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The Topologies of Uniform Convergence are the Topologies Generated by Collections of Lower Semicontinuous Seminorms

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Abstract

The topologies of uniform convergence are the topologies generated by collections of lower semicontinuous seminorms.

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1.0 Introduction

By a *lcs* (E, τ)[6, Abbreviation 8-1-12, p.105] we shall mean a separated *locally convex space*[6, First two lines of Section 7-1, p.91][2, Definition 2. 4.1, p.86]. Our terminologies shall be standard as found, for example, in [1], [5], [2] and [6]. We signify with /// the end or absence of a proof. Finally, the first number x in each square bracket, as in $[x, \cdots, \cdots]$, refers to reference number [x] listed at the end of the paper.

2.0 Seminorm Generators

All topological vector spaces are over $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$, the real field or the complex field. Of course, $\mathbb{R}$ and $\mathbb{C}$, with their usual topologies, are locally convex spaces in their own right. If Φ is a collection of topologies on a non-empty set E, say, by $\vee\Phi$, pronounced supΦ [6, Theorem1-6-8, p.11], we denote the *suprenum* of Φ. If E is a vector space and $p : E \to \mathbb{R}$ a seminorm on E, then following Wilanky[6, Definition 4-1-7, p.38], by σp we denote the pseudometric topology of p which is a vector topology by [6, Example 4-1-2, p.37]; indeed locally convex [1, Lemaa (37.10), p.149]. If P is a collection of seminorms on E, by σP we mean $\vee\{\sigma p : p \in P\}$. If τ is a vector topology and $\tau = \sigma P$, then we say that τ is *ge- nerated* by P[6, Theorem 7-2-2 and paragraph preceding Theorem 7-2-4, p.94] [5, Definition, p.107][1, Definition (37.9), p.149] and may write the locally convex space (E, τ) as $(E, \sigma P)$. If E is a non-empty set with τ_1, τ_2 topologies on E, by $\tau_1 \leq \tau_2$ we shall mean that τ_1 is coarser than τ_2.

3.0 Dual Pair

Let X and Y be vector spaces over the same field $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$. A mapping

$$B : X\text{x}Y \to \mathbf{K} \qquad \ldots(\Delta)$$

of the Cartesian product XxY into $\mathbf{K}$ is called a *bilinear form* or a *bilinear functional* if for every $x \in X$ and every $y \in Y$ the mappings

$$B_x : Y \to \mathbf{K} \quad \text{and} \quad B^y : X \to \mathbf{K}$$
$$z \mapsto B(x, z) \qquad\qquad z \mapsto B(z, y)$$

are linear functionals[1, Definition (38.1), p.155]. Bilinear forms abound. Of particular importance is the bilinear form from the Cartesian product Ex$E^\#$ of a vector space E over $\mathbf{K}$ and its vector space $E^\#$ of linear forms on it, called its *algebraic dual*: $B : E\text{x}E^\# \to \mathbf{K}$, $B(x, f) = f(x)$ for all $f \in E^\#$ and all $x \in E$. If (E, τ) is a topological vector space, the linear space of continuous linear forms on (E, τ), denoted E' or $(E, \tau)'$, is called its *continuous dual*. With $E^\#$ replaced by E' above, B is also a bilinear form.

If in (Δ) the bilinear form B has the property [1, Definition (38.1)(v) and (vi), p.155] that for $x \in X$, $B(x, y) = 0$ for all $y \in Y$ $\Rightarrow x =$ the zero of X, it is said to *separate the points* of X. Similarly, B is said to *separate the points* of Y if for $y \in Y$, $B(x, y) = 0$ for all $x \in X \Rightarrow y =$ the zero of Y. If B separates the points of X and the points of Y, then the triple (X, Y, B) is said to constitute a *separated dual pair* or simply a *dual pair* or a *dual system*[2, Definition 3.2.1, p.183] and this indicated by

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writing $< X, Y >$, and for $x \in X$, $y \in Y$, write $B(x, y)$ as $<x, y>$. For an instance, if (E, τ) is a lcs, then the canonical bilinear form $B : E\text{x}E' \to \mathbf{K}$, $B(x, f) = f(x)$, for all $x \in E$ and all $f \in E'$, gives a separated dual pair $< E, E'>$ [6, 8-2-2, p.107].

