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Game Theory and Quantum Theory, What Linkage?

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Abstract

In game theory, a game is essentially a conflict between two systems. In quantum theory the state (function) ψ embodies all that is quantum-mechanically relevant in a system. We show that some games between two interacting systems are encompassed by ψ too.

1.0 Introduction [1-5]

One of the most interesting and rich aspects of classical and of quantum physics is the challenge posed by the interaction term $I(P,Q)$ in the pertinent Hamiltonian, $H(P,Q)$, P,Q being the generalized momentum and position respectively, of the system. This term embodies the phenomenon of phase transition in statistical mechanics (quantum and classical), of entropy/information (exemplified by Maxwell deamon[5]) in thermodynamics. In Hamiltonian dynamics, we have ``chaos'' [4] because $I(P,Q)$ makes $H(P,Q)$ non-integrable, i.e. the number of constants of motion is less than the number of degrees of freedom. So, interaction is a challenge. And whatever area of knowledge that can shed some light on ``interaction'' must be welcomed in physics [1-5].

At the simplest level, we may consider the interaction between two similar subsystems, or two bodies such as two electrons, each of spin s in which we have $I(P,Q) = -Js_i s_j$, $J > 0$, where s can be spin-up or spin-down, i.e. as in Ising model of ferromagnetism,

$$I(P,Q) \equiv -J \sum_{i,j=1}^{N} s_i s_j, \quad s = 1 \text{ or } 0.$$

We shall see what quantum theory[2] says about game theory[1].

1.1 John von Neumann (1903-1957),

Mathematician and physicist, was one of the founders of Quantum Theory [2], a theory which has as its pillars, probability, uncertainty and quantization, von Neumann also pioneered Game Theory, a theory which has found great application in industrial, economic, political, diplomatic and military conflict situations, between any two or among more parties. Here, we shall explore if there is any link or symbiosis between Quantum Theory and Game Theory.

1.2 John Forbes Nash (1928 – 2015),

Mathematician and economist (1994 Nobel Laureate in Economics) ``took the baton'' from Neumann, as they were both at Princeton University and in Institute for Advanced Studies, Princeton, New Jersey, USA. The ``Nash equilibrium'' and ``The Beautiful Mind'' portray the legacy of Nash.

2.1 Game Theory

Game Theory is a mathematical language for analyzing and resolving a conflict situation between two or among many parties. Consider a two-party game between A and B, and assume it is a zero-sum conflict (i.e. A's gain is B's loss, and vice-versa). Assume that at steady-state A can play any of the strategies $A_1, A_2, \dots, A_m$ and B can play any of the strategies $B_1, B_2, \dots, B_n$, m and n finite., If $\mathbb{A}$ plays $A_i$and B plays B_j, the gain for A is a_{ij}. That is we have $m \times n$ game with payoff or gain or game matrix $\{a_{ij}\}$, a_{ij} real numbers. See Table 1.

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Table 1: $m \times n$ game matrix

B \ A	B_1	B_2		B_n
A_1	a_{11}	a_{12}		a_{1n}
A_2	a_{21}	a_{22}		a_{2n}
	$\vdots$	$\cdots$		$\vdots$
A_m	a_{m1}	a_{m2}		a_{mn}

For example, Table 2 is a 2×2 matrix

Table 2: $\boldsymbol{m = n = 2}$

B \ A	B_1	B_2
A_1	1	-1
A_2	-1	1

2.2 Optimal Strategy

We characterize a game as follows:

(i) The lower value of the game is defined by α,

$$\alpha_i \equiv \min_j a_{ij}, \quad \alpha \equiv \max_i, \quad \alpha \equiv \max_i \min_j a_{ij}$$

(ii) The upper value of the game is defined by β,

$$\beta_j \equiv \max_i a_{ij}, \quad \beta \equiv \quad \min_j \quad \max_i \ a_{ij}$$

(iii) The value of the game is ν, $\alpha \le \nu \le \beta$.

In Table 3, we have the game matrix extended by a last column α and by a last rowβ.

