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An Exact Parametric Solution of the Robinson-Trautman Equation of Petrov Type III

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Abstract

This paper is concerned with the construction of a parametric expression for type III Robinson-Trautman metrics using a direct approach. This addresses a long standing problem regarding the construction of (non-trivial) explicit Robinson-Trautman type III metrics.

1.0 Introduction

Robinson-Trautman solutions are algebraically special space-times constructed around a geodesic, shear-free, non-twisting but expanding null congruence. Such space-times have the canonical form given in [1] as

$$ds^2 = \frac{2r^2}{p^2}\, d\xi d\bar{\xi} - 2d\sigma dr - \left[2p^2\nabla\bar{\nabla}\ln p - 2r(\ln p)_{,\sigma} - \frac{2m(\sigma,\xi,\bar{\xi})}{r}\right]dr^2 \qquad (1.1)$$

where the function p is independent of r and the coordinate $\sigma = x^2$ labels the hypersurfaces. $r = x^1$ can be regarded as an affine parameter along the null geodesics lying in the hypersurfaces. The coordinate ξ is a complex stereographic-type coordinate such that $\xi = x^3 + ix^4$ and $\nabla = \frac{\partial}{\partial x^3} + i\frac{\partial}{\partial x^4} = 2\frac{\partial}{\partial \xi}$. The unknown functions $m = m(\sigma,\xi,\bar{\xi})$ and $p = p(\sigma,\xi,\bar{\xi})$ satisfies the equation

$$\nabla m = 0 \qquad (1.2)$$

and

$$p^2\nabla\bar{\nabla}\left(p^2\nabla\bar{\nabla}\ln p\right) - 3m(\ln p)_{,\sigma} + m_{,\sigma} = 0 \qquad (1.3)$$

respectively. A relabeling of the hypersurfaces as well as a relabeling of geodesics within the hypersurfaces is given by the coordinate transformations

$$\sigma' = \gamma(\sigma), \quad r' = \frac{r}{\dot{\gamma}} \qquad (1.4)$$

and

$$\xi' = f(\xi), \quad p'^2 = \left(\frac{\partial f}{\partial \xi}\right) = \left(\frac{\partial \bar{f}}{\partial \bar{\xi}}\right)p^2 \qquad (1.5)$$

respectively. Now the 2-surface ξ = constant and r = constant have line element

$$dl^2 = \frac{r^2}{2}p^{-2}\left[(dx^3)^2 + (dx^4)^2\right] = \frac{r^2}{2}p^{-2}\, d\xi d\bar{\xi} \qquad (1.6)$$

The Gaussian curvature of a 2-space with metric (1.6) is then

$$K = 2p^2\nabla\bar{\nabla}\ln p \qquad (1.7)$$

so that the equation satisfied by p becomes

$$\frac{1}{2}p^2\nabla\bar{\nabla}K - 3m(\ln p)_{,\sigma} + m_{,\sigma} = 0 \qquad (1.8)$$

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For solutions of Petrov type III or N, $m = 0$ in which case, equation (1.2) is identically satisfied and equation (1.8) reduces to

$$\nabla\overline{\nabla}K = 0 \tag{1.9}$$

The condition that these solutions be type III is $\nabla K \neq 0$ which gives

$$K = -3\left[f(\xi,\sigma) + \bar{f}(\bar{\xi},\sigma)\right] \quad f,_{\xi} \neq 0 \tag{1.10}$$

where f and $\bar{f}$ are complex conjugate functions of ξ *and* $\bar{\xi}$ respectively. Since σ is nowhere explicitly mentioned within the partial differential equation (1.10), one could ignore the dependence of f on σ so that the coordinate transformation $\xi \to f(\xi),\ p \to p\left|\frac{\partial f}{\partial \xi}\right|$, transforms equation (1.10) to the simpler form

$$K = 2p^2\nabla\overline{\nabla}\ln p = -12\left(\xi + \bar{\xi}\right) \tag{1.11}$$

In this case the function p satisfies the equation.