4.0 Barrels of A Dual Pair

Let $< X, Y >$ be a dual pair, assumed, he- nceforth, always separated, and τ a *locally convex* topology on X. τ is said to be *compatible with* $< X, Y >$ [6, Definition 8-2-8, p.108][2, Definition 3.4.1, p.198] provided $(X, \tau)' = \hat{Y}$, where $\hat{Y} = \{ \hat{y} : y \in Y, \hat{y}(x) = < x, y >, x \in X$ i.e, $\hat{y} = B^y\}$. By $\sigma(X, Y)$/ $\sigma(Y, X)$ is meant the weak topology by the maps $\hat{Y} / \hat{X}$, and it is compatible with $< X, Y >$, and is indeed the smallest compatible[6, Theorem 8-2-12, p.108][2, Last line of Example 3.4.2, p.198, and, first two lines of paragraph preceding Proposition 3.2.3, p.187].

If (E, τ) is a lcs, an absolutely convex, absorbing and closed subset of (E, τ) is called a *barrel* of (E, τ)[6, Definition 3-3-1, p.32][2, Definition 3.5.2, p.208].

We have, using Albert Wilansky's language of *duality invariant*[6, Theorem 8-4-1, p.114],

FACT 1 [2, Proposition 3.4.3, p.198] Being closed convex is a duality invariant. That is, if $< X, Y >$ is a dual pair, then the closed convex sets of X are the same for all locally convex topologies on X which are compatible with the dual pair. Or said this way : If $<X, Y>$ is a dual pair, then all compatible topologies for X have the same closed convex sets. ///

We have immediately from this that

FACT 2 [6, Remark 8-3-7, p.112] Being a barrel is duality invariant. That is, if $< X, Y >$ is a dual pair, then all compatible topologies for X have the same barrels. In particular, being a barrel of an lcs (E, τ) is a duality invariant. That is, all barrels of (E, τ) are the barrels of any other (E, τ^*) where τ^* is compatible with $< E, (E, \tau)' >$. ///

Hence, for a given dual pair $< X, Y >$, we may simply talk of *the barrels* of $< X, Y >$, since by FACT 2 any barrel of (X, τ) for compatible τ is also a barrel of (X, τ^*) for any other compatible τ^*.

5.0 Topology of Uniform Convergence and PO-LAR Topology

Let $< X, Y >$ be a dual pair and $\Re$ a collection of $\sigma(Y, X)$-bounded sets. i.e., bounded sets of the lcs $(Y, \sigma(Y, X))$. Then, the collection of polars (always absolute)[6, Definition 8-3-1, p.110][2, Definition 3.3.1, p.190] $\Re^0 = \{A^0 : A \in \Re\}$ is a colle- ction of absolutely convex absorbing subsets of X [2, Proposition 3.3.1(e) and (f), p.190 and 191] and so by [2, Proposition 2.4.6, p.88][5, Last paragraph, p.167] furnishes X with a locally convex topology $\tau_{uc(\Re)}$[Second paragraph, Section 3.4, p.195 of 2] called the *topology of uniform convergence* on the sets of $\Re$. If $\Re$ is the collection of all $\sigma(Y, X)$-bounded sets, $\tau_{uc(\Re)}$ is called the *strong topology* and denoted $\beta(X, Y)$. $\beta(Y, X)$ is also clear.

From the citation above [2, Proposition 2.4.6, p.88][5, Last paragraph, p.167] and [Sec- ond paragraph, Section 3.4, p.195 of 2] we have

Fact 1 A base of neighbourhoods of zero of $\tau_{uc(\Re)}$ is

$\{\lambda_1 A_1^0 \cap \lambda_2 A_2^0 \cap \ldots \cap \lambda_n A_n^0 : \lambda_1, \lambda_2, \ldots, \lambda_n > 0, A_1, A_2, \ldots, A_n \in \Re\}$. ///

Fact 2 [1, (37.28), p.154][2, last paragraph, p.195] Let $< X, Y >$ be a dual pair and $\Re$ a non-empty collection of non-empty $\sigma(Y, X)$-bounded sets. Then,

$\tau_{uc(\Re)} = \vee\{\sigma p_A : A \in \Re,$ p_A is a seminorm on X defined by $p_A(x) = \sup_{a\in A} \{|<x, a>|\}\}$.

