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EXTENSION OF SOME GRAVITATIONAL PHENOMENA IN SPHERICAL FIELDS BASED ON RIEMMANIAN GEOMETRY

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Abstract

An extended expression of the gravitational scalar potential exterior to homogeneous spherical mass is used to study some gravitational phenomena. New extended equations with more correction terms for gravitational singularity, gravitational length contraction, time dilation and spectral shift in spherical fields are obtained. The obtained extended expression for gravitational singularity introduces additional corrections terms that do not only depend on the mass but also on the radius of the astrophysical body under consideration and reduces to Schwarzschild's singularity to the order of c^0 and to the order of c^{-4} it contains additional correction terms. Also, the obtained expressions for gravitational length contraction and gravitational time dilation have additional correction terms. An explicit study of gravitational spectral shift using the world line element establishes the well-known concepts of gravitational red and blue shifts. The significance of the results obtained is that they reduce to well-known results in weak field approximations. This indicates that the Riemannian extension is mathematically sound and agrees with known astrophysical facts. The new expression for gravitational time dilation can be incorporated in the design of global positioning systems to improve their precession rate.

Keywords: Scalar Potential, Spherical Mass, Singularity, Length Contraction, Time Dilation and Spectral Shift

1.0 Introduction

Gravity is known to be the least understood of all the fundamental forces in nature. Interestingly, mass and space which are governed by gravitation are the basic building blocks of the universe. Thus far, the General Theory of Relativity has been the most fundamental and successful theorem of Physics that successfully explains the nature of gravity [1, 2].

Newton's theory of gravitation from inception was not able to explain some observed gravitational phenomena. These include the anomalous orbital precession of the orbit of the planets and the gravitational spectral shift by the Sun. Contrarily, Einstein's theory of gravitation which is popularly known as General Relativity gave outstanding explanations for these gravitational phenomena. Post - Newton and post - Einstein extensions of respective theories has undoubtedly produced far reaching results with correction terms [3 - 5].

Recently, a hybrid theory of gravitation that combines both Newton's dynamical theory and Einstein's geometrical theory into a Riemannian theory was developed [6, 7]. This theory has interesting generalizations of gravitational expressions which reduce to their respective expressions in the limit of Newton and Einstein theories.

A better understanding of the nature of mass and space will unravel things previously undreamed of. So far, studies of General Relativity and Gravity have yielded atomic clocks, guidance systems for space crafts and the Global Positioning Systems (GPS). We cannot foresee all that can come from a better understanding of space - time and mass - energy, but a theorem about these fundamental subjects must be thoroughly examined if we are to use it to our advantage [5, 7].

Gravitational singularity can be defined as a point where the quantities used to measure gravity become infinite. In other words, it is a location at which all the physical laws are indistinguishable from each other. The physical consequences of

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gravitational singularities are eminent and there are many reasons to conclude that the space time of the universe is singular. The case of gravitational collapse in Schwarzschild metric is widely studied and accepted [1, 2, 8]. However, the case of gravitational collapse should not be limited to Schwarzschild solution only. This research work expands on previous studies to obtain a post - Schwarzschild expression.

This research work will extend the gravitational length contraction, gravitational time dilation and gravitational spectral shift using a unique approach of Riemannian geometry introduced by [6, 7]. Our approach in this article unlike other attempts makes it possible for us to obtain physically interpretable theoretical values for gravitational phenomena in approximate gravitational fields exterior to bodies in the solar system. It is hoped that the obtained expression for gravitational time dilation can be incorporated into the contribution of gravitation in the design of Global Positioning System (GPS).

2.0 Theoretical Analysis

The general expression for the line element in the gravitational field of spherical massive body is given explicitly from the extended metric tensor by

$$c^2 d\tau^2 = -c^2\left\{1-\frac{2}{c^2}f(r,\theta,\phi,x^0)\right\}dt^2$$
$$-\left\{1-\frac{2}{c^2}f(r,\theta,\phi,x^0)\right\}^{-1}dr^2 - r^2 d\phi^2 - r^2 sin^2\theta d\phi^2 \qquad (1)$$

The extended Riemannian gravitational scalar potential exterior to a spherical astrophysical body $f_{(r)}$ is shown to be given explicitly as [9]

$$f_{(r)} = \frac{k}{r}\left\{1-\frac{k}{c^2R}\right\} - \frac{k^2}{c^2r^2} + \cdots \qquad (2)$$

where $k = GM$, G is the universal gravitational constant, M is the mass of the astrophysical body, c is the speed of light, R is the radius of the spherical astrophysical body and $r > R$ for the exterior field.

