\sigma p$: σp is a test topology on $E\}$. ///

Suppose $\mathfrak{R}$ is a collection of discs of lcs (E, τ). A sequence $(x_n)_{n=1}^{\infty}$ in E shall be called $\mathfrak{R}$-*null* provided $(x_n)_{n=1}^{\infty}$ is a null sequence in (E_B, σq_B) for some $B \in \mathfrak{R}$.

If E is a vector space, $\varnothing \neq B \subseteq E$ and $p : E \to \mathbb{R}$ a seminorm, we shall say that *p is bounded on B* provided $p(B)$ is a bounded subset of $\mathbb{R}$. If $(x_n)_{n=1}^{\infty}$ is a sequence in E, we shall say that *p is bounded on* $(x_n)_{n=1}^{\infty}$ if it is bounded on its range [|Compare the defini- tion : $(x_n)_{n=1}^{\infty}$ is *bounded* in the lcs (E, τ) provided its range is a bounded set of (E, τ). **FACT**: A convergent sequence is a bounded sequence. This FACT is employed in the proof of (v) $\Rightarrow$ (iv) of THEOREM 7 and THEOREM 3.3 |].

If (E, τ) is a lcs and $p : E \to \mathbb{R}$ a seminorm, we shall call p a *bounded seminorm* provided p is bounded on every bounded subset B of (E, τ)[| Compare Definition 4-4-2, p.47 of [9]|].

LEMMA 4[9, Problem 4-4-1, p.49] Let $p : (E, \tau) \to \mathbb{R}$ be a continuous seminorm on an lcs (E, τ). Then, p is a bounded seminorm. ///

We need a modification or do we say a corollary of [9, Theorem 4-4-1, p.47] to be able to prove one of the implications in the proof of THEOREM 7 below. We state it in

THEOREM 5 The following are equivalent for a set $S \neq \varnothing$ in a topological vector space (E, τ).

(i) S is bounded

(ii) For every sequence $(x_n)_{n=1}^{\infty}$ in S and every null sequence of scalars $(\varepsilon_n)_{n=1}^{\infty}$, we have that $(\varepsilon_n x_n)_{n=1}^{\infty}$ is null in (E, τ).

(iii) For every sequence $(x_n)_{n=1}^{\infty}$ is S, sequence $(\frac{1}{n^2} x_n)_{n=1}^{\infty}$ is null in (E, τ).

Proof That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is already given in [9, Theorem 4-4-1, p.47], and so we only need prove (iii) $\Rightarrow$ (i). *Proof* : Assume the opposite that S is not bounded. Hence, there exists a balanced neighbourhood of zero V, say, of (E, τ) that does not absorb S. And so, by LEMMA 1.1, $S \nsubseteq n^2V$ for all positive integer n. Therefore, there exists a sequence $(x_n)_{n=1}^{\infty}$ in S such that $\frac{1}{n^2} x_n \notin V$ for all n, and so $(\frac{1}{n^2} x_n)_{n=1}^{\infty}$ cannot be eventually in V. ///

Let $p : E \to \mathbb{R}$ be a seminorm on a vector space E. Then, the pseudometric topology σp of p on E is a locally convex topology and so (E, σp) is a locally convex space. p induces on E a pseudometric dp defined by $dp(a, b) = p(a - b)$, a , $b \in E$. A sub-set $B \neq \varnothing$ of E is said to be *metrically bounded* provided it is a bounded subset of the pseudometric space (E, dp). i.e., there exists $x_0 \in E$ and $\lambda > 0$ such that $dp(x, x_0) < \lambda$ for all $x \in B$. This is clearly equivalent to, employing the triangle inequality property of p, requiring that there exists $\lambda > 0$ such that

$$p(x) = p(x - \theta) = dp(x, \theta) < \lambda \text{ for all } x \in B \qquad \ldots(*)$$