Table 3: Extended Matrix

B \ A	B_1	B_2		B_n	α_i
A_1	a_{11}	a_{12}		a_{1n}	α_1
A_2	a_{21}	a_{22}		a_{2n}	α_2
	$\vdots$	$\cdots$			α_m
A_m	a_{m1}	a_{m2}		a_{mn}	
β_j	β_1	β_2		β_n	α \ β

An example is given in Table 4.

Table 4: Case $\alpha = 0,\ \beta = 0$

	B_1	B_2	B_3	α_i
A_1	-1	0	1	-1
A_2	0	0	0	0
				α_m
A_3	1	0	-1	
β_j	1	0	1	α \ β

2.3 Saddle:

If $\alpha = \beta\ (= \nu)$ then the game matrix has a saddle.
In Table 2, we can check $\alpha = -1$, $\beta = 1$; there is no saddle. In Table 4 there is a saddle at a_{22}. A saddle is an equilibrium point, or rather, a stable equilibrium point in that whatever alteration A (or B) makes to his strategy, the alteration can only hurt A (B) and help B (A), mutatis mutandis. Each playing a particular strategy is called ``playing pure strategy'' and the game may have no stable equilibrium. The stable equilibrium may lie in mixed strategy.

2.4 Mixed Strategy

Instead of each playing pure strategy, each can play mixed strategy. A plays strategy $A_i p_i$ of the time, $i = 1,2, \dots, m$

$$p_i \geq 0, \quad \sum_1^m p_i = 1.$$

and B plays B_j, q_j of the time, $j = 1,2, \dots, n,$

$$q_j \geq 0, \qquad \sum_1^n q_j = 1,$$

The average value of the game is

$$\nu_j = \sum_i p_i a_{ij}.$$

A's mixed strategy is denoted by

$$S_A = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ \\ p_1 & p_2 & \cdots & p_m \end{pmatrix}$$

and B's is denoted by

$$S_B = \begin{pmatrix} B_1 & B_2 & \cdots & B_n \\ \\ q_1 & q_2 & \cdots & q_n \end{pmatrix}$$

While the pure strategies involved in S_A called A's utility strategy or A's optimal mixed strategy and B's optimal mixed strategy are denoted by S_A^* and S_B^* respectively. The solution to a game has the property that if A (B) keeps to S_A^* (S_B^*) the gain remains unchanged and equals the value ν, no matter what B (A) does, provided B (A) keeps within S_B^* (S_A^*).

2.5 Fundamental Theorem of Game Theory

Proved by von Neumann, the fundamental theorem is: Every finite game (m and n finite) has at least one solution, either in pure strategy or in the range of mixed strategies.
For example, in a 2×2 matrix,

$$\begin{matrix} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{matrix}, \qquad S_A^* \equiv \begin{pmatrix} A_1 & A_2 \\ \\ p & 1-p \end{pmatrix}$$

If B plays B_1, then $\nu = pa_{11} + (1-p)a_{21}$ $\quad (a)$

If B plays B_2, then $\nu = pa_{12} + (1-p)a_{22}$ $\quad (b)$

From eqs (a) and (b) we obtain

$$p = (a_{22} - a_{21})/(a_{11} + a_{22} - a_{21} - a_{12}).$$

Putting p into eqn (a) or eqn (b), we have

$$\nu = (a_{11}a_{22} - a_{21}a_{12})/(a_{11} + a_{22} - a_{21} - a_{12}). \qquad (c)$$

2.6 Deletion Process

Before attempting to solve a game, it is necessary to eliminate superfluous strategies. That is, delete:
(i) a strategy that duplicates another,
(ii) a strategy that is inferior to another (i.e. of every element in row s is less than or equal to the corresponding element in row r, s should be deleted).
That way the matrix can be reduced to a simpler form. For example.

Table 5: Original 4 × 3 Matrix

	B_1	B_2	B_3
A_1	1	2	4
A_2	0	2	3
A_3	1	2	4
A_4	4	3	1

We see that A_3 clearly duplicatesA_1; so we delete A_3 to give Table 6.

Table 6:

	B_1	B_2	B_3
A_1	1	2	4
A_2	0	2	3
A_4	4	3	1

Again, every gain in A_1 is larger than or equal to its counterpart in A_2; delete A_2 to have Table 7.