Substituting equation (2) into (1) and neglecting higher terms of c^{-4} we obtain

$$c^2 d\tau^2 = c^2\left\{1-\frac{2}{c^2}\left[\frac{k}{r}\left\{1-\frac{k}{c^2R}\right\}-\frac{k^2}{c^2r^2}\right]\right\}dt^2$$
$$-\left\{1-\frac{2}{c^2}\left[\frac{k}{r}\left\{1-\frac{k}{c^2R}\right\}-\frac{k^2}{c^2r^2}\right]\right\}^{-1}dr^2 - r^2d\phi^2 - r^2sin^2\theta d\phi^2 \qquad (3)$$

Equation (3) is an extended Schwarzschild's line element.

It has been known that it is possible for a spherical body to have a point outside it at which Schwarzschild's metric has a singularity denoted by r_s and is given explicitly by the condition;

$$1-\frac{2GM}{c^2r_s} = 0 \qquad (4)$$

Examining equation (3), it can be seen that the metric becomes indefinite if the following condition is satisfied:

$$1-\frac{2GM}{c^2r_s}+\frac{2G^2M^2}{c^4Rr_s^2}+\frac{G^2M^2}{c^4r_s^2} = 0 \qquad (5)$$

Solving equation (5) gives an extended expression for Schwarzschild's singularity. This will obviously depend on the radius of the astrophysical body. This equation reduces to exactly Schwarzschild's singularity to the order of c^0 and to the order of c^{-4} it contains additional correction terms not found in Schwarzschild's terms.

By neglecting the power of c^{-4}, equation (5) reduces to

$$1-\frac{2GM}{c^2r_s} = 0 \qquad (6)$$

This is equivalent to Schwarzschild singularity condition (equation 4) and solving for r_s yields

$$r_s = \frac{2GM}{c^2} \qquad (7)$$

Equation (7) gives Schwarzschild's singularity which depends only on the mass of the astrophysical body. Thus, our extension introduces additional corrections terms that do not only depend on the mass but also depend on the radius of the astrophysical body under consideration.

3.0 Gravitational Length Contraction

The general expression for the space part of the line element in empty space is given in spherical polar coordinates as

$$ds^2 = dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \qquad (8)$$

If there are two points for which θ and ϕ are constant (along a radial line), then equation (8) reduces to

$$ds = dr \qquad (9)$$

Thus, in empty space, distance between two points is the same as the difference in the radial coordinates

On the contrary, in Schwarzschild's field, the space part of the metric is given by

$$ds^2 = \left(1 - \frac{2}{c^2} f\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \qquad (10)$$

Substituting equation (2) into (8) we obtain an extended space component of the metric as

$$ds^2 = \left\{1 - \frac{2}{c^2}\left\{\frac{GM}{r}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2r^2}\right\}\right\}^{-1} dr^2$$

$$-r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \qquad (11)$$

Thus, for points at which θ and ϕ are constant, equation (11) reduces to

$$ds = \left\{1 - \frac{2}{c^2}\left\{\frac{GM}{r}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2r^2}\right\}\right\}^{-\frac{1}{2}} dr \qquad (12)$$

Equation (12) is an extended expression for gravitational length contraction with additional correction terms. This equation reduces to exactly Einstein's equation to the order of c^{-2} and to the order of c^{-4} it contains additional correction terms not found in Einstein's equation.

By simplification and neglecting powers of c^{-4}equation (12) reduces to

$$ds = \left(1 - \frac{2GM}{c^2r}\right)^{-\frac{1}{2}} dr \qquad (13)$$

Equation (13) is the Schwarzschild's expression for gravitational length contraction and hence our extension satisfactorily reduces to Schwarzschild's expression.

4.0 Gravitational Time Dilation

It is well known that a clock at rest at a fixed point in the empty space, ($ds = d\theta = d\phi = 0$) keeps the same time as proper time [10].

