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On Partial W*-Quantum Dynamical Semigroups

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Abstract

A partial w*-quantum dynamical semigroup $\varphi_{\kappa,t}$ defined by composition of completely positive conjugate-bilinear maps is considered. We show that if $\varphi_{\kappa,t}$ satisfies some certain conditions then we can define a contraction semigroups of maps of linear operators P_t associated with $\varphi_{\kappa,t}$.

Keywords: Completely positive maps, partial w*-algebra, dynamical semigroups

1.0 Introduction

A representation theorem for completely positive maps on C *-algebras was first given by Stinespring (1955). An extension of Stinespring's theorem to partial *-algebras has been carried out by a number of authors [1-6] in order to overcome the rigid scheme of the C *-algebraic approach. In this paper we give an equivalent form of the generalization given in Bagarello et al. [2] of the Stinespring theorem. The equivalent form provide for us a suitable setting to define a partial w*-quantum dynamical semigroup $\varphi_{\kappa,t}$ by means of composition of completely positive conjugate-bilinear maps. We show that if partial w*-quantum dynamical semigroup $\varphi_{\kappa,t}$ satisfies some certain conditions then we can define a contraction semigroup P_t associated with $\varphi_{\kappa,t}$. The rest of the paper is arranged as follows; Section 2 gives the basic notions of the algebraic setting for partial *-algebra needed in the sequel. In Section 3 we recall some notions of completely positive invariant conjugate-bilinear map and also state an equivalent form of Bagarello et al. generalization. In Section 4, we define partial w*-dynamical semigroups $\varphi_{\kappa,t}$ by means of composition of completely positive maps and state the main result.

2.0 Algebraic Setting

A partial *-algebra is a quadruplet denoted by $(A, \Gamma, *, \cdot)$. This consists of an involutive complex linear space A with involution and a relation $\Gamma \subseteq A \times A$ endowed with a partial multiplication " $\cdot$ " on A. We have the following precise definition;

Definition 1. A partial *-algebra is a complex vector A with involution $x \to x^*$ and subset $\Gamma \subseteq A \times A$ such that

1. $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$
2. $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma, \ \forall \lambda, \mu \in \mathbb{C}$.
3. Whenever $(x, y) \in \Gamma$ there exists a product $x \cdot y \in A$ with the usual properties of multiplication: $x \cdot (y + \lambda z) = x \cdot y + \lambda (x \cdot z)$ and $(x \cdot y)^* = y^* \cdot x^*$ for $(x, y), (x, z) \in \Gamma$ and $\lambda \in \mathbb{C}$.

A partial *-algebra is in general ***non-associative.*** For$x \in A$,we denote the set of all left and right multipliers of x by $L(x)$ and $R(x)$ hence, a partial *-algebra, is said to be ***semi-associative*** partial *-algebra if and only if for $x, y \in A$ with $y \in R(x)$ implies that $y.z \in R(x)$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z, \forall z \in R(A)$. The study of partial *-algebras is largely dependent on several classes of multipliers introduced as follows: If C is an arbitrary subset of A, then

(i) $L(C) = \bigcap_{x \in C} L(C)$, the set of universal left multipliers of C.
(ii) $R(C) = \bigcap_{x \in C} R(C)$, the set of universal right multipliers of C.
(iii) $M(C) = L(C) \cap R(C)$, the set of universal multipliers of C.

A concrete realization of partial *-algebras arises as follows: Let D be a complex pre-Hilbert Space. Denote by $L^{\dagger}(D, H)$ the set of all closable linear operators X with domain $D(X) = D$ and the adjoint X^* with domain $D(X^*) \supset D$, each with range in $H. L^{\dagger}(D, H)$ is a complex linear space with respect to the usual notions of addition and scalar multiplication. This complex