THEOREM 6 [9, Second claim of Problem 4-4-1, p.49] Boundedness in the locally convex space (E, σp) is same as metric boundedness

Proof The family $\{\varepsilon Bp : \varepsilon > 0\}$[| $Bp = \{z \in E : p(x) < 1$|] is a base of neigh- bourhoods of zero of (E, σp). And so, by [2, (17.2), p.68] and LEMMA 1.1,

B is σp-bounded $\Leftrightarrow$ there exists $\lambda > 0$ such that $B \subseteq \lambda Bp$[= $\{z \in E : p(z) < \lambda\}$ =$\{z \in E : p(z - \theta) < \lambda\}$], which in turn is by (*) above

$\Leftrightarrow$ B is metrically bounded. ///

Now we have

THEOREM 7 Let $\mathfrak{R}$ be a collection of discs of the lcs (E, τ) such that $\bigcup_{B\in\mathfrak{R}} E_B$ spans E. Consider the inclusions i_{E_B} : (E_B, σq_B) $\to E$ and the inductive limit topology $\tau^{\mathfrak{R}}$ by these inclusions. Let $p : E \to \mathbb{R}$ be a seminorm on E. The following are equivalent.

(i) The restriction of p to E_B is σq_B-continuous for each $B \in \mathfrak{R}$.

(ii) p is $\tau^{\mathfrak{R}}$-continuous.

(iii) p is bounded on members of $\mathfrak{R}$.

(iv) p is bounded on $\mathfrak{R}$-null sequences.

(v) p maps $\mathfrak{R}$-null sequence to null sequences.

Proof We have $(E_B,\ \sigma q_B) \xrightarrow{i_{E_B}} (E, \tau^{\Re}) \xrightarrow{p} \mathbb{R}$.

(i) ⇔ (ii) : Immediate from LEMMA 2.

(ii) ⇒ (iii) : By COROLLARY 1.4, members of $\Re$ are $\tau^{\Re}$-bounded, and so this impli- cation is then immediate from LEMMA 4.

(iii) ⇒ (ii) : Suppose p is bounded on every member of $\Re$ and so there exists, by THEO- REM 6 and the paragraph preceding it, $\lambda_B > 0$ for every $B \in \Re$, such that $p(x) < \lambda_B$ for a- ll $x \in B$. And so, $B \subseteq \lambda_B Bp$ [| $Bp = \{x \in E : p(x) < 1\}$|]. From this follows that $Bp \supseteq \frac{1}{\lambda_B} B$, and by LEMMA 1.2, $Bp \cap E_B$ is a neighbourhood of zero of $(E_B,\ \sigma q_B)$ for every $B \in \Re$. By [9, Theorem 13-1-11, p.211], therefore, Bp is a neighbourhood of zero of $\tau^{\Re}$. Hence, by [9, Problem 4-5-1, p.55] $\sigma p \le \tau^{\Re}$ and p is $\tau^{\Re}$-continuous.

(v) ⇒ (iv): Immediate from definitions as convergent sequences in $(\mathbb{R}, |\ |)$ are bounded.

(iii) ⇒ (v): Suppose $(x_n)_{n=1}^{\infty}$ is null in $(E_B,\ \sigma q_B)$ for some $B \in \Re$. Therefore, by LEMMA 1.2, for $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $x_n \in \varepsilon B$ for all $n \ge N(\varepsilon)$.
Therefore, $p(x_n) \in \varepsilon p(B)$ for all $n \ge N(\varepsilon)$ and so $p(x_n) \in \varepsilon\Gamma p(B)$ for all $n \ge N(\varepsilon)$. [| ΓM = the absolutely convex hull of M |]. By hypothesis, $p(B)$ is bounded and so by [9, Problem 7-1-1, p.93 : the absolutely convex hull of a bounded set is bounded] $\Gamma p(B)$ is bounded. Hence, since ε was arbitrary, it follows from LEMMA 1.2 that $(p(x_n))_{n=1}^{\infty}$ is null in $(\mathbb{R}_{\Gamma p(B)},\ \sigma q_{\Gamma p(B)})$, where $\mathbb{R}$ is the reals. By LEMMA 1.5, therefore, $(p(x_n))_{n=1}^{\infty}$ is null in $(\mathbb{R}, \sigma|\cdot|)$.

