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THE WEBB TOPOLOGY

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Abstract

(i) We show that a seminorm on a topological vector space is sequentially continuous if and only if it is sequentially continuous at the zero of the space.

(ii) We associate with a separated locally convex space (V_K, τ) a topology, τ^+, which we call its Webb topology, (John H.Webb [4]) and give a sequential description of the seminorm generators of τ^+,.

Keywords: seminorm, sequentially continuous.

1. LANGUAGE AND NOTATION

Our language and notation shall be pretty standard as found, for example in the great classics [1] and [2].
$\mathbb{N} = \{1, 2, \ldots\ldots\}$– the natural numbers,
$\mathbb{R} = (\mathbb{R}, +, \cdot, 0,1)$ – the real numbers,
$\mathfrak{C} = (\mathfrak{C}, +, \cdot, 0,1)$ – the complex numbers,
$K = (K, +, \cdot, 0,1)$ – $= \mathbb{R}$ or $\mathfrak{C}$.
We assume familiarity with elements of *General Topology*(**GT**) and with the elements of *Topological Vector Spaces* (**TVS**) from which we freely use needed results. We signify by /// the end or absence of a proof.
Our vector space $(V, +, \theta)_K = V_K$ with *zero* θ shall have K as its field of scalars.
Let $X \neq \emptyset$. We write $(x_i)_{i\in(I, \leq)}$ for a *net in X based on the directed set* $(I, \leq)$, and so a *sequence in X* shall be written $(x_n)_{n\in(\mathbb{N}, \leq)}$.
Let (X, τ) be a topological space and $x_0 \in X$. By $Nx_0(\tau)$ we denote the collection of all the neighbourhoods of x_0, called the *neighbourhood system of* x_0, or , the *filter of neighbourhoods of* x_0. By a *local base* of *neighbourhood* at x_0 is meant a subfamily $\mathcal{B}x_0(\tau)$ of $Nx_0(\tau)$ such that for every $V \in Nx_0(\tau)$ there exists $U \in \mathcal{B}x_0(\tau)$ such that $U \subseteq V$. A net $(x_i)_{i\in(I, \leq)}$ in (X, τ), *converges* to x_0 if it is eventually in every $V \in Nx_0(\tau)$. And we write
$x_i \xrightarrow{\tau} x_0$.
Let $((V, +, \theta))_K, \tau) = (V_K , \tau)$ be a topological vector space. A net in (V_K , τ) converging to θ shall be called a *null net.*

FACT 1(TVS) Let $((V, +, \theta)_K, \tau) = (V_K , \tau)$ be a topological vector space, $a \in V_K$ and $(x_i)_{i\in(I, \leq)}$ a net in (V_K , τ). Then,
(i) $x_i \xrightarrow{\tau} a \Rightarrow x_i - a \xrightarrow{\tau} \theta$
(ii) $x_i \xrightarrow{\tau} \theta \Rightarrow x_i + a \xrightarrow{\tau} a$
(ii) $x_i \xrightarrow{\tau} a \Leftrightarrow x_i - a \xrightarrow{\tau} \theta$. ///

FACT 2 (TVS) Let V_K be a vector space, and $p : V_K \to \mathbb{R}$ a seminorm. Then, for $x, y \in V_K$,
$|p(x) - p(y)| \leq p(x - y)$. ///
Let (X, τ) and (X', τ') be topological spaces and $x_0 \in X$. A map (function) $f : (X, \tau) \to (X', \tau')$ is said to be *continuous*

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((τ, τ')-*continuous*) *at* x_0 if for every $V \in N_{f(x_0)}(\tau')$ we have $f^{-1}(V) \in N_{x_0}(\tau)$. And f is said to be *continuous* ((τ, τ'))-*continuous*) if it is continuous at all $x \in X$.