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linear space becomes a partial *-algebra when equipped with an involution defined for $X \in L^{\dagger}(D,H)$ that is the map $X \to X^{\dagger} = X^{*} \upharpoonright D$ and a weak partial multiplication
$X \cdot Y = X^{\dagger *}Y$ defined whenever X is a weak left multiplier of Y that is, if and only $YD \subset D(X^{\dagger *})$ and $X^{\dagger}D \subset D(Y^{*})$ for $(X, Y) \in \Gamma$,
where $\Gamma = \left\{(X,Y) \in \left(L^{\dagger}(D,H)\right)^{2} : YD \subset D(X^{\dagger *}),\ X^{\dagger}D \subset D(Y^{*})\right\}$.
The quadruplet $(L^{\dagger}(D,H),\ \Gamma,\ \dagger,\ \cdot)$ is called a partial *-algebra and is denoted by $L_{w}^{\dagger}(D,H)$. A subalgebra M of $L_{w}^{\dagger}(D,H)$ is a partial O*-algebra on D. See, Antoine et al. [1].
We have the notion of a state ω on M which is a sequilinear map $\omega: M \times M \to \mathbb{C}$, such that $\omega(e,e) = 1$, where $\mathbb{C}$ is the set of complex numbers and e is the unit element in M. On M, there are various topologies that can be define, in particular we have the strong* τ_{s^*}, the weak τ_{w} and the $\sigma - weak\tau_{\sigma w}$ topologies respectively.We have the following
Definition2.Let M be an arbitrary $\tau_{\sigma w}$-closed, unital partial O*-algebra, endowed with a partial multiplication defined for $(X, Y) \in \Gamma$, we have $X \cdot Y = X^{\dagger *}Y$ such that $YD \subset D(X^{\dagger *})$ and $X^{\dagger}D \subset D(Y^{*})$. Now for any $X \in M$, if M satisfies the following conditions;

(1) $D = \bigcap_{X \in M} D(X^{\dagger})$
(2) $R(M)$ is τ_{s^*} dense in M

Then M will be called a partial w*-algebra.

3.0 Completely Positive Invariant Conjugate-bilinear Map

In this section, we state an equivalent form of the completely positive conjugate-bilinear map which will be used to define the notion of a partial w*-dynamical semigroups. Before we state the result in this section, we have the following definitions from Bagarello's et al. [2]. Let M be a partial w*-algebra with unit e and A a vector space .We denote by $S(A)$ the involutive vector space of all sesquilinear forms φ on $A \times A$ with involution $\varphi(\xi,\eta)^{\dagger} = \overline{\varphi(\eta,\xi)}$, $\xi,\eta \in A$.
Definition 3. A map $\Phi: D(\Phi) \times D(\Phi) \to S(A)$ is said to be conjugate- bilinear if

(i) $D(\Phi) \subset M$
(ii) $\Phi(X,Y)^{\dagger} = \Phi(Y, X)$
(iii) $\Phi(X, \alpha Y + \beta Z) = \alpha\Phi(Y, X) + \beta\Phi(Y, Z)$, $\forall X,Y,Z \in D(\Phi), \forall \alpha,\beta \in \mathbb{C}$.

Definition 4. A conjugate-bilinear map $\Phi: D(\Phi) \times D(\Phi) \to S(A)$ is said to quasi-invariant if there exists a subspace B_{Φ} of $D(\Phi)$ such that

(i) $B_{\Phi} \subset R(M)$
(ii) $MB_{\Phi} \subset D(\Phi)$
(iii) $\Phi(\mathrm{a}X, Y) = \Phi(X,\ a^{*}Y)$, $\forall\, a \in M$, $\forall X,Y \in B_{\Phi}$
(iv) B_{Φ} satisfies the density condition; $X \in D(\Phi)$, $\forall\, \xi \in A$, there exists a net $\{X_n\} \subset B_{\Phi}$ such that $\lim_{n\to\infty} \Phi(X_n - X, X_n - X)(\xi,\xi) = 0$
(v) It is said to be *invariant*, if $\Phi(a^{*}X, bY) = \Phi(X,(ab)Y)$, $\forall\, a,b \in M$, $a \in L(b)$, $\forall X,Y \in B_{\Phi}$.