(iv) ⇒ (iii): This is the place where we need THEOREM 5. Assume that p is bounded on $\Re$-null sequences. Suppose $B \in \Re$. Let $(y_n)_{n=1}^{\infty}$ be a sequence in $p(B)$, and so $y_n = p(x_n)$ for some $x_n \in B$. Since $x_n \in B$, $\frac{1}{n} x_n \in \frac{1}{n} B$. Let $\varepsilon > 0$. By a property of the real numbers $\mathbb{R}$ [1, Corollary 2.5, p.40] there exists a positive integer $N(\varepsilon)$ such that $\frac{1}{N(\varepsilon)} < \varepsilon$. By another property [1, Exercise 2.1.15, p. 30], $\frac{1}{n} < \frac{1}{N(\varepsilon)}$ for all $n > N(\varepsilon)$. And so, for all $n > N(\varepsilon)$, by [2, (17.2), p.68].

$\frac{1}{n} x_n \in \frac{1}{n} B \subseteq \frac{1}{N(\varepsilon)} B \subseteq \varepsilon B$.

Since ε was arbitrary, it follows from LEMMA 1.2 that the sequence $(\frac{1}{n} x_n)_{n=1}^{\infty}$ is null in $(E_B,\ \sigma q_B)$. By hypothesis, therefore, p is bounded in $(\frac{1}{n} x_n)_{n=1}^{\infty}$. i.e., $\{\ p(\frac{1}{n} x_n)\ :\ n = 1, 2, 3, \ldots\}$ is a bounded set of $\mathbb{R}$, and so from elementary Analysis, $\frac{1}{n} p(\frac{1}{n} x_n) \to 0$ as $n \to \infty$. That is, $\frac{1}{n^2} p(x_n) \to 0$ as $n \to \infty$. By THEOREMS 5 and 6, taking cognizance of the metric locally convex space $(\mathbb{R}, \sigma|\cdot|)$, p is bounded on B. ///

Since the collection of seminorms continuous on the lcs (E, τ) generates its topology τ, we have

COROLLARY 8 Let $\Re$ be a collection of discs of (E, τ) such that $\bigcup_{B\in\Re} E_B$ spans E. Then,

$\tau^{\Re} = \vee\{\sigma p : p$ is a seminorm on E, p is bounded on members of $\Re\}$
$= \vee\{\sigma p : p$ is a seminorm on E, p is bounded on $\Re$-null sequences$\}$
$= \vee\{\sigma p : p$ is a seminorm on E, p maps $\Re$-null sequence to null sequences$\}$
$= \vee\{\sigma p : p$ is a seminorm on E, the restriction of p to $(E_B, \sigma q_B)$ is continuous for each disc $B \in \Re\}$ (That is, the restriction of p to E_B is σq_B-continuous). ///

COROLLARY 9

$\tau^b = \vee\{\sigma p : p$ is a seminorm on E, p is a bounded seminorm$\}$
$= \vee\{\sigma p : p$ is a seminorm on E, p is bounded on local null sequences$\}$
$= \vee\{\sigma p : p$ is a seminorm on E, p maps local null sequence to null sequences$\}$
$= \vee\{\sigma p : p$ is a seminorm on E, the restriction of p to $(E_B, \sigma q_B)$ is continuous for each disc $B\}$.