Proof We give this for clarity. Let $A \in \Re$. Define $p_A(x) = \sup_{a\in A} \{|<x, a>|\}$, $x \in X$. One checks that p_A is a seminorm [2, last paragraph, p.195]. *Proof* : Let $x \in X$. We first show that $p_A(x) < \infty$. $<x, a> = \hat{x}(a)$. $(Y , \sigma(Y, X))' = \hat{X}$. By [6, Theorem 4-4-3,p.57], $\hat{x}(A)$ is a bounded set of real numbers and so $\sup_{a\in A} |< x, a >| = \sup_{a\in A} |\hat{x}(a)| < \infty$. Clearly, $p_A(x) \geq 0$. Let $x, y \in X$. Then, $p_A(x + y) = \sup_{a\in A} |< x+y, a >| = \sup_{a\in A} |< x, a > + < y, a >| \leq \sup_{a\in A}\{|< x, a >| + |< y, a >|\} \leq \sup_{a\in A} |< x, a >| + \sup_{a\in A} |< y, a)| = p_A(x) + p_A(y)$.

Similarly, $p_A(\lambda x) = |\lambda|\, p_A(x)$, $\lambda \in K$. By FACT 1 the finite intersections of the sets λA^0, where $\lambda > 0$, and $A \in \Re$ form a base of neighbourhoods of zero of $\tau_{uc(\Re)}$. Clearly, for $\lambda > 0$, and $x \in X$,

$\sup_{a\in A} \{|<x, a>|\} = p_A(x) \leq \lambda \iff x \in \lambda A^0$.

From this follows, by taking finite intersections, that the topology generated by the p_A's, $\vee\{\sigma p_A$:

$A \in \Re\}$, has the same base of neighbourhoods of zero as $\tau_{uc(\Re)}$. Hence, $\tau_{uc(\Re)} = \vee\{\sigma p_A : A \in \Re\}$.
///

Albert Wilansky in [6, Definition 8-5-1, p.118] calls a non-empty collection $\Im$ of non-empty $\sigma(Y, X)$-bounded subsets of Y, $< X, Y >$ a dual pair, a *polar family* if the following two conditions are satisfied:

(i) For $A, B \in \Im$, there exists $C \in \Im$ such that $A \cup B \subseteq C$, and

(ii) For $D \in \Im$, there exists $E \in \Im$ such that $E \supseteq 2D$.

Clearly, from (i) follows that

(i)′ For $A_1, A_2, \ldots, A_n \in \Im$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, there exists $C \in \Im$ such that $A_1 \cup A_2 \cup \ldots \cup A_n \subseteq C$.

The collection of polars $\{A^0 : A \in \Im\}$ is an additive[6, Definition 4-2-1, p.40] filterbase of absolutely convex absorbing subsets of X[6, Theorem 8-5-3, p.119] and so by [6, Theorem 4-3-5, p.45] form a base of neighbourhoods of zero for a locally convex topology $\tau_{p(\Im)}$ on X called a *polar topology* of $<X, Y>$. Of course, $\Im$ being a collection of $\sigma(Y, X)$-bounded sets, it also furnishes X with a topology of uniform convergence $\tau_{uc(\Re)}$. We have

Claim 3 $\tau_{p(\Im)} = \tau_{uc(\Im)}$, and so $\tau_{p(\Im)}$ is a topology of uniform convergence. That is, a polar topology is a topology of uniform convergence.

Proof: From the preceding paragraph

$$\{A^0 : A \in \Im\} \qquad \ldots(\Delta)$$

is a base of neighbourhoods of zero of $\tau_{p(\Im)}$. By FACT 1, the collection of sets

$$\{\lambda_1 A_1^0 \cap \lambda_2 A_2^0 \cap \ldots \cap \lambda_n A_n^0 : \lambda_1, \lambda_2, \ldots, \lambda_n > 0, A_1, A_2, \ldots, A_n \in \Re, n \in \mathbb{N}\} \qquad \ldots(\Delta\Delta)$$

is a base of neighbourhoods of zero of $\tau_{uc(\Im)}$. Clearly, $(\Delta) \subseteq (\Delta\Delta)$, and so

$$\tau_{p(\Im)} \leq \tau_{uc(\Im)} \qquad \ldots(*)$$

We establish the revere inequality of (*). Consider the typical member

$$\lambda_1 A_1^0 \cap \lambda_2 A_2^0 \cap \ldots \cap \lambda_n A_n^0 \qquad \ldots(\Sigma)$$

of $(\Delta\Delta)$. By [1, (17.2), p.68], if $\lambda = \min\{½, \lambda_1, \lambda_2, \ldots, \lambda_n\}$, then