A subspace B_{Φ} satisfying the above requirements is called a *core*for B_{Φ}.
Definition 5.A conjugate-bilinear map $\Phi: D(\Phi) \times D(\Phi) \to S(A)$ is said to be positive if $\Phi(X, X) \geq 0$ ($\Phi(X, X)(\xi,\xi) \geq 0$, for every $\xi \in A$) each $X \in D$: The map Φ is said to be completely positive if, for each $n \in \mathbb{N}$ $\sum_{i,j=1}^{n} \Phi(X_i, X_j)(\xi_i,\xi_j) \geq 0$, $\forall \{X_1, \dots X_n\} \subset D(\Phi)$, $\forall\{\xi_1 \dots \xi_n\} \subset A$.
We will now give an equivalent form for the completely positive conjugate-bilinear map given in definition 5.
Theorem 1. Let M be a partial w*-algebra and A a vector space. Let Φ, Θ be a completely positive invariant conjugate-bilinear maps with $\Phi \subset \Theta$, then there exists a completely conjugate-bilinear map κ defined on the core B of Φ, such that for all $a,b \in M$ and $X_i, X_j \in B$ we have,$\sum_{i,j=1}^{n} \langle i_{\Phi}(\xi_i),\ \Theta_{\kappa}(aX_i, bY_j) i_{\Phi}(\eta_j)\rangle_{H_{\Phi}} = \sum_{i,j=1}^{n} \Phi(aX_i, bY_j)(\xi_i,\eta_j)$,
where i_{Φ} is a composite map and $\Theta \upharpoonright_{D(\Phi)} = \Theta_{\kappa}$, $\xi_i,\eta_j \in A$.
Proof :For a completely positive map Φ with domain $D(\Phi) \subset M$, let $D(\Phi) \otimes A$ denote the algebraic tensor product of $D(\Phi)$ and A. Define a subspace N_{Φ} of $D(\Phi) \otimes A$ with respect to the map Φ as follows, let$\xi = \sum_{i=1}^{n} X_i \otimes \xi_i$ and $\eta = \sum_{j=1}^{n} Y_j \otimes \eta_j \in D(\Phi) \otimes A$ then we have $N_{\Phi} = \{\xi,\eta \in D(\Phi) \otimes A : \langle\xi,\eta\rangle_{\Phi} = 0\}$. With this we define the linear map λ_{Φ} from $D(\Phi) \otimes A$ into $D(\Phi) \otimes A \setminus N_{\Phi}$ by $\lambda_{\Phi}(\xi) = \xi + N_{\Phi}$. We call $\lambda_{\Phi}(\xi)$ the coset of $D(\Phi) \otimes A \setminus N_{\Phi}$ containing ξ and this induces an inner product on $\lambda_{\Phi}(D(\Phi) \otimes A)$ given by $\langle\lambda_{\Phi}(.), \lambda_{\Phi}(.)\rangle$. Let H_{Φ} denotes the completion of
$\lambda_{\Phi}(D(\Phi) \otimes A)$ with respect to the inner product $\langle\lambda_{\Phi}(.), \lambda_{\Phi}(.)\rangle$. Since the core B of Φ satisfies the density condition by definition, we have that $\lambda_{\Phi}(B \otimes A)$ is dense $\lambda_{\Phi}(D(\Phi) \otimes A)$ and hence dense in H_{Φ}. Now define a map π_0 on M with $\lambda_{\Phi}(B \otimes A) = D(\pi_0)$ by

$$\pi_0(a)\lambda_{\Phi}(\xi) = \lambda_{\Phi}(a\xi) = \sum_{i=1}^{n} \lambda_{\Phi}(aX_i \otimes \xi_i)$$

where $a \in R(M)$ and $\xi \in A$. The map is a well defined *-representation. Let π_B denotes its closure, thus λ_Φ is a strongly cyclic vector representation of M for π_B. Now define an injective map $i: A \to (B \otimes A)$ by $\sum_{i=1}^{n} i(\xi_i) = \sum_{i=1}^{n}(e \otimes \xi_i)$, for $e \in B$, $\xi_i \in A$. And using the map $\lambda_\Phi: (B \otimes A) \to H_\Phi$, we define a composite $i_\Phi: A \to H_\Phi$ $\sum_{i=1}^{n} i_\Phi(\xi_i) = \sum_{i=1}^{n} \lambda_\Phi \circ i(\xi_i)$.Applying the composite map, we define a conjugate-bilinear map $\kappa: B \times B \to H_\Phi$ by