3 $\tau^{buc} = \beta^*(E, E')$ Let (E, τ) be a lcs. For the definition of $\beta^*(E, E')$ we need some clarification. Let $\varnothing \ne X$ and $\Re$ a family of subsets of X. A subfamily $\Re^*$ of $\Re$ is called a *fundamental subfamily* of $\Re$ if for every $B \in \Re$ there exists $B^* \in \Re^*$ such that $B^* \supseteq B$[3, 4th paragraph, p.109][6, Paragraph preceding Example 13.21, p. 144].

Example 1 Let (E, τ) be a lcs and $\Re$ the collection of all the bounded sets of (E, τ). Then, the collection D of all the discs of (E, τ) is a fundamental subfamily of $\Re$. *Proof* : [9, Problem 7-1-1, p.93 : the absolutely convex hull of a bounded set is bounded]. ///

THEOREM 2 Let (E, τ) be a lcs and $\Re$ a family of bounded sets of (E, τ). If $\Re^*$ is a fundamental subfamily of $\Re$, then $\tau_{uc(\Re)} = \tau_{uc(\Re^*)}$.

Proof For $B \in \Re$, there exists $B^* \in \Re^*$ such that $B^* \supseteq B$, and so, clearly, employing the notation of [4, FACT 4.2], $p_{B^*} \geq p_B$. Hence, by [2, Lemma (37.11), p.149] $\sigma p_{B^*} \geq \sigma p_B$. Therefore, $\tau_{uc(\Re)}$ does not have more generators (subbase) than $\tau_{uc(\Re^*)}$, and so

$$\tau_{uc(\Re^*)} \geq \tau_{uc(\Re)} \qquad \ldots(\Delta)$$

But $\Re^* \subseteq \Re$, and so

$\cup\{\sigma p_{B^*} : B \in \Re^*\} \subseteq \cup\{\sigma p_B : B \in \Re\}$.

Hence, again, this time, $\tau_{uc}(\Re)$ has a bigger subbase than $\tau_{uc(\Re^*)}$. Therefore,

$$\tau_{uc(\Re^*)} \leq \tau_{uc(\Re)} \qquad \ldots(\Delta\Delta)$$

From (Δ) and $(\Delta\Delta)$ follows that

$$\tau_{uc(\Re^*)} = \tau_{uc(\Re)}. \ ///$$

Now for the lcs (E, τ) consider the dual pair $<E, E'>$. If $\Re$ = the $\beta(E', E)$-bou- nded subsets of E' and $\Re^*$ the absolutely convex $\beta(E', E)$-bounded subsets of E', then, by the preceding THEOREM 2, and [9, Problem 7-1-1, p.93], $\tau_{uc(\Re^*)} = \tau_{uc(\Re)}$. Denote this topology[7, Paragraph preceding 0.3.1, p.2][9, Remark 10-1-3, p.150][3, Exercise 3.6.5, p.220] by $\beta^*(E, E')$.

Let (E, τ) be a lcs and consider the dual pair $<E, E'>$. Consider the finest topology of uniform convergence τ^{buc} w.r.t the dual pair $<E, E'>$, having same bounded sets as τ. *CLAIM* : τ^{buc} exists. *Proof of CLAIM* : Immediate from [4, Theorem 4.7] and [9, theorem 4-4-5.p.48, noting that τ itself is a topology of uniform con- vergence w.r.t $<E, E'>$]. /// Clearly, all the neighbouhoods of zero of τ^{buc} are τ-bor- nivores. Indeed, being a topology of uniform convergence, τ^{buc} has a base of neighbour- hoods of zero consisting of τ-barrels[4, FACT 4.4] which are of necessity τ-bornivores. Well-known is that $\beta^*(E, E')$ has a base of neighbourhods of zero comprisi-ng *all* the bornivore barrels of (E, τ)[7, Observation 3.1.5(c), p.82][9, Lemma 10-1-5 p.150]. Hence $\beta^*(E, E')$ and τ have same bounded [|Note by [3, Exercise 3.6.5(a), p.220] that $\tau \leq \beta^*(E, E')$|] sets and

$$\tau^{buc} \leq \beta^*(E, E') \qquad \ldots(*)$$

But, $\beta^*(E, E')$ being a topology of uniform convergence, by the maximality of τ^{buc},

$$\beta^*(E, E') \leq \tau^{buc} \qquad \ldots(**)$$

By (*) and (**)

$\tau^{buc} = \beta^*(E, E')$.