FACT 3 (GT) Let (X, τ) and (X', τ') be topological spaces and $x_0 \in X$. The map $f : (X, \tau) \to (X', \tau')$ is continuous at x_0 if and only if for every net $(x_i)_{i\in(I, \leq)}$ in (X, τ) converging to x_0, the net $(f(x_i)_{i\in(I,\leq)})$ in (X', τ') converges to $f(x_0)$. ///

FACT 4 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ and $((V', +, \theta')_K, \tau') = (V_K', \tau')$ be topological vector spaces. A linear map f : $(V_K, \tau) \to (V_K', \tau')$ is continuous if and only if it is continuous at θ. ///

FACT 5 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space and $p : (V_K, \tau) \to \mathbb{R}$ a seminorm. Then, p is continuous if and only if p is continuous at θ. ///

Let (X, τ) and (X', τ') be topological spaces and $x_0 \in X$. A map $f : (X, \tau) \to (X', \tau')$ is said to be *sequentially continuous* at x_0 if for every sequence $(x_n)_{n\in(\mathbb{N}, \leq)}$ in (X, τ) converging to x_0, the sequence $(f(x_n))_{n\in(\mathbb{N},\leq)}$ in $X', \tau')$ converges to $f(x_0)$. The map f is simply called *sequentially continuous* if it is sequentially continuous at every $x \in X$.

Immediate from FACT 3, therefore, is

FACT 6 (GT) *Continuity implies sequential continuity* Let (X, τ) and (X', τ') be topological spaces and $x_0 \in X$. and f : $(X, \tau) \to$
(X', τ') a map. Then,

(i) f is continuous at x_0 implies it is sequentially continuous at x_0,

and

(ii) f is continuous implies it is sequentially continuous. ///

If $p : V_K \to \mathbb{R}$ is a seminorm on a vector space V_K, it induces a psendometric

$dp : V_K x V_K \to \mathbb{R}$

$(v, w) \mapsto p(v - w)$

on V_K. The pseudometric topology, τ_{dp} of dp, is called the *topology of the seminorm p*. We here simplify write τ_p for τ_{dp}.

FACT 7 (TVS) [3] τ_p is a vector topology, and so, (V_K, τ_p) is a topological vector space. ///

Example 8 Consider the real field $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1)$, and the vector space $\mathbb{R}_\mathbb{R}$. The absolute value function

$|\;| : \mathbb{R} \to \mathbb{R}$

$x \mapsto |x|$

is a seminorm (indeed, a norm) on $\mathbb{R}_\mathbb{R}$. The topology $\tau_{|\;|}$ is called $\mathbb{R}$'s *usual topology*. And we here denote it $\tau_\mathbb{R}$.

Observation 9 It is important to note, in what follows, that *sequential convergence in* $(\mathbb{R}_\mathbb{R}, \tau_\mathbb{R})$ is nothing other than the sequential conver- gence in $\mathbb{R}$ of Elementary Real Analysis — $x_n \to x$ as $n \to \infty$ iff when ever given $\varepsilon > 0$, there exists a positive integer N such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

FACT 10 (TVS) Let $p : V_K \to \mathbb{R}$ be a seminorm on the vector space V_K. Let $(x_i)_{i\in(I,\leq)}$ be a net in V_K and $x \in V_K$. Then,

$x_i \xrightarrow{\tau_p} x \Leftrightarrow p(x_i - x) \xrightarrow{\tau_\mathbb{R}} 0$. ///

COROLLARY 11 Let $(x_n)_{n\in(\mathbb{N},\leq)}$ be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. Then,

$x_n \xrightarrow{\tau_\mathbb{R}} x \Leftrightarrow |x_n - x| \to 0$ as $n \to \infty$. ///

FACT 12 (GT) Let $X \neq \varnothing$ and τ_1, τ_2 topologies on X. If

(i) $\tau_1 \leq \tau_2$

and

(ii) τ_1 is separated,

then, τ_2 is also separated. ///

FACT 13 (TVS) Let V_K, be a vector space, and $\varnothing \neq A \subseteq V_K$. Then, A is absolutely convex if and only if $\lambda A + \mu A \subseteq A$ for $\lambda, \mu \in K$, $|\lambda| + |\mu| \leq 1$.

In particular, if A is absolutely convex, then

$\frac{1}{2}A + \frac{1}{2}A \subseteq A$. ///

Definition 14 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space. Call, an absolutely convex absorbing $U \subseteq V_K$ in which every τ-null sequence eventually lies, a *sequential neighbourhood of zero*.