$$\sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \quad \kappa(X_i, Y_j) i_\Phi(\eta_j)\rangle_{H_\Phi} = \langle \sum_{i=1}^{n} \lambda_\Phi(X_i \otimes \xi_i), \sum_{j=1}^{n} \lambda_\Phi(Y_j \otimes \eta_j)\rangle_{H_\Phi}$$
$$= \langle \lambda_\Phi(\xi), \lambda_\Phi(\eta)\rangle_{H_\Phi}$$

For $X_i, Y_i \in B$ and $\xi_i, \eta_j \in A$. Let the extension of κ to $D(\Phi)$ be denoted still by κ, then for $a, b \in M$ and $\forall X_i, Y_i \in B$ and $\xi_i, \eta_j \in A$. Let$\xi_a = \sum_{i=1}^{n} aX_i \otimes \xi_i$,$\eta_b = \sum_{i=1}^{n} bY_i \otimes \eta_i$, then we define the map $\kappa: D(\Phi) \times D(\Phi) \to H_\Phi$ by

$$\sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \quad \kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi} = \langle \lambda_\Phi(\xi_a), \lambda_\Phi(\eta_b)\rangle_{H_\Phi}.$$

Let Θ be a completely positive conjugate-bilinear map with domain $D(\Theta)$ containing $D(\Phi)$ such that its restriction to $D(\Phi)$ is denoted by Θ_κ .

That is the map $\Theta_\kappa: D(\Phi) \times D(\Phi) \to H_\Phi$ given by

$\sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \Theta_\kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi} = \sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi}$.

Thus we have

$$\sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \quad \Theta_\kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi} = \langle \lambda_\Phi(\xi_a), \lambda_\Phi(\eta_b)\rangle_{H_\Phi}$$
$$= \langle \sum_{i=1}^{n} \lambda_\Phi(aX_i \otimes \xi_i), \sum_{j=1}^{n} \lambda_\Phi(bY_j \otimes \eta_j)\rangle_{H_\Phi}$$
$$= \sum_{i,j=1}^{n} \langle \pi_B(a) \lambda_\Phi(X_i \otimes \xi_i), \quad \pi_B(b) \lambda_\Phi(Y_j \otimes \eta_j)\rangle_{H_\Phi}$$
$$= \sum_{i,j=1}^{n} \Phi(aX_i, bY_j)(\xi_i, \eta_j)$$

This conclude the proof.

The next lemma considers the ordering of completely positive maps with respect to their cores But before then, we have the following,

Definition 6 . Let $\mathcal{C}$ be the set of all cores B^i, $i = 1,2, \dots, n$, for the completely positive map Φ , endowed with the order $\subset$. This induces an order on a family $\{\kappa_{B^i}\}_{i=1}^{n}$ of completely positive conjugate-bilinear maps defined on the corresponding cores as follows; $\kappa_{B^1}, \kappa_{B^2}$ be defined on B^1, B^2 respectively, then $\kappa_{B^1} \subset \kappa_{B^2}$ iff $B^1 \subset B^2$ and we denote this by $\kappa_{B^1} \preccurlyeq \kappa_{B^2}$. Let $B^{\mathcal{L}}$ denote the largest of the cores in $\mathcal{C}$ for the completely positive map Φ, then we have $\kappa_B \preccurlyeq \kappa_{B^{\mathcal{L}}} \forall B \in \mathcal{C}$.

Notation : We write $\kappa_{B^{\mathcal{L}}}$ as κ and κ_B as κ'. Then we have the following

Theorem 2. The map Θ_κ is maximal for any family $\{\kappa_{B^i}\}_{i=1}^{n}$ of completely positive conjugate-bilinear defined on the set of cores in $\mathcal{C}$.