Now, consider a clock at rest at a fixed point $(r,\ \theta,\ \phi)$ in the Schwarzschild's gravitational field, the extended metric after substituting equation (2) will become

$$c^2 d\tau^2 = -c^2 \left\{1 - \frac{2}{c^2}\left\{\frac{GM}{r}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2r^2}\right\}\right\} dt^2 \qquad (14)$$

Solving for dt we obtain

$$dt = \left\{1 - \frac{2}{c^2}\left\{\frac{GM}{r}\left(1 - \frac{GM}{c^2R}\right) - \frac{G^2M^2}{c^2r^2}\right\}\right\}^{-\frac{1}{2}} d\tau \qquad (15)$$

By expansion, we get

$$dt = \left(1 - \frac{GM}{c^2r} + \frac{G^2M^2}{c^4Rr^2} + \frac{G^2M^2}{c^4r^2}\right) d\tau \qquad (16)$$

Equation (18) is a new extended expression for gravitational time dilation with additional correction terms. This equation reduces to Schwarzschild's equation for time dilation to the order of c^{-2} and to the order of c^{-4} it contains additional correction terms. By neglecting the powers of c^{-4} we obtain

$$dt = \left(1 - \frac{GM}{c^2r}\right) d\tau \qquad (17)$$

Equation (17) is Schwarzschild's expression for time dilation.

5.0 Gravitational Spectral Shift

Consider a beam of light moving from a source or emitter E at a fixed point in the gravitational field of a spherical body to an observer or a receiver R at a fixed point in the gravitational field.

Let the coordinates of the emitter be $\left(E_t, E_r, E_\theta, E_\phi\right)$ and those of the receiver R be $\left(R_t, R_r, R_\theta, R_\phi\right)$.

Light moves along a null geodesic given by $d\tau = 0$ (Einstein's equation of motion of a photon); implying that the line element for a photon reduces to

$$c^2 \left(1 - \frac{2}{c^2} \mathrm{f}\right) dt^2 = \left(1 - \frac{2}{c^2} \mathrm{f}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \qquad (18)$$

Substituting equation (2), into equation (18) gives

$$c^2\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}dt^2=\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}^{-1}dr^2 -r^2d\theta^2-r^2sin^2\theta d\phi^2 \quad (19)$$

where, $k=GM$

Let v be any parameter that can be used to trace the path of the photon, then equation (19) can be written as

$$c^2\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}\left(\frac{dt}{dv}\right)^2 =\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}^{-1}\left(\frac{dr}{dv}\right)^2 -r^2\left(\frac{d\theta}{dv}\right)^2-r^2sin^2\theta\left(\frac{d\phi}{dv}\right)^2 \quad (20)$$

This implies that

$$\frac{dt}{dv}=\frac{1}{c}\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}^{-\frac{1}{2}}ds \quad (21)$$

By expansion and neglecting the powers of c^{-4}, we obtain

$$\frac{dt}{dv}=\frac{1}{c}\left(1-\frac{2k}{c^2r}\right)^{-\frac{1}{2}}ds \quad (22)$$

where,

$$ds^2=\left\{1-\frac{2}{c^2}\left\{\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right\}\right\}^{-1}\left(\frac{dr}{dv}\right)^2 -r^2\left(\frac{d\theta}{dv}\right)^2-r^2sin^2\theta\left(\frac{d\phi}{dv}\right)^2 \quad (23)$$

By integration equation (22) within the interval (R_t,E_t) on the left hand side and (R_v,E_v) on the right hand side

$$R_t-E_t=\frac{1}{c}\int_{E_v}^{R_v}\left\{\left\{1-\frac{2}{c^2}\left(\frac{k}{r}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r^2}\right)\right\}^{-\frac{1}{2}}ds\right\}dv \quad (24)$$

Similarly by expansion and neglecting power of c^{-4} equation (24) re duces to

$$R_t-E_t=\frac{1}{c}\int_{E_v}^{R_v}\left\{\left(1-\frac{2k}{c^2r}\right)^{-\frac{1}{2}}ds\right\}dv \quad (25)$$

The integration on the right hand side is the same for all signals of light, thus for two signals of light

$$R_t^{(1)}-E_t^{(1)}=R_t^{(2)}-E_t^{(2)} \quad (26)$$

Equation (26) can be written as

$$\Delta R_t=R_t^{(2)}-R_t^{(1)}=E_t^{(2)}-E_t^{(1)}=\Delta E_t \quad (27)$$

Thus coordinate time difference of two signals at the point of emission equals that at the point of reception, but by gravitational time dilation, the proper time intervals for two beams or signals are given by

$$\Delta E_\tau=\left\{1-\frac{2}{c^2}\left(\frac{k}{r_E}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r_E^2}\right)\right\}^{\frac{1}{2}}\Delta E_t \quad (28)$$

and

$$\Delta R_\tau=\left\{1-\frac{2}{c^2}\left(\frac{k}{r_R}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r_R^2}\right)\right\}^{\frac{1}{2}}\Delta R_t \quad (29)$$

