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A NOTE ON THE CONTINUOUS DUAL OF THE WEBB TOPOLOGY

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Abstract

For a separated locally convex space (V_K, τ), the continuous dual of the Webb topology τ^+ is the sequential dual of τ.

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Subject Classification *General Topology*(GT), *Topological Vector Spa- ces*

1 LANGUAGE AND NOTATION

Our language and notation shall be standard as found in [1],[2], [3] and [4]. By $\mathbb{N}$ we denote the *natural numbers* 1, 2, ….., and by $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1)$/ $\mathfrak{C} = (\mathfrak{C}, +, \cdot, 0, 1)$ the field of the real numbers/ complex numbers. The absolute value/ modulus $|\ |$ / $|\ |$ puts on $\mathbb{R}$ / $\mathfrak{C}$ a metric $d_{|\ |}$/ $d_{|\ |}$ whose topology is denoted $\tau_{\mathbb{R}}$ /$\tau_{\mathfrak{C}}$ and called the *usual topology of* $\mathbb{R}$ / $\mathfrak{C}$ in this paper. So, $(\mathbb{R}, \tau_{\mathbb{R}})$/($\mathfrak{C}$, $\tau_{\mathfrak{C}}$) is a topological space. By K we mean $\mathbb{R}$ or $\mathfrak{C}$, and so $\tau_{\mathsf{K}} = \tau_{\mathbb{R}}$ / $\tau_{\mathfrak{C}}$, and $((\mathsf{K}, +, \cdot, 0, 1)), \tau_{\mathsf{K}}) = (\mathsf{K}, \tau_{\mathsf{K}})$ is a topological space.

Our *vector space*, $(V, +, \theta)_{\mathsf{K}} = V_{\mathsf{K}}$ is an additive Abelian group with scalar multiplication by K, and, an additive identity θ called its *zero*.

A field is a vector space over itself, and so $\mathsf{K} = (\mathsf{K}, +, \cdot, 0, 1)$ is a vector space over itself, of course, with 0 as its zero. So, the notation K_{K} is unambiguous, just as calling a *linear map* $f : V_{\mathsf{K}} \to \mathsf{K}$ from a vector space V_{K} into its field of scalars K, a *linear functional*, is in order.
We assume familiarity with the elements of *General Topology* (**GI**) and *Topological Vector Spaces* (**TVS**). The reader is advised to read [1] before reading this paper. We signify by /// the end or absence of a proof.
Let (X, τ) be a topological space and $x_0 \in X$. We denote by $\mathscr{N}_{x_0}(\tau)$ the filter of neighbourhoods of x_0, also called the neighbourhood system of x_0 ($\equiv$ the collection of all the neighbourhoods of x_0).

FACT 1 (**GT**) Let (X_1, τ_1) and (X_2, τ_2), be topological spaces and $f : (X_1, \tau_1) \to (X_2, \tau_2)$ a map. Then, if f is continuous it is sequentially continuous. ///

A map $f : V_{\mathsf{K}} \to \mathsf{K}$ of a vector space V_{K} into its field of scalars K is called a *functional*. Since $\mathsf{K} = \mathsf{K}_{\mathsf{K}}$ is also a vector space, if f is linear, it is called a *linear functional*.

FACT 2 $(\mathsf{K}_{\mathsf{K}}, \tau_{\mathsf{K}}) = (\mathsf{K}, \tau_{\mathsf{K}})$ is a topological vector space. ///

Let (V_{K}, τ) be a topological vector space. A *continuous linear functional* on (V_{K}, τ) is a linear functional $f : (V_{\mathsf{K}}, \tau) \to (\mathsf{K}, \tau_{\mathsf{K}})$ that is continuous. Similarly, a *sequentially continuous linear functional* on (V_{K}, τ) is a linear functional $f : (V_{\mathsf{K}}, \tau) \to (\mathsf{K}, \tau_{\mathsf{K}})$ that is sequentially continuous. We shall denote by $(V_{\mathsf{K}}, \tau)'$ the collection of all continuous linear functionals on (V_{K}, τ), and call this collection the *continuous dual* of (V_{K}, τ). Similarly, the collection $(V_{\mathsf{K}}, \tau)^+$, of all sequentially contin- uous linear functional on (V_{K}, τ) is called the *sequential dual* of (V_{K}, τ).

Let (V_{K}, τ) be a separated (Hausdorff) locally convex space. The *Webb topology*, τ^+[1], associated with τ, is the finest locally convex topology on V_{K} having the same convergent sequences, with same limits, as τ. We call (V_{K}, τ) a *Webb space* if $\tau = \tau^+$.

Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space, and $(x_n)_{n\in(\mathbb{N},\leq)}$ a sequence in V_{K}. If $(x_n)_{n\in(\mathbb{N},\leq)}$ τ-converges to θ, we call it *a null sequence*. If $\varnothing \neq U \subseteq V_{\mathsf{K}}$ and there exists $N \in \mathbb{N}$, such that $x_n \in U$ for all $n \geq N$, we say that $(x_n)_{n\in(\mathbb{N},\leq)}$ is *eventually in U*. Let $\varnothing \neq W \subseteq V_{\mathsf{K}}$. W is called a *sequential neighbourhood of zero* [1] if it is balanced, convex absorbing and every null sequence is eventually in it.