Proof: Let $B \in \mathcal{C}$ be an arbitrary core such that $B \subset B^{\mathcal{L}}$, then from definition we have $\kappa' \subset \kappa$. Let $a, b \in M$, $X_i, Y_i \in B$ and $i_\Phi(\xi_i), i_\Phi(\eta_j) \in H_\Phi$ with $\xi_i, \eta_j \in A$ then we have

$\sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \Theta_{\kappa'}(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi} = \sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \kappa'(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi}$.

$$= \sum_{i,j=1}^{n} \langle \pi_B(a) \lambda_\Phi(X_i \otimes \xi_i), \quad \pi_B(b) \lambda_\Phi(Y_j \otimes \eta_j)\rangle_{H_\Phi}$$
$$\leq \sum_{i,j=1}^{n} \langle \pi_{B^{\mathcal{L}}}(a) \lambda_\Phi(X_i \otimes \xi_i), \quad \pi_{B^{\mathcal{L}}}(b) \lambda_\Phi(Y_j \otimes \eta_j)\rangle_{H_\Phi}$$
$$= \sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \quad \kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi}$$

$= \sum_{i,j=1}^{n} \langle i_\Phi(\xi_i), \Theta_\kappa(aX_i, bY_j) i_\Phi(\eta_j)\rangle_{H_\Phi}$.

Hence $\Theta_{\kappa'} \leq \Theta_\kappa$ and since $B \in \mathcal{C}$ is an arbitrary core implies that Θ_κ is maximal.

4.0 Partial w*-Dynamical Semigroups

In this section we let Θ_κ take values in a partial w*- algebra M. We defined a dynamical semigroup of linear mappings $\varphi_{\kappa,t}$ and show that under some certain conditions on $\varphi_{\kappa,t}$ we can define a contraction semigroups of maps of linear operators P_t associated with $\varphi_{\kappa,t}$. We have the following

Definition 7.For $\in \mathbb{R}_+$, with $0 \le t < \infty$. Let Mbe a partial w*- algebra on D and let $\Theta_\kappa : D(\Phi) \times D(\Phi) \to M$ be a completely positive conjugate-bilinear map that is associated with a one parameter completely positive conjugate-bilinear map $t \ni \mathbb{R}_+ \to V_t(X,Y) \in D(\Phi)$ with $X, Y \in M$. By being associated, we mean that their composition is well defined and closed. Then for any $X, Y \in M$we define a completely positive conjugate-bilinear map $\Theta_\kappa^t : M \times M \to M$ by

$$\Theta_\kappa^t(X,Y) = \Theta_\kappa\big(e, V_t(X,Y)\big) = \Theta_\kappa \circ V_t(X,Y)$$

Remark: If $V_0 = e$, we have $\Theta_\kappa^0 = \Theta_\kappa \circ V_0 = \Theta_\kappa \circ e = \Theta_\kappa$ and if write $V_t \circ V_s = V_{t+s}$, then $\Theta_\kappa^{t+s}(X,Y) = \Theta_\kappa^t \circ \Theta_\kappa \circ V_s(X,Y) = \Theta_\kappa^t \circ V_t\big(e, V_s(X,Y)\big)$

$$= \Theta_\kappa^t \circ V_t \circ V_s(X,Y) = \Theta_\kappa^t \circ V_{t+s}(X,Y)$$

$\forall\ X, Y \in M$ and $0 \le t < \infty, 0 \le s < \infty$.

We have an adapted version of a definition of a partial w*-quantum dynamical semigroup from(Ekhaguere , 1993) .

Definition 8.A family $\{\varphi_{\kappa,t}\}_{t\in\mathbb{R}_+}$of completely positive conjugate-bilinear maps on $M \times M$

(a) $\varphi_{\kappa,0} = \Theta_\kappa$

(b) $\varphi_{\kappa,t} \circ \varphi_{\kappa,s} = \varphi_{\kappa,t+s,}$ for arbitrary $t, s \ge 0$

(c) $\varphi_{\kappa,t}$ is σ-weakly continuous on $M \times M$ with respect to the σ-weakly topology $\tau_{\sigma w}$ is called a partial w*-quantum dynamical semigroup.