$\lambda A_1^0 \subseteq \lambda_1 A_1^0$, $\lambda A_2^0 \subseteq \lambda_2 A_2^0$, …, $\lambda A_n^0 \subseteq \lambda_n A_n^0$,

and so

$$(\Sigma) \supseteq \lambda A_1^0 \cap \lambda A_2^0 \cap \ldots \cap \lambda A_n^0$$
$$= \lambda(A_1^0 \cap A_2^0 \cap \ldots . \cap A_n^0)$$
$$= \lambda\left(\bigcap_{k=1}^{n} A_k^0\right).$$

By the condition (i)′ (defining a polar family) there exists $A \in \Im$ such that $A \supseteq \bigcup_{k=1}^{n} A_k$, and so

$$A^0 \subseteq \left(\bigcup_{k=1}^{n} A_k\right)^0 = \bigcap_{k=1}^{n} A_k^0$$

Hence, $\lambda\left(\bigcap_{k=1}^{n} A_k^0\right) \supseteq \lambda A^0$. That is, $(\Sigma) \supseteq \lambda A^0$. Clearly, λA^0 is a neighbourhood of zero of $\tau_{p(\Im)}$, and so (Σ) is a neighbouhood of zero of $\tau_{p(\Im)}$. Since (Σ) was arbitrary, it follows that $\tau_{p(\Im)} \geq \tau_{uc(\Im)}$ which is the reverse inequality of (*) that we set out to prove. ///

We have a

CRUCIALFACT 4 [6, 8-5-8, p.120]. Let $<X, Y>$ be a dual pair. A locally convex topology τ on X is a polar topology of $< X, Y >$ if and only if τ has a base of neighbour- hoods of zero which are barrels of $< X, Y >$. ///

FACT 5 Let $< X, Y >$ be a dual pair.

(i) If B is a barrel of $< X, Y >$, so is αB for any non-zero scalar α. [For *absorbing*, note that $\frac{x}{\alpha} \in \lambda B$ for all $|\lambda| \geq \mu_x \Leftrightarrow x \in \lambda(\alpha B)$ for all $|\lambda| \geq \mu_x$][2, Definition 2.6.1, p. 108].

(ii) If $B_1, B_2, \ldots, B_n$, $n \in \mathbb{N}$, are barrels of $< X, Y >$, so is $B_1 \cap B_2 \cap \ldots \cap B_n$.

(iii) If $\alpha_1, \alpha_2, \ldots, \alpha_n$, $n \in \mathbb{N}$, are non-zero scalars and $B_1, B_2, \ldots, B_n$ are barrels of $< X, Y >$, so is $\alpha_1 B_1 \cap \alpha_2 B_2 \cap \ldots \cap \alpha_n B_n$.

Proof (i) and (ii) are easily verified. [|For absolute convexity apply directly the FACT [3, p.4] : A is absolutely convex $\Leftrightarrow rA + sA \subseteq A$ for r, s scalars, $|r| + |s| \leq 1$. /// Note also that an intersection of closed sets is closed |] (iii) is immediate from (i) and (ii). ///

CLAIM 6 A topology of uniform convergence is a polar topology.

Proof 1: Let $< X, Y >$ be a dual pair. Suppose $\Re$ is a non-empty collection of non-empty $\sigma(Y, X)$-bounded sets. By FACT 2, employing the notation there, for $A \in \Re$ the closed unit disc of p_A,

$Ucd\text{-}p_A = \{x \in X : p_A(x) \leq 1\}$

is the polar A^0 of A. By [2, Proposition 3.3.1, p.190/191], A^0 is a barrel of $< X, Y >$. By FACT 1 and FACT 5(iii), therefore, $\tau_{uc(\Im)}$ has a base of neighbourhoods of zero which are barrels of $< X, Y >$. By the CRUCIAL FACT 4, therefore, $\tau_{uc(\Im)}$ is a polar topology of $< X, Y >$. ///