Using equations (27) to (29), we get

$$\frac{\Delta R_t}{\Delta E_t}=\frac{\left\{1-\frac{2}{c^2}\left(\frac{k}{r_R}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r_R^2}\right)\right\}^{\frac{1}{2}}}{\left\{1-\frac{2}{c^2}\left(\frac{k}{r_E}\left(1-\frac{k}{c^2R}\right)-\frac{k^2}{c^2r_E^2}\right)\right\}^{\frac{1}{2}}} \quad (30)$$

Hence, by simplification and neglecting powers of c^{-4} equation (30) reduces to

$$\frac{\Delta R_\tau}{\Delta E_\tau} = \left(\frac{1-\frac{2k}{c^2 r_R}}{1-\frac{2k}{c^2 r_E}}\right)^{\frac{1}{2}} \quad (31)$$

Consider the emission of either a peak or a crest of the light wave as one event. Let n be the number of peaks emitted in a proper time interval ΔE_τ then by definition, the frequency of the light relative to the emitter, υ_E is given by

$$\upsilon_E = \frac{n}{\Delta E_\tau} \quad (32)$$

Similarly, since the n umber of cycles n is invariant

$$\upsilon_R = \frac{n}{\Delta E_R} \quad (33)$$

Consequently, from equation (30)

$$\frac{\upsilon_R}{\upsilon_E} = \left\{1 - \frac{2}{c^2}\left(\frac{k}{r_E}\left(1 - \frac{k}{c^2 R}\right) - \frac{k^2}{c^2 r_E^2}\right)\right\}^{\frac{1}{2}} \left\{1 - \frac{2}{c^2}\left(\frac{k}{r_E}\left(1 - \frac{k}{c^2 R}\right) - \frac{k^2}{c^2 r_E^2}\right)\right\}^{-\frac{1}{2}} \quad (34)$$

This reduces to

$$\frac{\upsilon_R}{\upsilon_E} = \left(1 - \frac{2k}{c^2 r_E}\right)^{\frac{1}{2}} \left(1 - \frac{2k}{c^2 r_R}\right)^{-\frac{1}{2}} \quad (35)$$

By expansion and assuming that

$$c^2 r_E \ggg 2k \text{ and } c^2 r_R \ggg 2k \quad (36)$$

Then,

$$\frac{\upsilon_R}{\upsilon_E} \approx 1 - \frac{k}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right) \quad (37)$$

Alternatively,

$$z = \frac{\Delta\upsilon}{\upsilon_E} = \frac{\upsilon_R - \upsilon_E}{\upsilon_E} = \frac{k}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right) \quad (38)$$

or

$$z \approx \frac{k}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right) \quad (39)$$

From the analysis, it follows that if the source is nearer the body than the receiver, then

$$\frac{1}{r_E} > \frac{1}{r_R} \quad (40)$$

Hence, $\Delta\upsilon < 0$, which shows a reduction in the frequency of the light. In this case the light is said to have undergone a red shift (moves towards red in the visible spectrum). Otherwise, light undergoes a blue shift. Similarly, $\frac{\Delta\lambda}{\lambda_E} \approx \frac{k}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right)$

6.0 Conclusion

We have in this research work shown how to extend some gravitational phenomenon (Schwarzschild's singularity, gravitational length contraction, time dilation and spectral shift) in the gravitational field of static homogeneous spherical mass using an extended Newtonian gravitational scalar potential exterior. It is interesting and instructive to note that our extended Schwarzchild's singularity introduces additional corrections terms that do not only depend on the mass but also depend on the radius of the astrophysical body under consideration. It is also interesting and instructive to note that our extension of the gravitational length contraction, time dilation and spectral shift satisfactorily reduces to Einstein's expression to the order of c^{-2}. It contains post Einstein's correction terms to all orders of c^{-4} which are open up for theoretical development and experimental investigations and applications. It is obvious that to the order of c^{-2}, the Riemannian dynamical theory of gravitation is in perfect agreement with Einstein's geometrical theory of gravitation.

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