Theorem 3. A family $\{\varphi_{\kappa,t}\}_{t\in\mathbb{R}_+}$of completely positive conjugate-bilinear maps on $M \times M$ of the form$\langle \xi , \varphi_{\kappa,t}(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa \circ V_t(X,Y)\eta \rangle$ (1)

is a partial w*-quantum dynamical semigroup for $X, Y \in M$ and $\xi , \eta \in$ D.

Proof:We show that definition 8 is satisfied

$\langle \xi , \varphi_{\kappa,0}(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa \circ V_0(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa^0(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa(X,Y)\eta \rangle$ and for the second property, that is $\varphi_{\kappa,t} \circ \varphi_{\kappa,s} = \varphi_{\kappa,t+s,}$ we have the following

$\langle \xi , \varphi_{\kappa,t+s}(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa \circ V_{t+s}(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa^{t+s}(X,Y)\eta \rangle$

$= \langle \xi , \Theta_\kappa^t(e, \Theta_\kappa^s(X,Y)\eta \rangle = \langle \xi , \Theta_\kappa \circ V_t(e, \Theta_\kappa \circ V_s(X,Y)\eta \rangle$

$$= \langle \xi , \Theta_\kappa \circ V_t \circ \Theta_\kappa \circ V_s(X,Y)\eta \rangle = \langle \xi , \varphi_{\kappa,t} \circ \varphi_{\kappa,s}(X,Y)\eta \rangle$$

It is obvious that completely positive maps are σ-weakly continuous, thus the third property in the definition is satisfied, hence is a partial w*- quantum dynamical semigroup.

Definition 9.The †- invariant subspace$(M \times M)^\dagger$ of $M \times M$ is given by

$(M \times M)^\dagger = \{(X,Y) \in D(\Phi) \times D(\Phi):\ \Theta_\kappa^\dagger(X,Y) = \Theta_\kappa(X,Y)\}$.

In what follows ω is a *GNS- representable state* with a self adjoint *-representation π_ω, see Antoine et al. [1].

Theorem 4. For a *GNS- representable state*ω on $M \times M$, if $\{\varphi_{\kappa,t}\}_{t\in\mathbb{R}_+}$is a partial w*-quantum dynamical semigroup on $M \times M$ satisfying the following conditions

(1) $\pi_\omega(\varphi_{\kappa,t}^\dagger) \in L\Big(\pi_\omega(\varphi_{\kappa,t})\Big)$

(2) $\Big(\varphi_{\kappa,t} \circ \Theta_\kappa^\dagger(X,Y)\Big) \cdot \Big(\varphi_{\kappa,t} \circ \Theta_\kappa(X,Y)\Big) \le \varphi_{\kappa,t}(\Theta_\kappa^\dagger(X,Y), \Theta_\kappa(X,Y))$

(3) $\varphi_{\kappa,t} \circ \omega = \omega$

then the linear map P_t defined by $P_t \pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0} = \pi_\omega\Big(\varphi_{\kappa,t}(X,Y)\Big)\lambda_{\xi_0}$ (2)

is a semigroup of contractions on the pre-Hilbert space into its completions for any $(X,Y) \in (M \times M)^\dagger$, where λ_{ξ_0} is a strongly cyclic vector representation of M for π_ω.