Proof 2: Suppose $\Re$ is a non-empty collection of non-empty $\sigma(Y, X)$-bounded sets. Let $sc\Re = \{\lambda V : \lambda \in \mathbf{K}, V \in \Re\}$ be the collection of scalar multiples of members of $\Re$, and FU($sc\Re$) the collection of finite unions of members of $sc\Re$. By [2, Proposition 3.4.2(c) and (b), p.196], $\tau_{uc(\Re)} = \tau_{uc(sc\Re)}$, and $\tau_{uc(sc\Re)} = \tau_{uc(FU(sc\Re))}$, and so

$$\tau_{uc(\Re)} = \tau_{uc(FU(sc\Re))} \qquad \ldots(\Delta)$$

One checks easily that FU($sc\Re$) is a polar family as

(i) $(\lambda_1 A_1 \cup \lambda_2 A_2 \cup \ldots \cup \lambda_n A_n) \cup (\mu_1 B_1 \cup \mu_2 B_2 \cup \ldots \cup \mu_m B_m)$
$\subseteq \lambda_1 A_1 \cup \lambda_2 A_2 \cup \ldots \cup \lambda_n A_n \cup \mu_1 B_1 \cup \mu_2 B_2 \cup \ldots \cup \mu_m B_n$, and

(ii) $2\lambda_1 A_1 \cup 2\lambda_2 A_2 \cup \ldots \cup 2\lambda_n A_n \supseteq 2(\lambda_1 A_1 \cup \lambda_2 A_2 \cup \ldots \cup \lambda_n A_n)$,
the A's and B's members of $\Re$ and the λ's and μ's scalars. By CLAIM 3,

$$\tau_{p(FU(sc\Re))} = \tau_{uc(FU(sc\Re))} \qquad \ldots(\Delta\Delta)$$

(Δ) and ($\Delta\Delta$) give

$$\tau_{uc(\Re)} = \tau_{p(FU(sc\Re))}. \ ///$$

CLAIM 3 and CLAIM 6 assure us that a polar topology of a dual pair $< X, Y >$ is a topology of uniform convergence and vice-versa.

We have from CLAIM 3, CLAIM 6, FACT 4 and FACT 5, the following result which is of independent interest.

THEOREM 7 Let $< X, Y >$ be a dual pair. The supremum $\vee_{\alpha \in I} \tau_\alpha$, of a collection $\{\tau_\alpha : \alpha \in I\}$ of topologies on X of uniform convergence w.r.t. $< X, Y >$, is a topology on X of uniform convergence. ///

We proceed to prove our

6.0 Main Theorem

That the topologies of uniform convergence are the topologies generated by collections of lower semicontinuous seminorms.

Let (X, τ) be a topological space, $f : (X, \tau) \to \mathbb{R}$ a real function and $a \in X$. f is said to be *lower semicontinuous at a* [4, Exercise 4, p.132] if for each real α with $\alpha < f(a)$ there exists a neighbourhood $V_{\alpha a}$ of a such that $f(x) > \alpha$ for all $x \in V_{\alpha a}$. f is called *lower semicontinuous* if it is lower semicontinuous at every $a \in X$. We have from [4, Exercise 4(b) p.132] that f is lower semcontinuous if and only if the set $\{x \in X : f(x) > \alpha\}$ is open in (X, τ) for each $\alpha \in \mathbb{R}$. Immediate from this is

FACT 1 [2, Exercise 3.6.1(a), p.219][6, Problem 8-3-113, p.113]. For topological space (X, τ), real-valued $f : (X, \tau) \to \mathbb{R}$ is lower semicontinous if and only if the set $\{x \in X : f(x) \leq \alpha\}$ is closed in (X, τ) for each $\alpha \in \mathbb{R}$. ///

Now we have

THEOREM 2 [2, Exercise 3.6.1(b), p.219] Let (E, τ) be a lcs and $p : E \to \mathbb{R}$ a seminorm on E. Then, the closed unit disc of p, $Ucd\text{-}p = \{x \in E : p(x) \leq 1\}$ is a barrel of the dual pair $<E, E'> \Leftrightarrow p$ is lower semicontinous for any compatible topology on E for the dual pair $< E, E'>$.