FACT 15 (TVS) Let $(V, +, \theta)_K = V_K$ be a vector space, and $\mathcal{B}$ a filterbase in V_K of absolutely convex absorbing sets such that for every $V \in \mathcal{B}$ there exists $U \in \mathcal{B}$ such that $U + U \subseteq V$. Then, there exists a unique topology, τ, on V_K, making (V_K, τ) a locally convex space, and for which $\mathcal{B}$ is a local base of balanced convex neighbourhood of θ. ///

FACT 16 (TVS) Let $(V, +, \theta)_K = V_K$ be a vector space, and suppose τ_1 and τ_2 are vector topologies on V_K. Then, $N_\theta(\tau_1) \subseteq N_\theta(\tau_2) \Rightarrow \tau_1 \leq \tau_2$. ///

Let $X \neq \varnothing$, and suppose Φ is a collection of topologies on X. Let ζ be a topology on X such that

$\zeta \geq \tau$ for all $\tau \in \Phi$ ……(Δ)

There exists a coarsest among such ζ of (Δ); this coarsest is called the *supremum* of Φ and denoted $\vee\Phi$.

FACT 17(GT) Let $X \neq \varnothing$, Φ a collection of topologies on X, $x \in X$, and $(x_i)_{i\in(I, \leq)}$ a net in X. Then,

$x_i \xrightarrow{\vee\Phi} x$ if and only if $x_i \xrightarrow{\tau} x$ for each $\tau \in \Phi$. ///

FACT 18 (TVS) Let V_K be a vector space, and Φ a collection of topologies on V_K.

(i) If Φ comprises vector topologies, then $\vee\Phi$ is a vector topology on V_K.

(ii) If Φ comprises locally convex topologies, then $\vee\Phi$ is a locally convex topology. ///

Notation 19 Let $(V, +, \theta)_K = V_K$ be a vector space, and p a seminorm on V_K. Define

$p(\leq 1) \equiv \{\nu \in V_K : p(\nu) \leq 1\}$.

FACT 20 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space, and $p : (V_K, \tau) \to (\mathbb{R}, \tau_{\mathbb{R}})$ a seminorm on V_K. Then, p is continuous if and only if $p(\leq 1) \in N_\theta(\tau)$. ///

Let V_K, be a vector space, and p a seminorm on V_K. In the paragraph preceding FACT 7, we denoted by τ_p the (psseudometric) topology of p. Let P be a collection of seminorms on V_K. We shall here denote by τ_P the supremum $\vee\{\tau_p : p \in P\}$.

FACT 21 (TVS) Let (V_K, τ) be a locally convex space. Then,

(i) There exists a collection P of seminorms on V_K such that $\tau = \tau_P$, and

(ii) In (i), P can be taken to be the collection of all τ-continous seminorms. ///

Language 22 The (i) of the above FACT 21 is usually phrased by saying that, *a locally convex topology can be generated by a collection of seminorms*. The (ii) is usually phrased thus: *A locally convex topology τ can be generated by the collection P of all τ-continuous seminorms.*

2 SEMINORM, SEQUENTIAL CONTINUITY Here we stare and establish the (i) of the *Abstract*. In what follow, in (Δ^1). 0 is the zero of the reals $(\mathbb{R}, +, \cdot, 0, 1)$

THEOREM 1 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space, and $p : (V_K, \tau) \to (\mathbb{R}, \tau_{\mathbb{R}})$ a seminorm on (V_K, τ). Then, p is sequentially continuous if and only if it is sequentially continuous at θ.

Proof The implication $\Rightarrow$ is immediate from definition. For the implication $\Leftarrow$, we have the

Hypothesis p is sequentially continuous at θ.

We want to show that p is everywhere sequentially continuous. So:

Suppose $x_0 \in V_K$ and $(x_n)_{n\in(\mathbb{N}, \leq)}$ is a sequence in (V_K, τ) converging to x_0. That is, $x_n \xrightarrow{\tau} x_0$, and so by 1.1, …..(*)

$x_n - x_0 \xrightarrow{\tau} \theta$.