Table 7:

	B_1	B_2	B_3	α_i
A_1	1	2	4	1
A_4	4	3	1	1
β_j	4	3	4	

From Table 7, we obtain $\alpha = \max_i \alpha_i = 1$, and $\beta = \min \beta_j = 3$.

3.0 Quantum Theory

All of the foregoing is classical or deterministic game, or the phenomenological aspect of games. How about their noumena, the thing-in-itself (Kant)? In other words, how about the game at the reality level, i.e. at the quantum level? What light does game theory shed on quantum theory? Or more appropriately, what light does the "more real'' quantum theory shed on game theory? Let us consider the interaction of two systems F and G which are in quantum states $|f>$ and $|g>$ respectively, in pertinent Hilbert space. (As an example we may consider the Ising model of ferromagnetism, interaction

$$I \equiv -J \sum_{r}^{N} \sum_{t=1}^{N} s_r s_t, \; J > 0\,.$$

We determine metrix$|f> \otimes <g|$ by direct product of $|f>$ and $<g|$

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} \bigotimes \; g_1^* \, g_2^* \quad \cdots \quad g_n^* \; \equiv \begin{pmatrix} f_1 g_1^* & f_1 g_2^* & \cdots & f_1 g_n^* \\ f_2 g_1^* & & & f_2 g_1^* \\ \vdots & & & \\ f_m g_1^* & & & f_m g_1^* \end{pmatrix} = M + iN,$$

Where $M \equiv Re\,|f> \otimes <g|$, $N \equiv Im\,|f> \otimes <g|$.

To illustrate numerically

$$|f> \equiv \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad <g| \equiv (1 \quad 1 \quad 1 \quad 0)$$

$$|f> \otimes <g| \;=\; \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array}\begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Using the deletion process stated earlier,

$$M \equiv \begin{array}{c} \\ F_1 \\ F_2 \end{array}\begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}\end{array}$$

Because F_4 duplicates F_2 , (delete F_4); and F_3 is a duplicate of F_1 , we have

$$F_1 \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

to give $\alpha = 0,\ \beta = 0$. which gives a saddle. In general, $M \equiv Re|f> \otimes <g|$. If the f's and g's are real, then $Re|f> \otimes <g|$ has its row r a multiple of row 1. By the deletion process we obtain a $1 \times n$ matrix which has a saddle. For if d =min g_i and D is maxg_i,
then we have the following Table:

Table 8:

	G_1	G_2……						G_n	α_i
F_1	d_1	d_2	…	d	…	D		d_n	d
β_j	d_1	d_2	....	d	…	D	….	d_n	

$\alpha = d,\ \beta = d.$
Thus, $\alpha = \beta = d = \nu$.

4.0 Comments and Conclusion

In game theory, $\{a_{ij}\}$ are given data, and p's and q's (in mixed strategies) are a number of independent data.
In quantum theory, $\{f_i g_j^*\}$ are given by ψ, $|f><g|$, the state or probability amplitude of the system. In fact the system and the ψ are one and the same. For to know ψ is to know the system, and vice-versa. And the p's, q's depend on ψ too as shown by eq.(c). So, ψembodies everything, including the game teased out of the system. Our $|g>$ may be $O|h>$ where O is a linear operator on $|h>$ to signify some interaction.
As $m \to \infty$, the saddle and thermodynamic equilibrium might be the same. Like economists, physicists may still have a lot to say about game theory..

5.0 References

[1] For Introduction to Game Theory, see Ye, S, Venttsel, Elements of Game Theory. (Mir Publishers, Moscow, 1980)) Translated from Russian by Vladimir Shokurov, 68pp.

[2] For Quantum Theory, any good book on Quantum Mechanics. (There are several).

[3] For difference between Phenomenon and Noumenon, consult any good philosophy book on Modern Philosophers – I. Kant, D. Hume, etc.

[4] Other ideas such as "Chaos", Ising Model of ferromagnetism, are in Current Literature.

[5] "Maxwell daemon" is well-discussed by Norbert Wiener, Cybernetics MIT Press and John Wiley & Sons, Inc., New York, 1961, 212pp.