Proof By [3, Proposition 1.4.6, p.13][2, Last Paragraph, p.88][6, Problem 2-1-101, p.17] the closed unit disc of p, $Ucd\text{-}p = \{x \in E : p(x) \leq 1\}$ is absolutely convex and absorbing for any seminorm p at all. Therefore, from FACT 1 $\Leftarrow$ is now immediate.

Now for $\Rightarrow$. We verify FACT 1 for p in place of f there.

I. If $\alpha < 0$, then $\{x \in E : p(x) \leq \alpha\} = \varnothing$. The empty set is a closed set in any topological space.

II. Suppose $\alpha = 0$. By hypothesis, $Ucd\text{-}p = \{x \in E : p(x) \leq 1\}$ is a barrel of he dual pair $< E, E'>$. And so, $\frac{1}{n} Ucd\text{-}p = \frac{1}{n}\{x \in E : p(x) \leq 1\} = \{x \in E : p(x) \leq \frac{1}{n}\}$ is a barrel for all $n \in \mathbb{N}$, of the dual pair $< E, E'>$, by FACT 4.5(i), and hence closed for all $n \in \mathbb{N}$ in any compatible topology on E of the dual pair. Since the intersection of closed sets is closed, so is

$$\bigcap_{n=1}^{\infty} \tfrac{1}{n} Ucd\text{-}p = \bigcap_{n=1}^{\infty} \{x \in E : p(x) \leq \tfrac{1}{n}\}.$$

But

$$\bigcap_{n=1}^{\infty}\{x \in E : p(x) \le \tfrac{1}{n}\} = \{x \in E : p(x) = 0\}.$$

So, $\{x \in E : p(x) \le 0\} = \{x \in E : p(x) = 0\}$ is closed.

III. Suppose $\alpha > 0$. *Ucd-p* being by hypothesis a barrel of the dual pair $<E, E'>$ is closed in any compatible topology on E for $<E, E'>$, and so, by FACT 4.5(i), α*Ucd-p* being still a barrel is closed. But

$\alpha Ucd\text{-}p = \{x \in E : p(x) \le 1\} = \{x \in E : p(x) \le \alpha\}$.

Thus, we have shown that for all $\alpha \in \mathbb{R}$, and in any compatible topology on E for the dual $<E, E'>$, $\{x \in E : p(x) \le \alpha\}$ is closed, and so by FACT 1, p is lower semicontinuous. ///

We now have

FACT 3 Lower semicontinuous seminorms are duality invariant. And so we can talk of the lower semicontinuous seminorms of a dual pair.

Proof Immediate! But a proof will only be repetitive. Perhaps for emphasis, let $< X, Y >$ be a dual pair, and suppose τ^* is a topology on X compatible with this dual pair. Let p be a seminorm on X. By THEOREM 2, p is (X, τ^*)-lower semi-continuous $\Leftrightarrow$ $Ucd\text{-}p = \{x \in E : p(x) \le 1\}$ is a barrel of (X, τ^*). If τ^{**} is any other compatible topology on X for the dual pair $< X, Y >$, $Ucd\text{-}p = \{x \in E : p(x) \le 1\}$is also a barrel of (X, τ^{**}) being a barrel of (X, τ^*) already [FACT 3.1], and so by THEOREM 2 again p is (X, τ^{**})-lower semicontinuous. Hence, the lower semicontinuity of p is $< X, Y >$-duality invariant. ///

Now, the

MAIN THEOREM 4 Let $< X, Y >$ be a dual pair. The topologies of uniform conver-gence w.r.t $<X, Y >$ are the topologies generated by collections of lower semi continuous seminorms of $<X, Y >$.

Proof Let P be a collection of lower semicontinuous seminorms of $< X, Y >$. By THEOREM 2, therefore, $Ucd\text{-}p = \{x \in E : p(x) \le 1\}$, for $p \in P$, is a barrel of $< X, Y >$. i.e., is a barrel of (X, τ) for any topology τ on X compatible with $< X, Y >$. And so by FACT 4.5(i), $\varepsilon Ucd\text{-}p$, for $\varepsilon > 0$, is a barrel of $<X, Y >$. But $\{\varepsilon Ucd\text{-}p : \varepsilon > 0\}$ is a base of neighbourhoods of zero of σp and so, since $\sigma P = \vee\{\sigma p : p \in P\}$, the sets