So, we have

THEOREM 3 $\tau^{buc} = \beta^*(E, E')$ is the finest topology on E of uniform convergence w.r.t $<E, E'>$ having same bounded sets as τ. Clearly, $\beta^*(E, E')$ is the finest topology of un- iform convergence having same local null sequences as τ.[By the definition of a local null sequence – See third paragraph of 5]. ///

By a *local sequential neighbourhood of zero* of lcs (E, τ) shall be meant an absolutely convex subset of E in which every local null sequence eventually lies.

FACT 4 [7, Proposition 5.1.3(ii), p.151] Let (E, τ) be a lcs. The sequence $(x_n)_{n=1}^{\infty}$ in E is τ-local null if and only if there exists an unbounded sequence $(\alpha_n)_{n=1}^{\infty}$ of real scalars, $\alpha_n > 0$ for all n, such that $(\alpha_n x_n)_{n=1}^{\infty}$ is a τ-null sequence. ///

THEOREM 5 Suppose B is a barrel of lcs (E, τ). The following are equivalent.

(i) B is a local sequential neighbourhod of zero of (E, τ).

(ii) B is a bornivore.

That is, the bornivere barrels are the barrels that are local sequential neighbourhoods of zero

Proof(i) ⇒ (ii): Immediate from [5, LEMMA 1].

(ii) ⇒ (i): Suppose the barrel B is a bornivore. Let $(x_n)_{n=1}^{\infty}$ be a local null sequence and so, by FACT 4, there exists an increasing unbounded sequence $(\alpha_n)_{n=1}^{\infty}$, $\alpha_n > 0$ for all n, such that $(\alpha_n x_n)_{n=1}^{\infty}$ is a null sequence, and so

$$\{\alpha_n x_n : n \in \mathbb{N}\} \qquad \ldots(\Delta)$$

is a bounded set. Since B is a bornivoure, it absorbs (Δ) and so by LEMMA 1.1 for some $\lambda > 0$,

$\alpha_n x_n \in \lambda B$, for all n.

That is,

$x_n \in \frac{\lambda}{\alpha_n} B$, for all n.

For some positive integer N, $\frac{\lambda}{\alpha_n} < 1$, for all $n \geq N$. Hence by [2, (17.2), p.68],

$x_n \in B$, for all $n \geq N$, and so B is a local sequential neighbourhood of zero . ///

OBSERVATIONS 6 If in the lcs (E, τ) the sequence $(x_n)_{n=1}^{\infty}$ is local null, then for scalar $\lambda > 0$ the sequence $(\lambda x_n)_{n=1}^{\infty}$ is also local null. *Proof* : by FACT 4 $(\alpha_n x_n)_{n=1}^{\infty}$ is null for some unbounded increasing sequence $(\alpha_n)_{n=1}^{\infty}$ of real positive scalars. The sequence $(\frac{1}{\lambda}\alpha_n)_{n=1}^{\infty}$ is also clearly increasing unbounded. Clearly $(\frac{1}{\lambda}\alpha_n)\lambda = \alpha_n$, and so

$((\frac{\alpha_n}{\lambda})\lambda x_n)_{n=1}^{\infty} = (\alpha_n x_n)_{n=1}^{\infty}$ is local null.

