Transactions of the Nigerian Association of Mathematical Physics
***Volume 1, (November, 2015*), pp 89 – 90**
© Trans. of NAMP

A Characterization of Local Completeness

Sunday Oluyemi

Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, P.M.B 4000, Ogbomoso, Nigeria.

Abstract

A separated locally convex space is locally complete if and only if all the precompact sets are strongly bounded.

Keywords: Precompact sets, strongly bounded sets, locally complete space
2010 Amer Math.Soc. Subject Classification 46A03, 46A08, 46A17, 46A20

1.0 Introduction

All topologies are assumed separated and by a *lcs* (E, τ) shall be meant a separated locally convex space with continuous dual E'. The field of scalars is $\mathbf{K} = \mathbb{R}$, the real numbers or $\mathbb{C}$, the complex numbers[1, First paragraph, p.47]. All dual pairs $< X, Y >$[7, Section 2. We shall freely use the language and notation of [7]] shall be separated, with polars taken w.r.t them absolute, and, by $\sigma(X, Y)$ and $\beta(X, Y)$ shall, respectively, be meant the *weak topology* and the *strong topology* on X of the dual pair. If p is a seminorm on the linear space X, following Wilansky in [8], we shall denote its pseudometric topology by σp. The topology τ_1 on X being finer than τ_2 on X is indicated by $\tau_1 \geq \tau_2$, while the subspace topology of τ_1 induced on $\emptyset \neq A \subseteq X$ is denoted $\tau_1|A$. /// signifies the end or absence of a proof.

For lcs (E, τ), an absolutely convex absorbing closed subset B of E is called a *barrel* of (E, τ). A subset W of E absorbing all bounded sets is called a *bornivore*, [8, Definition 4-4-6, p. 48], while $P \subseteq E$ is called a *precompact set* if for every neighbourhood of zero V, of (E, τ), there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of E such that $P \subseteq \bigcup_{i=1}^{n} (x_i + V)$ [8, First paragraph of Section 6-4, p.83][4, Theorem 2.10.2(c), p.145].

An absolutely convex bounded subset B of the lcs (E, τ) is called a *disc*; [3, Definition 3.2.1, p.82] if E_B is the linear span of B in E, then B is absorbing in E_B and the Minkowski functional q_B of B in E_B is a norm, and so, (E_B, q_B) is a normed space[3, Proposition 3.2.2., p.82][4, Proposition 3.5.6(a), p.207]. A sequence $(x_n)_{n=1}^{\infty}$ in E that converges to $x \in E$ in the normed space (E_{B^*}, q_{B^*}), for some disc B^* of (E, τ), is said to *locally converge to x in* (E, τ); if $x = \theta$, the zero of E, $(x_n)_{n=1}^{\infty}$ is called a *local null sequence*. The sequence $(z_n)_{n=1}^{\infty}$ in E is called a *locally Cauchy sequence* [3, Definition 5.1.1, p.151] of (E, τ) if it is Cauchy in (E_B, q_B) for some disc B of (E, τ); and (E, τ) called a *locally complete space*[5, p.8][3, Definition 5.1.5, p.152] if every locally Cauchy sequence locally converges. Hans Jarchow noted in [5] that (E, τ) being locally complete is equivalent to requiring that for every closed disc B of (E, τ), (E_B, q_B) is a Banach space[3, Proposition 5.1.6, p.152]. Again, by [3, Proposition 5.1.6, p.152], every closed disc B of (E, τ) implying (E_B, q_B) Banach is equivalent to: Every bounded subset of E is included in a Banach disc[| Disc B is a *Banach disc* if (E_B, q_B) is a Banach space |]. Carlos Bosch and Jan Kucera employed the two pages of [2] to show that a space (E, τ) meets this last condition [they called such a space *fast complete*] if and only if its bounded sets are strongly bounded. Albert Wilansky, however, in [8, Definition 10-4-3, p.158] named a space meeting this Bosch-Kucera characterization a *Banach-Mackey space*, and showed that[8, Theorem 10-4-7, p.158] a space is Banach-Mackey if and only if every barrel is a brornivore. We here add to all these characterization two others the first of which is

THEOREM 1 Lcs (E, τ) is locally complete if and only if every precompact set is strongly bounded (i.e., $\beta(E, E')$-bounded).

Before giving a proof of THEOREM 1 we note that a contribution of this paper is that

(i) While the Bosch-Kucera characterization requires our examining all bounded sets for strong boundedness, THEOREM 1 prescribes that we restrict our examination to the smaller class of precompact sets[8, Lemma 6-4-1, p.83].

A second contribution is that

Corresponding author: Sunday Oluyemi, Tel.: +2348102016571

(ii) This is the first characterization, known to the author, relating local completeness with precompact sets, and as a consequence we are able to relate the topology of precompact convergence[4, p.234] with the strong* topology $\beta(X', X)$[8, Section 10-1, last two lines of second paragraph, p.149 and Remark 10-1-3, first line, p.150] for a locally complete space, thus coming up with another characterization of local completeness, our THEOREM 2 below.

For the proof of THEOREM 1, we need some lemmas.

LEMMA 1 [8, Problem 8-6-114, p.126] A set absorbing all local null sequences (i.e., absorbing their range) is a bornivore. ///

LEMMA 2 Local null sequences are also ordinarily null.