$$2p^2(\ln p),_{\xi\bar{\xi}} = -3\left(\xi + \bar{\xi}\right) \tag{1.12}$$

Equation (1.12) is the Robinson equation of Petrov type III. It has a known but rather trivial solution given in [1] as

$$p = \left(\xi + \bar{\xi}\right)^{\frac{3}{2}} \tag{1.13}$$

(1.13) is the only known exact solution to (1.12) available for study. Its rather trivial nature makes it unsuitable for adequately describing type III Robinson -Trautman space-times. Finding exact solutions to type III radiative space-times centered on non-twisting congruencies are difficult to obtain. A great deal of attention therefore, has been focused on studying radiative space-times in the linear approximation, .As a result of this, only very few researchers have dealt with the complete nonlinear problem, [2]. This has caused many authors to apply well known analytic methods such as the Wahlquist-Estabrook method and the Lie group method to this equation with the hope of obtaining new solutions (see for example [2], [3], [4], [5], [6], [7], [8]) and the references therein). These attempts have however not yielded the desired results. Only those solutions that can be constructed from (1.13) by arbitrary re-parameterization of the coordinates ($\xi, \bar{\xi}$) are currently available for study. For such solutions, new coordinates ξ"s and functions $p'' s$ are introduced in the form

$$\xi' = \xi'(\xi,\sigma) \tag{1.14}$$

and

$$p'^2 = \left(\frac{\partial \xi'}{\partial \xi}\right)\left(\frac{\partial \bar{\xi}'}{\partial \bar{\xi}}\right)p'^2 \tag{1.15}$$

which can be used to generate an infinite class of type III solutions. This however can not constitute all the type III solutions unless the most general solutions to (1.12) is found [3]. Hence it is an outstanding problem in general relativity to find the most general solutions to (1.12). The lack of exact solutions and the non existence of a suitable analytic method for the construction of new solutions to (1.12) have necessitated the development of suitable mathematical techniques that can lead to the construction of exact solutions to (1.12). In this paper, avoiding the sophisticated tools of differential geometry and using the direct method given in [9], equation (1.12) is reduced into a generalized Endem-Fowler equation. The Endem - Fowler equation is then reduced by means of an admissible functional transformation into Abel's equation of the first kind which is solved (parametrically) without imposing any further restrictions on the equation.

2.0 The generalized Endem-Fowler equation

Following the substitution, $V = p^{-2}$, equation (1.12) becomes

$$VV_{\xi\bar{\xi}} - V_{\xi}V_{\bar{\xi}} - 3\left(\xi + \bar{\xi}\right)V^3 = 0 \tag{2.1}$$

We seek similarity solutions to (2.1) in the form

$$V\left(\xi,\bar{\xi}\right) = \alpha\left(\xi,\bar{\xi}\right) + \beta\left(\xi,\bar{\xi}\right)W\left(Z\left(\xi,\bar{\xi}\right)\right) \tag{2.2}$$

where $\alpha\left(\xi,\bar{\xi}\right), \beta\left(\xi,\bar{\xi}\right)$ and $Z(\xi,\bar{\xi})$ are differentiable functions. Substituting (2.2) into (2.1) and collecting coefficients of like derivatives and powers of W yields the equation

$$\begin{aligned}&\beta^2 Z_\xi Z_{\bar\xi} WW'' + \alpha\beta Z_\xi Z_{\bar\xi} W'' + \beta^2 Z_{\xi\bar\xi} WW' - \beta^2 Z_\xi Z_{\bar\xi} W'^2 \\ &+ \left(\alpha\beta_{\bar\xi} Z_\xi + \alpha\beta Z_{\xi\bar\xi} + \alpha\beta_\xi Z_{\bar\xi} - \beta\alpha_\xi Z_{\bar\xi} + \alpha_\xi \beta Z_{\bar\xi}\right) W' \\ &+ \left(\alpha\beta_{\xi\bar\xi} + \beta\alpha_{\xi\bar\xi} - \beta_\xi \alpha_{\bar\xi} - \alpha_\xi \beta_{\bar\xi} - 6\left(\xi + \bar\xi\right)\alpha^2\beta\right) W \\ &+ \left(\beta\beta_{\xi\bar\xi} - \beta_\xi\beta_{\bar\xi} - 2\alpha\beta^2 - 6\left(\xi + \bar\xi\right)\alpha\beta^2\right) W^2 \\ &- 3\beta^3\left(\xi + \bar\xi\right) W^3 - \left(\alpha\alpha_{\xi\bar\xi} - \alpha_\xi\alpha_{\bar\xi} - 3\left(\xi + \bar\xi\right)\alpha^3\right) = 0\end{aligned} \tag{2.3}$$