$$\left.\begin{array}{l}\varepsilon_1 Ucd\text{-}p_1 \cap \varepsilon_2 Ucd\text{-}p_2 \cap \ldots \cap \varepsilon_n Ucd\text{-}p_n : p_1, p_2, \ldots \\ \qquad \ldots, p_n \in P, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0, n \in \mathbb{N} \}\end{array}\right\} \ldots(\Delta)$$

constitute a base of neighbourhoods of zero of σP. By FACT 4.5(iii), members of (Δ) are also barrels of $< X, Y >$. Thus, σP has a base of neighbourhoods of zero which are barrels of $<X, Y>$. By FACT 4.4, therefore, σP is a polar topology and so by CLAIM 4.3, a topology of uniform convergence. This concludes the proof of the implication $\Leftarrow$.

For $\Rightarrow$ suppose $\Re$ is a non-empty collection of non-empty $\sigma(Y, X)$-bounded sets of Y, and so consider the topology $\tau_{uc(\Re)}$ on X of uniform convergence on the sets of $\Re$. By FACT 4.2,

$\tau_{uc(\Re)} = \vee\{\sigma p_B : B \in \Re, p_B \text{ is the seminorm defined by } p_B(x) = \sup_{b \in B}\{|< x, b >|\}\}$, …(TUC)

and, for $B \in \Re$,

$Ucd\text{-}p_B = B^0$

By [2, Proposition 3.3.1(e) and (f); p.190], B^0 is a barrel of $<X, Y>$, and so by THEOREM 2, p_B is a lower semicontinuous seminorm of $<X, Y>$. Hence, $\tau_{uc(\Re)}$ is generated by a collection $\{ p_B : B \in \Re\}$ of lower semicontinous seminorms of $<X, Y>$. ///

FACT 5 For a dual pair $<X, Y>$, there is a finest topology of uniform convergence, which is σP, where P is **the** collection of **all the** lower semicontinuous seminorms of $< X, Y >$.

Proof Immediate from MAIN THEOREM 4. ///

FACT 6 [1, (37.28)(iii), p.154][2, Definition 3.4.2, p.201][6, Paragraph between 8-5-4 and 8-5-5, p.119]. $\beta(X, Y)$ is the finest topology of uniform convergence of the dual pair $< X, Y >$.

Proof If $\Re$ and $\Re'$ are non-empty collections of non-empty $\sigma(X, Y)$-bounded sets, and $\Re \subseteq \Re'$, then it is clear from (TUC) preceding FACT 5 that $\tau_{uc(\Re)} \le \tau_{uc(\Re')}$

Immediate from FACT 5, FACT 6 and CLAIMS 4.3 and 4.6 is

IMPORTANT CONSEQUENCE 7 For a dual pair $< X, Y >$,

$\beta(X, Y) = \vee\{\sigma p : p \text{ is a lower semicontinuous seminorm of } < X, Y >\}$.

That is, $\beta(X, Y)$ is generated by the collection of **all** lower semicontinuous seminorms. ///

7.0 Barrelledness And Lower Semicontinuous SEM-INORMS

We recall that a lcs is called a *barrelled space* if every barrel of the space is a neighbourhood of zero[6, Definition 9-3-1, p.136]. The only theorem that we wish to prove in this section is

THEOREM 1 A lcs is barrelled if and only if every lower semicontinuous seminorm is continuous.

A proof of this theorem is difficult to locate in the literature. Indeed, Albert Wilansky and John Horvath stated this theorem as exercises in their books [6, Problem 9-3-105, p.139][2, Exercise 3.6.1(c), p.219]. We furnish two proofs here. First some lemmas.

LEMMA 2 [6, Lemma 7-2-1, p.94][5, Lemma 11.2, p.106]. Let p be a seminorm on the topological vector space (E, τ). Then, p is continuous $\Leftrightarrow$ $Ucd\text{-}p = \{x \in E : p(x) \le 1\}$ is a neighbourhood of zero in (E, τ). ///

LEMMA 3 Let B be a barrel of the lcs (E, τ) and q_B the Minkowski functional [2, paragraph following Example 2.4.22, p.94][5, Section II.12, p.111-114][(*)of the proof of Theorem (37.4) of [1], p.146] of B. Then,

$$Ucd\text{-}q_B \{x \in E : q_B(x) \le 1\} = B \qquad \ldots(*)$$

and since B is a closed set, $Ucd\text{-}q_B$ is a closed set in (E, τ), and so by THEOREM 5.2, q_B is a lower semicontinuous seminorm on (E, τ).