Hence, the sequence $(\lambda x_n)_{n=1}^{\infty}$ is local null, again by FACT 4. *COROLLARY* : If V is a local sequential neighbourhood of zero, then λV, for scalar $\lambda > 0$, is also a local sequential neighbourhood of zero. *Proof* : Suppose $(x_n)_{n=1}^{\infty}$ is local null in (E, τ), and so, by the preceding $(\frac{1}{\lambda}x_n)_{n=1}^{\infty}$ is local null. Hence, for some positive integer $N(\lambda)$,

$\frac{1}{\lambda}x_n \in V$ for all $n \geq N(\lambda)$,

from which follows that

$x_n \in \lambda V$ for all $n \geq N(\lambda)$.///

Let (E, τ) be a lcs and q a seminorm on E. Call q a *local sequentially continuous seminorm* if $(q(x_n))_{n=1}^{\infty}$ is a null sequence whenever $(x_n)_{n=1}^{\infty}$ is locally null in (E, τ). It is clear from THEOREM 2.7 that the local sequentially continuous seminorms are the bounded seminorms

We have

THEOREM 7 The following are equivalent for the lcs (E, τ).

I Every lower semicontinuous local sequentially continuous seminorm is continuous

II. Every barrel which is a local sequential neighbourhood of zero of (E, τ) is a neighbourhood of zero of (E, τ)

Proof 1⇒II: Assume **I**. Let B be a barrel of (E, τ) which is a local sequential neighbourhood of zero of (E, τ). By [4, Lemma 6.3], $B = \{x \in E : q_B(x) \leq 1\}$, where q_B is the Minkowski functional of B. By [4, Theorem 5.2], therefore, q_B is lower semicontinuous. Since B is a local sequential neighbourhood of zero of (E, τ), by the COROLLARY in OBSERVATIONS 6, $\varepsilon B = \{x \in E : q_B(x) \leq \varepsilon\}$, for any $\varepsilon > 0$, is a local sequential neighbourhood of zero of (E, τ). From this clearly follows that q_B is local sequentially continuous. By hypothesis, therefore, q_B is τ-continuous. Hence, by [8, Lemma II.11.2, p.106], B is a neighbourhoood of zero in (E, τ).

I1⇒ I: Assume **II**. Let q be a lower semicontinuous local sequentially continuous semi- norm on (E, τ). Then, by [4, Theorem 5.2], $Ucd\text{-}q = \{x \in E : q(x) \leq 1\}$is a barrel and so since q is local sequentially continuous, $Ucd\text{-}q$ is a local sequential neighbourhood of zero. By hypothesis, therefore, $Ucd\text{-}q$ is a neighbourhood of zero of (E, τ) and so again by [8, Lemma II.11.2, p.106], q is continuous. ///

Call lcs (E, τ) a *quasibarrelled* space if $\tau = \beta^*(E, E') = \tau^{buc}$

COROLLARY 8 For lcs (E, τ), the following are equivalent.

(i) (E, τ) is quasibarrelled [i.e., $\tau = \beta^*(E, E')$].

(ii) Every bornivore barrel is a nieghbourhood of zero of (E, τ).

(iii) Every barrel which is a local sequential neighbourhood of zero of (E, τ) is a neighbourhood of zero of (E, τ).

(iv) Every lower semicontinuous local sequentially continuous seminorm is continuous.

Proof (i) ⇔ (ii) is by [7, Observation 3.1.5(c), p.82].

(ii) ⇔ (iii) is by THEOREM 5.

(iii) ⇔ (iv) is by THEOREM 7. ///

We have

THEOREM 9 For lcs (E, τ),

$\tau^{buc} = \beta^*(E, E') = \vee(\sigma p : p$ is a lower semicontinous local sequentially continuous seminorm on $(E, \tau)\}$.

$= \vee(\sigma p : p$ is a lower semicontinous bounded seminorm$\}$.