FACT 3 [1] Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a separated locally convex space. The sequential neighbourhoods of zero constitute a τ^+-local base of neighbourhoods of θ. ///

FACT 4 (TVS) Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space, $U \in \mathscr{N}_\theta(\tau)$, $\lambda \in \mathsf{K}$, $\lambda \neq 0$. Then, also, $\lambda U \in \mathscr{N}_\theta(\tau)$. ///

Definition 5 We remind the reader of the following definitions for use in the next section. Let V_{K} be a vector space and $\varnothing \neq A \subseteq V_{\mathsf{K}}$.

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(i) A is said to be *absorbing* if for every $x \in V_K$ there exists $\varepsilon = \varepsilon(x) > 0$ such that $\lambda x \in A$ for all $\lambda \in K$, $|\lambda| \leq \varepsilon$.

(ii) A is said to be *balanced* if $\lambda A \subseteq A$ for all $\lambda \in K$, $|\lambda| \leq 1$.

(iii) A is said to be *convex* if $\lambda A + \mu A \subseteq A$ for $\lambda, \mu \geq 0$, $\lambda + \mu = 1$.

2 THE CONTINUOUS DUAL OF τ^+ Let (V_K, τ) be a separated locally convex space. We here describe $(V_K, \tau^+)'$.

Notation 1 Let V_K be a vector space and $p : V_K \to \mathbb{R}$ a seminorm on V_K. Define

$p(\leq 1) \equiv \{x \in V_K : p(x) \leq 1\}$.

FACT 2 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space and $p : (V_K, \tau) \to \mathbb{R}$ a seminorm.
Then,
p is continuous
$\Leftrightarrow$
$p(\leq 1) \in \mathcal{N}_\theta(\tau)$. ///

Notation 3 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space and $f : (V_K, \tau) \to (K, \tau_K)$ a linear functional on (V_K, τ). Define
$|f|(\leq 1) \equiv f^{-1}(\overline{B_{d|\,|}(0,1)})$

where $\overline{B_{d|\,|}(0,1)}$ is the closed unit Ball of radius 1 cenrted on 0. *Note* In $\mathbb{R}$, $\overline{B_{d|\,|}(0,1)} = [-1, 1]$.

THEOREM 4 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space and $f : (V_K, \tau) \to ((K_K, +, 0), \tau_K)$ a linear functional. Then, f is continuous if and only if $|f|(\leq 1) \in \mathcal{N}_\theta(\tau)$.

Proof The **implication** $\Rightarrow$ is immediate, since $\overline{B_{d|\,|}(0,1)}$ is a τ_K-neighbourhood of 0, and so, f is continuous implies

$|f|(\leq 1) = f^{-1}(\overline{B_{d|\,|}(0,1)}) \in \mathcal{N}_\theta(\tau)$.

For the implication $\Leftarrow$ first note that a τ_K-local base of neighbourhoods of 0 is
$\{\varepsilon\overline{B_{d|\,|}(0,1)} : \varepsilon > 0\}$
and that
$f^{-1}(\varepsilon\overline{B_{d|\,|}(0,1)}) = \varepsilon f^{-1}(\overline{B_{d|\,|}(0,1)}) = \varepsilon|f|(\leq 1)$.

By hypothesis, $|f|(\leq 1) \in \mathcal{N}_\theta(\tau)$, and so by 1.4, $\varepsilon|f|(\leq 1) \in \mathcal{N}_\theta(\tau)$. Hence, we have

$f^{-1}(\varepsilon\overline{B_{d|\,|}(0,1)}) \in \mathcal{N}_\theta(\tau)$ for every $\varepsilon > 0$.

And from this follows that f is continuous. ///
Our advertised *note on the continuous dual* was stated and proved by John Webb in [4]. We here restate it and give the simple proof in detail with the help of the preceding THEOREM 4.[| The *continuous dual* of a vector topology uniquely determines it |].

The Note Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space. Then

$(V_K, \tau^+)' = (V_K, \tau)^+$ …..(*)

Proof Let $f \in (V_K, \tau^+)'$. By 1.1, therefore, $f \in (V_K, \tau^+)^+$. Since τ^+ and τ have same convergent sequences, it therefore follows that $f \in (V_K, \tau)^+$. Thus, we have shown that

$(V_K, \tau^+)' \subseteq (V_K, \tau)^+$ ….(Δ^1)

Hypothesis $f \in (V_K, \tau)^+$.

CLAIM $f \in (V_K, \tau^+)'$.

Proof of CLAIM One checks almost trivially that $|f|(\leq 1)$ is a τ-seque- ntial neighbourhood of zero. And so by 1.3, $|f|(\leq 1) \in \mathcal{N}_\theta(\tau^+)$. By THEOREM 4, therefore, $f \in (V_K, \tau^+)'$. Thus, we have shown that

$(V_K, \tau)^+ \subseteq (V_K, \tau^+)'$ …..(Δ^2)
Clearly, (Δ^1) and (Δ^2) give (*). ///

COROLLARY 6 If (V_K, τ) is a Webb space, then

$(V_K, \tau)' = (V_K, \tau)^+$. ///

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