Proof Only (*) need be proved. The proof is a careful adaptation of some parts of the proof of [1, Theorem (37.4), p.146]. Let $x \in E$ and since B is absorbing, being a barrel, let

$\mathbb{R}^+_B(x) = \{\alpha > 0 : x \in \alpha B\}$

and so $q_B(x) = \inf \mathbb{R}^+_B(x)$.

Suppose $x_0 \in B$. Then, $x_0 \in B = 1 \cdot B$, from which follows that $q_B(x_0) \le 1$, and so $x_0 \in \{x \in E : q_B(x) \le 1\} = Ucd\text{-}q_B$. This proves the inclusion $\supseteq$. For $\subseteq$, suppose $x_0 \in Ucd\text{-}q_B = \{x \in E : q_B(x) \le 1\}$. Hence, $q_B(x_0) \le 1$. i.e., $\inf \mathbb{R}^+_B(x_0) \le 1$. If $\beta > 1 \ge \inf \mathbb{R}^+_B(x_0) = q_B(x_0)$, then there exists $\mu \in \mathbb{R}^+_B(x_0)$ such that $\inf \mathbb{R}^+_B(x_0) \le \mu < \beta$. And so, $x_0 \in \mu B$ and since by [1, (17.2), p.68] $\mu B \subseteq$

βB, it follows that $x_0 \in \beta B$. Hence, $\beta^{-1} x_0 \in B$. But β was arbitrary. And we have thus shown that

$\beta^{-1} x_0 \in B$ for all $\beta > 1$.

By the continuity of the partial maps[2, Last sentence of the first paragraph p.74] $\beta^{-1} x_0 \to x_0$ as $\beta \to 1$, and so x_0 belongs to the closure of B, which is B since B is closed. ///

Now to a

Proof of THEOREM 1 <u>$\Rightarrow$:</u> Let (E, τ) be a barreled lcs and p a seminorm on (E, τ). Suppose p is lower semicontinuous. By THEOREM 5.2, $Ucd\text{-}p = \{x \in E : p(x) \le 1\}$is a barrel, and so since (E, τ) is a barrelled space, $Ucd\text{-}p$ is a neighbourhood of zero. By LEMMA 2, therefore, p is continuous.

<u>$\Leftarrow$:</u> Suppose every lower semicontinous seminorm $p : (E, \tau) \to \mathbb{R}$ on the lcs (E, τ) is continuous. Let B be a barrel of (E, τ). By LEMMA 3, its Minkowski functional q_B is lower semicontinuous, and so, by hypothesis, continuous. Since also by LEMMA 3,

$Ucd\text{-}q_B = \{x \in E : q_B(x) \le 1\} = B$,

and by the continuity of q_B, $\{x \in E : q_B(x) \le 1\}$ is a neighbourhood of zero[LEMMA 2] of (E, τ), it follows that B is a neighbourhood of zero of (E, τ). Since B was arbitrary, (E, τ) is barrelled. ///

Another Proof of THEOREM 1 By [6, Theorem 9-3-10, p.138] lcs (E, τ) is barreled $\Leftrightarrow$ $\tau = \beta(E, E')$. The forward implication in THEOREM 1 now follows from IMPORTANT CONSEQUENCE 5.7 and [6, Problem 4-5-1, p. 55]. For the reverse implication, suppose that every lower semicontinuous seminorm p on lcs (E, τ) is continuous. Again by [6, Problem 4-5-1, p.55, $\sigma p \le \tau$. By the IMPORTANT CONSEQUENCE 5.7, therefore, $\beta(E, E') \le \tau$. But, in general, $\tau \le \beta(E, E')$, and so, $\tau = \beta(E, E')$. Hence, by [6, 9-3-10, p.138] again, (E, τ) is barrelled. ///

While THEOREM 6.1 is well-known, even with a proof difficult to locate in the literature, IMPORTANT CONSEQUENCE 5.7 and our MAIN THEOREM 5.4 are unknown. We discuss elsewhere some consequences of our MAIN THEOREM.

8.0 References

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