Proof: Now for a *GNS- representable state*ω we have thetriple $(\pi_\omega, H_\omega, \lambda_{\xi_0})$. We will show that the linear map P_t defined in equation (2) is a contraction semigroup of linear maps on a pre-Hilbert space into its completion, where λ_{ξ_0} is a strongly cyclic vector representation of M for π_ω. Thus we have

$$P_0 \pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0} = \pi_\omega\Big(\varphi_{\kappa,0}(X,Y)\Big)\lambda_{\xi_0} = \pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}$$

And

$$P_t P_s \pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0} = P_t \pi_\omega\left(\varphi_{\kappa,s}(X,Y)\right)\lambda_{\xi_0}$$
$$= \pi_\omega\big(\varphi_{\kappa,t}(e,\varphi_{\kappa,s}(X,Y))\big)\lambda_{\xi_0}$$
$$= \pi_\omega\left(\varphi_{\kappa,t+s}(X,Y)\right)\lambda_{\xi_0} = P_{t+s}\,\pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}$$

Hence, it is a semigroup. To show that it is a contraction, we have the following

$$\left\|P_t\pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}\right\|^2_{H_\omega}$$
$$= \langle \pi_\omega\left(\varphi_{\kappa,t}(X,Y)\right)\lambda_{\xi_0}, \pi_\omega\left(\varphi_{\kappa,t}(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left(\varphi^\dagger_{\kappa,t}(X,Y)\right)\cdot\ \pi_\omega\left(\varphi_{\kappa,t}(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left((\ \Theta^t_\kappa\ )^\dagger(X,Y)\right)\cdot\ \pi_\omega\big(\ \Theta^t_\kappa\ (X,Y)\big)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left((\ \Theta^t_\kappa\ )^\dagger\circ{\Theta_\kappa}^\dagger(X,Y)\right)\cdot\ \pi_\omega\big(\ \Theta^t_\kappa\ \circ\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}\rangle_{H_\omega}$$

since $(X,Y) \in\ (M\times M)^\dagger$, we have

$$= \langle \lambda_{\xi_0}, \pi_\omega\left((\ \Theta^t_\kappa\ )^\dagger\circ\Theta_\kappa(X,Y)\right)\cdot\ \pi_\omega\big(\ \Theta^t_\kappa\ \circ\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\big(\ (\Theta^t_\kappa\ )^\dagger(e,\Theta_\kappa(X,Y))\big)\cdot\ \pi_\omega\big(\ \Theta^t_\kappa\ (e,\Theta_\kappa(X,Y))\big)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\big(\varphi^\dagger_{\kappa,t}(e,\Theta_\kappa(X,Y))\big)\cdot\ \pi_\omega\big(\varphi_{\kappa,t}(e,\Theta_\kappa(X,Y))\big)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\big(\varphi^\dagger_{\kappa,t}(e,\Theta_\kappa(X,Y))\big)\cdot\ \big(\varphi_{\kappa,t}(e,\Theta_\kappa(X,Y))\big)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left(\varphi^\dagger_{\kappa,t}\circ\ \Theta_\kappa(X,Y)\right)\cdot\ \left(\varphi_{\kappa,t}\circ\ \Theta_\kappa(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left(\varphi_{\kappa,t}\circ\ \Theta^\dagger_\kappa(X,Y)\right)\cdot\ \left(\varphi_{\kappa,t}\circ\ \Theta_\kappa(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$\leq \langle \lambda_{\xi_0}, \pi_\omega\left(\varphi_{\kappa,t}(\Theta^\dagger_\kappa(X,Y),\Theta_\kappa(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \langle \lambda_{\xi_0}, \pi_\omega\left(\varphi_{\kappa,t}(\Theta_\kappa(X,Y),\Theta_\kappa(X,Y)\right)\lambda_{\xi_0}\rangle_{H_\omega}$$
$$= \omega\left(e,\varphi_{\kappa,t}(\Theta_\kappa(X,Y),\Theta_\kappa(X,Y)\right)$$
$$= \omega\circ\varphi_{\kappa,t}\big(\Theta_\kappa(X,Y),\Theta_\kappa(X,Y)\big)$$
$$= \omega\big(\Theta_\kappa(X,Y),\Theta_\kappa(X,Y)\big) = \left\|\pi_\omega\big(\Theta_\kappa(X,Y)\big)\lambda_{\xi_0}\right\|^2_{H_\omega}$$

Since $\omega\circ\varphi_{\kappa,t} = \omega$, this shows that the semigroup is a contraction semigroup. This concludes the proof.

5.0 References

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