To transform equation (2.3) into an ordinary differential equation for *W***,** it is necessary that the ratios for different derivatives and powers of *W* be functions of *Z* only. Taking the WW'' as the normalizing coefficient, the coefficient of W^3 yields the constraint

$$\beta^2 Z_\xi Z_{\bar\xi} \Gamma(Z) = -3\beta^3\left(\xi + \bar\xi\right) \tag{2.4}$$

where $\Gamma(Z)$ is a function to be determined. Assume

$$\Gamma(Z) = 1 \tag{2.5}$$

and β to be a non zero constant (i.e. $\beta = -\frac{1}{3}$) so that $Z_\xi Z_{\bar\xi}$ takes the form

$$Z_\xi Z_{\bar\xi} = \left(\xi + \bar\xi\right) \tag{2.6}$$

Integrating equation (2.6) with respect to ξ *and* $\bar\xi$ yield

$$Z = \frac{2}{3}\left(\xi + \bar\xi\right)^{\frac{3}{2}} \tag{2.7}$$

where we have taken the constants of integration to be zero. The coefficient of WW' yields the following constraint

$$\beta^2 Z_\xi Z_{\bar\xi} \Gamma(Z) = \beta^2 Z_{\xi\bar\xi} \tag{2.8}$$

where $\Gamma(Z)$ is to be determined. Using equation (2.6) and (2.7), $\Gamma(Z)$ can be written in this case as

$$\Gamma(Z) = \frac{1}{3Z} \tag{2.9}$$

The coefficient of W'' yields

$$\beta^2 Z_\xi Z_{\bar\xi} \Gamma(Z) = \alpha\beta Z_\xi Z_{\bar\xi} \tag{2.10}$$

where $\Gamma(Z)$ is to be determined. Making use of the equation (2.6) and the freedom as remarked in [9] we get

$$\alpha\left(\xi, \bar\xi\right) = 0 \tag{2.11}$$

The coefficient of W^2 gives

$$\beta^2 Z_\xi Z_{\bar\xi} \Gamma(Z) = \beta\beta_{\xi\bar\xi} - \beta_\xi \beta_{\bar\xi} - 2\alpha\beta^2 - 6\left(\xi + \xi\right)\alpha\beta^2 \tag{2.12}$$

so that

$$\Gamma(Z) = 0 \tag{2.13}$$

Equation (2.3) therefore simplifies to

$$WW'' + \frac{1}{3Z} WW' - W'^2 + W^3 = 0 \tag{2.14}$$

where β is an arbitrary non zero constant. This can be written as

$$P' + \frac{1}{3Z} P + e^{\int PdZ} = 0 \tag{2.15}$$

where $P = P(Z)$ is defined by

$$P = \frac{W'}{W} \tag{2.16}$$

Set

$$P = Z^{-\frac{1}{3}}\, Q(Z) \tag{2.17}$$

Equation (2.15) can therefore be written as

$$Q' = -Z^{\frac{1}{3}} e^{\int Z^{\frac{1}{3}} Q dz} \tag{2.18}$$

or

$$Q'(\xi) = -\, e^{\left\{\left(\frac{3}{4}\right)^{\frac{1}{2}} \int \frac{Q(\xi)}{\sqrt{\xi}} d\xi\right\}}, \quad \xi = \frac{3}{4} Z^{\frac{4}{3}} \tag{2.19}$$

which can be expressed as

$$Q''(\xi) = \left(\frac{3}{4}\right)^{\frac{1}{2}} \xi^{-\frac{1}{2}}\, Q(\xi) Q'(\xi) \tag{2.20}$$

or in the more convenient form

$$Y''(\xi) = \xi^{-\frac{1}{2}}\, Y(\zeta) Y'(\zeta) \tag{2.21}$$

Equation (2.21) is a generalized Emden-Fowler equation [10]. Given a nonlinear ODE of the generalized Emden-Fowler type, it is shown in [11] that there exist admissible functional transformations that can lead to the construction of exact parametric solutions to (2.21). We introduce the transformations

$$U = \zeta^{-1} Y, \qquad k = \zeta^{-\frac{1}{2}} \frac{Y'}{Y} \tag{2.22}$$

The total derivatives corresponding to the newly introduced variables are from equation (2.22) given by