Proof For a disc B of lcs (E, τ), by [3, Proposition 3.2.2., p.82], $\tau|E_B \leq \sigma q_B$.///

LEMMA 3 A bornivore barrel is a barrel absorbing precompact sets.

Proof From LEMMA 2, local null sequences are null. Null sequences are precompact sets (i.e., the range of a null sequence is precompact) and so local null sequences are precompact. The $\Leftarrow$ claim of this lemma is now immediate from LEMMA 1. For the $\Rightarrow$ claim, it suffices to note that a prcompact set is a bounded set. ///

Now to the

Proof of THEOREM 1 The forward implication $\Rightarrow$ is clear, since by local completeness τ and $\beta(E, E')$ have same bounded sets[the Bosch-Kucera characterization] and precompact sets are bounded.

For the implication $\Leftarrow$ observe that the barrels of (E, τ) constitute a base of neighbourhoods of zero of $\beta(E, E')$[3, Observation 3.1.5, p.82] and so all $\beta(E, E')$-bounded sets are the subsets of E absorbed by all the barrels of (E, τ). Therefore, if every precompact set is $\beta(E, E')$-bounded, then the precompact sets are absorbed by all the barrels of (E, τ) and so by LEMMA 3 all the barrels of (E, τ) are bornivores. And the two sentences preceding the statement of THEOREM 1 complete the proof. ///

Let $< X, Y >$ be a dual pair. Wilansky in [8, Definition 8-5-1, p.118] calls a non-empty collection $\mathfrak{R}$ of non-empty $\sigma(Y, X)$-bounded subsets of *Y a polar family*[7, Paragraph preceding CLAIM 4.3] if

(i) for $A, B \in \mathfrak{R}$, there exists $C \in \mathfrak{R}$ such that $A \cup B \subseteq C$, and

(ii) for $D \in \mathfrak{R}$, there exists $E \in \mathfrak{R}$ such that $E \supseteq 2D$;

and in [8, Definitions 8-5-9, p.120] calls the polar family $\mathfrak{R}$ a *saturated polar family* if subsets of double polars, A^{oo}, of members A of $\mathfrak{R}$ are also members of $\mathfrak{R}$. We denote by $\tau_{p(\mathfrak{R})}$ the polar topology on X determined by $\mathfrak{R}$. [It is denoted $\tau_{\mathfrak{R}}$ by [8] on its page 119].

LEMMA 4 $\tau_{p(\mathfrak{R})}$ is also the topology on X of uniform convergence $\tau_{uc(\mathfrak{R})}$, on the sets of $\mathfrak{R}$.

Proof [7, CLAIM 4.3]. ///

LEMMA 5[6, the LEMMA of p.13] Let (E, τ) be a lcs.

(a) The collection of precompact sets is a saturated polar family.

(b) The collection of $\beta(E, E')$-bounded sets is a saturated polar family. ///

LEMMA 6 [8, Theorem 8-5-11, p.120] Let $< X, Y >$ be a dual pair and $\mathcal{F}$ and $\mathcal{B}$ polar families in Y with $\mathcal{F}$ saturated. Then, $\tau_{uc(\mathcal{F})} \geq \tau_{uc(\mathcal{B})}$ if and only if $\mathcal{F} \supseteq \mathcal{B}$. ///

If $\mathcal{B}$ is the collection of precompact sets of the lcs (E, τ), considering the dual pair $< E, E' >$, $\tau_{uc(\mathcal{B})}$ is called the *topology of precompact convergence* on E'[4, Definition 3. 9.2, p.234] pc(E', E); if $\mathfrak{R}$ is the collection of the $\beta(E, E')$-bounded sets, $\tau_{uc(\mathfrak{R})}$ is denoted $\beta^*(E', E)$[4, Exercise 5, p.220][8, Remark 10-1-3, p.150].

Considering LEMMAS 5 and 6, if $\mathcal{B}$ is the collection of precompact sets of the lcs (E, τ) and $\mathcal{F}$ the collection of strongly bounded sets (i.e., ($\beta(E, E')$-bounded sets) and considering the dual pair $< E, E' >$, it is immediate from THEOREM 1, therefore, that

THEOREM 2 Lcs (E, τ) is locally complete if and only if pc$(E', E) \leq \beta^*(E', E)$. ///

2.0 Reference

[1] Sterling K Berberian, Lectures in Functional Analysis and Operator Theory, GTM 15, Springer-Verlag, World Publishing Corporation, Beiging, China, 1974

[2] Carlos Bosch and Jan Kucera, A Necessary and sufficient condition for w*-bounded sets to be strongly bounded, Proc. AMS, vol, 101, No.3, (1987), 453-454.

[3] Pedro PerezCarreras and Jose Bonet, Barrelled Locally Convex Spaces, North-Holland Mathematics Studies, Vol, 131, North-Holland 1987.

[4] John Horvath, Topological Vector Spaces and Distribution I, Addison-Wesley, 1966.

[5] Hans Jarchow, On generalizations of DF-spaces, Analele Universitatii Din Craiova, Seria Matamatica, Fizica-Chimie, vol. VIII (1980), 8 – 17.

[6] Sunday Oluyemi, A characterizations of BTB Space, Science Focus, LAUTECH, (Nigeria) vol. 2 (2003) 13 – 14.

[7] Sunday Oluyemi, The topologies of uniform convergence are the topologies generated by collections of lower semicontinuous seminorms.

[8] Albert Wilansky, Modern Methods in Topological Vector Spaces, McGraw-Hill, 1978