By the *Hypothesis*¸ therefore,

$p(x_n - x_0) \xrightarrow{t_R} p(\theta) = 0$ …..(Δ^1)

By 1.2,

$|p(x_n) - p(x_0)| \le p(x_n - x_0)$ …..(Δ^2)

Clearly, by 1.9, (Δ^1) and (Δ^2) give

$p(x_n) \xrightarrow{t_R} p(x_0)$ …..(Δ^3)

Clearly, (*) and (Δ^3) show that p is sequentially continuous at x_0. ///

Note 2 The preceding THEOREM 1 is an analogue of the popular.

THEOREM Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ and (V_K', τ') be topolo- gical vector spaces, and $f : (V_K, \tau) \to (V_K', \tau')$ a linear map.

Then, f is (τ, τ')–continuous if and only if it is (τ, τ')-continuous at θ./\/\/

Note 3 An application of THEOREM 1 appears in the proof of the forward implication $\Rightarrow$, of THEOREM 3.5.

3 THE WEBB TOPOLOGY Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space, and $\mathcal{B}$ the collection of all the sequential neighbourhoods of zero (1.14). One sees easily that $\mathcal{B}$ is a filterbase in V_K, and that if $V \in \mathcal{B}$ there exists $U \in \mathcal{B}$ such that $U + U \subseteq V$. By 1.15, there exists a unique topology, τ^+, making (V_K, τ^+) a locally convex space, and for which $\mathcal{B}$ is a local base of neighbourhoods of θ.

Before stating our first theorem, we trivially note that

FACT 1 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space. Then,

(i) an absolutely convex $U \in N_\theta(\tau)$ is a sequential neighbourhood of zero, and

(ii) (V_K, τ) has a local base of neighbourhoods of θ with members sequential neighbourhoods of zero. ///

THEOREM 2 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space. Then, $\tau \le \tau^+$, and so τ^+ is also separated.

Proof Denote by $\mathcal{B}$ the τ-sequential neighbourhoods of zero. By FACT 1(ii), (V_K, τ) has a local base of neighbourhoods, $\mathcal{R}_\theta(\tau)$, of θ, with members sequential neighbourhoods of zero. Hence,

$\mathcal{R}_\theta(\tau) \subseteq \mathcal{B} \subseteq N_\theta(\tau^+)$.

Hence, $\mathcal{R}_\theta(\tau) \subseteq N_\theta(\tau^+)$, and since $N_\theta(\tau^+)$ is a filter, it follows that

$N_\theta(\tau) \subseteq N_\theta(\tau^+)$.

By 1.16, therefore, $\tau \le \tau^+$. That τ^+ is also separated is by 1.12. ///

THEOREM 3 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space. Then,

(i) τ and τ^+ have same convergent sequences,

(ii) τ^+ is the finest locally convex topology on V_K having same convergent sequences with same limits, as τ.

Proof (i): That a τ^+-convergent sequence is τ^--convergent is immediate since $\tau \le \tau^+$. Now suppose the sequence $(x_n)_{n\in(\mathbb{N}, \le)}$ in V_K τ-converges to $x \in V_K$. That is,

$x_n \xrightarrow{\tau} x$ …..(ρ^1)

By 1.1., therefore, (ρ^1) gives

$x_n - x \xrightarrow{\tau} \theta$ …..(ρ^2)

By the definition of a sequential neighbourhood of zero it follows from (ρ^2) that

$x_n - x \xrightarrow{\tau^+} \theta$ …..(ρ^3)

Clearly, 1.1 and (ρ^3) again give

$x_n \xrightarrow{\tau^+} x$

(ii): I Let Φ be the collection of all the separated locally convex topologies on V_K having same convergent sequences, with same limits, as τ. Then, by now familiar arguments,
(i) $\vee\Phi$ is separated and locally convex,
(ii) $\vee\Phi$ has same convergent sequences, with same limits, as τ.
II Immediate from **I** is that

$$\tau^+ \le \vee\Phi$$

III Let $\mathcal{B}_\theta(\vee\Phi)$ be a local base of neighbourhoods of θ of $\vee\Phi$, and suppose

$$x_n \xrightarrow{\tau} \theta \qquad \text{…..}(\Sigma)$$

By **I(ii)**, also,
$x_n \to \theta$ in $\vee\Phi$.