$$dU = U\left[-\zeta^{-1} + k\zeta^{\frac{1}{2}}\right] d\zeta \tag{2.23}$$

and

$$dk = \left[\zeta^{-\frac{1}{2}} \frac{Y''}{Y} - \left(\frac{1}{2}\zeta^{-\frac{3}{2}} + \zeta^{-\frac{1}{2}} \frac{Y'}{Y}\right) \frac{Y'}{Y}\right] d\zeta \tag{2.24}$$

respectively. Using (2.22), equation (2.21) can be written as

$$Y'' = kYU\zeta \tag{2.25}$$

or equivalently

$$\zeta^{-\frac{1}{2}} \frac{Y''}{Y} = \zeta^{\frac{1}{2}}\, kU \tag{2.26}$$

Thus equation (2.24) , upon the use of (2.26), becomes

$$dk = \zeta^{-\frac{1}{2}} k \left[U - \left(\frac{1}{2}\zeta^{-\frac{3}{2}} + k\right)\right] d\zeta \tag{2.27}$$

Therefore

$$\frac{dU}{dk} = \frac{U\left[k - \zeta^{-\frac{3}{2}}\right]}{\left[U - \left(\frac{1}{2}\zeta^{-\frac{3}{2}} + k\right)\right]}, \quad U - \left(\frac{1}{2}\zeta^{-\frac{3}{2}} + k\right) \neq 0 \tag{2.28}$$

so that U is determined from the equation

$$\left[kU + \left(g(k) - k\right)k\right] \frac{dU}{dk} \left[k + 2g(k)\right] U \tag{2.29}$$

Note that we have made the substitution

$$g(k) = -\frac{1}{2}\zeta^{-\frac{3}{2}} \tag{2.30}$$

Equation (2.29) is the Abel ODE of the second kind [12]. For the solution of equation (2.29) there are two possibilities for which there exist a solution.

Case (i); $g(k)=k$

Corresponding to this choice, equation (2.29) has a solution

$$U(k)=3k \tag{2.31}$$

which corresponds to the known solution (1.31)

Case (ii); $g(k)=\mu k^{-2}$ where μ is an arbitrary non zero constant.

Corresponding to this choice, equation (2.29) has a solution

$$U(k)=-\mu k^{-2}+k\pm\sqrt{\left(\mu k^{-2}-k\right)^2-2v} \tag{2.32}$$

where v is an arbitrary constant. Based on equations (2.30), a functional relation between k *and* ζ is found to be

$$g(k)=\mu k^{-2}=-\frac{1}{2}\zeta^{-\frac{3}{2}} \tag{2.33}$$

which provides a parametric relation namely $\zeta = \zeta(k,C)$. To obtain the parametric expression for Y, we see that (2.22) and (2.23) gives

$$\frac{dY}{d\zeta}=k\zeta^{\frac{1}{2}}Y \tag{2.34}$$

and

$$\frac{dU}{d\zeta}=\frac{dU}{dk}\frac{dk}{d\zeta}=-\zeta^{-2}Y+\zeta^{-1}\frac{dY}{d\zeta}. \tag{2.35}$$

respectively, where $\dfrac{dk}{d\zeta}$ is calculated as

$$\frac{dk}{d\zeta}=\frac{3}{4}k\zeta^{-1} \tag{2.36}$$

upon differentiating (2.33).. Inserting (2.34) into (2.35), we have that the exact parametric solution to (2.21) is

$$Y=\frac{1}{\left[-\zeta^{-\frac{3}{2}}+k\right]}\frac{dU}{dk}\frac{dk}{d\zeta} \tag{2.37}$$

where

$$\frac{dU}{dk}=-\left(2\mu k^{-3}+1\right)\left[-1\pm\frac{\mu k^{-2}-k}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}}\right] \tag{2.38}$$

which gives

$$Y=-\frac{3}{4}\zeta^{-\frac{1}{2}}\left[-1\pm\frac{\mu k^{-2}-k}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}}\right] \tag{2.39}$$

3.0 Analytic solution to (1.12)

We have constructed the exact parametric solution to the generalized Endem-Fowler equation (2.21). We now present, using a proposition, the analytic solution to (1.12).

Proposition 1. Given a function $P=\left(\frac{3}{4}\zeta\right)^{-\frac{1}{4}}Y$ with independent variable $z=\left(\frac{3}{4}\zeta\right)^{\frac{3}{4}}$

$$\int P\,dz=\ln k\pm\frac{1}{2}\ln\left(2\left(\mu k^{-2}-k+\sqrt{\left(\mu k^{-2}-k\right)^2}-2v\right)\right)\pm\frac{3}{2}\ln Y \tag{3.1}$$

where Y is a function defined in equation (2.38) and k is a parameter given by (2.33) μ and v are arbitrary non zero constants.