Proof By [4, FACT 4.5 and THEOREM 5.2] and [COROLLARY in OBSERVATIONS 6, noting that $\mu B = |\mu| B$ for balanced B and scalar μ] and the proof of II of THEOREM 7, any topology on E generated by a collection, P, say, of lower semicontinuous local sequentially continuous seminorms has a base of barrels that are local sequential neighbourhoods of zero. And so, by THEOREM 5 and [7, Observa- tion 3.1.5(c), p.82] such a topology is coarser than $\beta^*(E, E') = \tau^{buc}$. Hence, if

$\tau^* = \vee\{\sigma p : p$ is a lower semicontinous local sequentially continuous seminorm on $E\}$,
then

$$\tau^* \leq \tau^{buc} \qquad \ldots(\Delta)$$

By [4, Main Theorem 5.4], τ^{buc} is generated by a collection Q of lower semicontinuous seminorms. For $q \in Q$, $Ucd\text{-}q = \{x \in E : q(x) \leq 1\}$ is a barrel [4,
Theorem 5.2] and a neighbourhood of zero of $\sigma p \leq \tau^{buc}$, and hence a neighbourhood of zero of $\tau^{buc} = \beta^*(E, E')$. And so by [7, Observation 3.1.5(c), p.82], contains a borni- vore barrel, and so is a bornivore barrel, and so by THEOREM 5 is a local sequential neighbourhood of zero of (E, τ). Hence, by the COROLLARY in OBSERVATIONS 6, $\varepsilon Ucd\text{-}q = \{x \in E : q(x) \leq \varepsilon\}$, for any $\varepsilon > 0$, is a local sequentially neighbourhood of zero of (E, τ), and so q is local sequentially continuous. Therefore, $\sigma q \leq \tau^*$ for all $q \in Q$, from which follows that

$$\tau^{buc} \leq \tau^* \qquad \ldots(\Delta\Delta)$$

By (Δ) and $(\Delta\Delta)$, $\tau^{buc} = \tau^*$. And this proves the second equality. The third follows from the observation in the paragraph preceding THEOREM 7. ///

A corollary of the just mentioned observation in the preceding proof and Coro- llary 8 ((i) $\Leftrightarrow$ (iv)) is

COROLLARY 10 [9, Problem 10-1-109, p.152] Lcs (E, τ) is quasibarrelled $\Leftrightarrow$ every bounded lower semicontinuous seminorm on (E, τ) is continuous. ///

4.0 Another Characterization of Local Compl-eteness

First recall from the paragraph preceding THEOREM 1 of [5] that lcs (E, τ) is locally complete $\Leftrightarrow$ every barrel is a bornivore barrel. By {7, Obser- vation 3.1.5(b) and (c) p.82] we therefore have

THEOREM 1 Lcs (E, τ) is locally complete $\Leftrightarrow \beta^*(E, E') = \beta(E, E')$. ///

Now, we have the characterization advertised by the title of the paper.

THEOREM 2 Lcs (E, τ) is locally complete $\Leftrightarrow^1$ Every lower semincontinuous seminorm is local sequentially continuous $\Leftrightarrow^{2-}$ Every lower semicontinuous seminorm is bounded.

Proof $\Leftrightarrow^2$ is the last sentence preceding THEOREM 3.7.

For $\Leftrightarrow^1$ we consider first the forward implication $\Rightarrow$: Assume (E, τ) locally complete, and so by THEOREM 1, $\beta^*(E, E') = \beta(E, E')$. By [4, Theorem 5.7], THEOREM 3.9 and [9, Problem 7-2-105, p.97], therefore, it follows that every lower semicontinuous seminorm is local sequentially continuous. For the reverse implication $\Leftarrow$ suppose every lower semicontinuous seminorm is local sequentially continuous. Noting that in general $\beta^*(E, E') \leq \beta(E, E')$, it follows from THEOREM 3.9 and [4, Theorem 5.7] that $\beta(E, E') \leq \beta^*(E, E')$ and so $\beta^*(E, E') = \beta(E, E')$. By THEOREM 1, then, (E, τ) is locally complete. ///

5.0 References

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