Hence, if $U \in \mathcal{B}_\theta(\vee\Phi)$, then, the sequence $(x_n)_{n\in(\mathbb{N},\ \le)}$ is eventually in U. By (Σ), therefore, U is a τ-sequential neighbourhood of θ. Hence, we have shown that $\mathcal{B}_\theta(\vee\Phi) \subseteq N_\theta(\tau^+)$, from which follows that

$$N_\theta(\vee\Phi) \subseteq N_\theta(\tau^+) \qquad \text{…..}(\nabla)$$

From (∇) and by now familiar arguments, follows that

$$\vee\Phi \le \tau^+ \qquad \text{…..}(\nabla\nabla)$$

IV Clearly, **II** and the preceding $(\nabla\nabla)$ give that τ^+ is the finest separated locally convex topology on V_K having same convergent sequences, with same limits, as τ. ///

Definition 4 We shall call a separated locally convex space (V_K, τ) a *Webb space* if $\tau = \tau^+$.
The *Webb topology* τ^+ . of a separated locally convex space (V_K, τ) has been extensively studied by John H. Webb in [4] and by Albert Wilansky in [2], and also by R.F. Snipes in [5] who calls what we here call a Webb space a *C-sequential space*.

THEOREM 5 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space, and Φ the collection of all separated locally convex topologies on V_K having same convergent sequences with same limits as τ. Then, $\tau^+ = \vee\Phi$.
Proof See the proof of THEOREM 3(II) above. ///
THEOREM 6 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a separated locally convex space, and p a seminorm on V_K. Then, p is τ^+-continuous if and only if it is τ-sequentially continuous.
Proof ⇒: *Hypothesis* p is τ^+-continuous.
By 1.6, therefore, p is τ^+-sequentially continuous. Hence,

$$x_n \xrightarrow{\tau^+} \theta \Rightarrow p(x_n) \xrightarrow{\tau_R} p(\theta) \qquad \text{…..}(\rho^5)$$

But by THEOREM 3(i)

$$x_n \xrightarrow{\tau^+} \theta \Rightarrow x_n \xrightarrow{\tau^+} \theta,$$

from which follows by (ρ^5) that

$$x_n \xrightarrow{\tau} \theta \Rightarrow p(x_n) \xrightarrow{\tau_R} p(\theta) = 0.$$

And so, p is τ-sequentially continuous at θ. By 2.1, therefore, p is τ-seq- uentially continuous.
⇐: *Hypothesis* p **is** τ-sequentially continuous.
We want to show that p is τ^+-continuous. Since by definition, the τ-sequential neighbourhoods of zero constitute a τ^+-local base of neighbourhods of θ, it suffices by 1.20 to show that $p(\le 1)$ is a τ-seq- uential neighbourhood of zero. So, suppose

$$x_n \xrightarrow{\tau} \theta \qquad \text{…..}(\rho^6)$$

By *Hypothesis*, it follows from (ρ^6) that

$$p(x_n) \xrightarrow{\tau_R} p(\theta) = 0$$

and so eventually $(x_n)_{n\in(\mathbb{N},\leq)}$ shall lie in $p(\leq 1)$. One checks easily that $p(\leq 1)$ is absolutely convex and absorbing. ///
A corollary of THEOREM 6 and 1.21(ii) is

THEOREM 7 Let (V_K, τ) be a separated locally convex space. Then, $\tau^+ = \vee\{\tau_p : p$ is a seminorm on V_K, p is τ-sequentially continuous$\}$. ///
A corollary of THEOREM 6 and THEOREM 7 is

THEOREM 8 A separated locally convex space is a Webb space if and only if the sequentially continuous seminorms are the continuous seminorms.///

Remark 9 We furnish elsewhere an application of THEOREM 8.

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