Proof. Using (2.34), (2.35), (2.36), (2.39) and (2.38), it is easy to show that;

$$\int P dz = \int \frac{1}{k} dk \pm \int \frac{-\mu k^{-3}+1}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}} dk \tag{3.2}$$

The second integral in (3.2) can be written as

$$\int \frac{-\mu k^{-3}+1}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}} dk = \frac{1}{2}\int \frac{d\chi}{\sqrt{\chi^2-2v}} + \frac{3}{2}\int \frac{dk}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}} \tag{3.3}$$

where $\chi = \mu k^{-2} - k$. Also

$$\int \frac{dk}{\sqrt{\left(\mu k^{-2}-k\right)^2-2v}} = \pm \int \zeta^{\frac{1}{2}} k d\zeta = \pm \int \frac{dY}{Y} \pm \ln Y \tag{3.4}$$

A description of type III space times is given the next section using differential invariants of order one.

4.0 Curvature Invariants for Type III Robinson-Trautman Space-Times

The local properties of the gravitational field can often be described using the curvature tensor and its covariant derivatives which are often calculated to different order of approximation. These properties show up in scalars formed from contracting their tensor products. The importance of these variants exceeds the classification purpose: they also provide a measure of the amplitude of the gravitational field as well as help to study the regularly of the field. In this section, we shall study the solutions obtained in the previous section in terms of an invariant of order one using only first order derivative of the Weyl tensor and the expression for the parametric equation calculated.

$$J_1 = 6^2 \left|4\psi_3\right|^4 \left(\theta_s . \bar{\theta}^s\right)^2 \tag{4.1}$$

where

$$\theta_s = -\tau\eta_s - \kappa l_s - r m_s + \sigma \bar{m}_s \tag{4.2}$$

and η_s , l_s , $m_s + \bar{m}_s$ are null tetrad. τ, κ, r, σ are four of the twelve complex Newman-Penrose spin coefficients. Then in general we have

$$\theta_s \bar{\theta}^s = \tau\bar{\kappa} + \bar{\tau}\kappa - \left(r\bar{r} + \sigma\bar{\sigma}\right) \tag{4.3}$$

but since the field is geodesic and shear free this means $\kappa = 0$ and $\sigma = 0$ which makes (4.3) reduce to

$$\theta_s \bar{\theta}^s = -\left|r\right|^2 \tag{4.4}$$

The variant J_1 takes the form

$$J_1 = 6^2 \left|4 r \psi_3\right|^4 \tag{4.5}$$

For type III Robinson-Trautman spacetimes we have

$$\psi_3 = -\frac{1}{2}\frac{P}{r^2} K_{,\xi} \tag{4.6}$$

where

$$K = -3\left(\xi + \bar{\zeta}\right) \tag{4.7}$$

hence

$$\psi_3 = \pm \frac{3}{2r^2\beta^{\frac{1}{2}}} \left[k\left(2\left(\mu k^{-2} - k + \sqrt{\left(\mu k^{-2}-k\right)^2 - 2v}\right)\right)^{\pm\frac{1}{2}} Y^{\pm\frac{3}{2}} \right]^{-\frac{1}{2}} \tag{4.8}$$

The calculations of the invariant s J_1 therefore yields

$$J_1 = 6^6 \left|r\right|^2 \beta^{-2} \left|k\right|^{-1} \left[\left|2\left(\mu k^{-2} - k + \sqrt{\left(\mu k^{-2}-k\right)^2 - 2v}\right)\right|^{\pm 1} \left|Y\right|^{\pm 3} \right]^{-1} \tag{4.9}$$

The invariant becomes singular at some point

5.0 Conclusion

In this paper the Robinson-Trautman equation of Petrov type III was transformed into a second order nonlinear ordinary differential equation of the generalised Emden-Fowler type (2.21). The equation was then solved parametrically using an approach given in [13]. The solution obtained is new and has not been previously reported in the literature for the Robinson-Trautman equation. We show also that there exist a non vanishing invariant of the first order. The invariant can be used for analyzing singularities in type III vacuum spacetimes with expansion.

6.0 References

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