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ON DIFFERENTIABILITY OF THE RESTRICTION AND THE IDT

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Abstract

An observation on the differentiability of the restriction, f | I, implying the differentiability of f, is noted, and applied in a proof of the Inverse Differentiation Theorem (The IDT).

Keywords: continuous at *a*, differentiable at *a*, inverse, Con- tinuity of he Inverse Theorem, invertible, The Inverse Differentiation Theorem (the IDT).

1. LANGUAGE AND NOTATION

Our language (summed up already in our keywords and phrases) and notation shall be pretty stan- dard as found in Bartle-Sherbert [1]. We signify by /// the end or absence of a proof.
Our concern in this paper is *Elementary Real Analysis* **(ERA)**, and so, our functions are *real functions*

$f\ : A \to \mathbb{R}, \varnothing \neq A \subseteq \mathbb{R}$ ……(RealFun)

For ease of reference, and perhaps also fixing notation, we recall some needed results of **ERA**.

THEOREM 1 Let $\varnothing \neq A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$ continuous at *a*. Then,

(i) If $f(a) > 0$, there exists $\delta > 0$ such that
$f(x) > 0$ for all $x \in A \cap N_\delta(a)$.

(ii) If $f(a) < 0$, there exists $\delta > 0$ such that
$f(x) < 0$ for all $x \in A \cap N_\delta(a)$.

(iii) If $f(a) \neq 0$, there exists $\delta > 0$ such that
$f(x) \neq 0$ for all $x \in A \cap N_\delta(a)$.
and

(iv) If $f(a) \neq 0$, there exists $\lambda > 0$ and $\delta > 0$ such that
$|f(x)| > \lambda$ for all $x \in A \cap N_\delta(a)$. ///

We give some interpretations of the preceding THEOREM 1 for $A = I$ an interval.

INTERP. 2 Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$. Suppose

(i) a is a left/right endpoint of I,
(ii) f is continuous at a,
and
(iii) $f(a) \neq 0$.

Then,

(α) There exists a subinterval J of I
such that
(β) a is a left/right endpoint of J,
(γ) $f(x) \neq 0$ for al $x \in J$. ///

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INTERP. 3 Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$. Suppose
(i) a interior to I,
(ii) f continuous at a,
and
(iii) $f(a) \neq 0$.
Then,
(α) There exists a subinterval J of I
such that
(β) a is interior to J,
and
(γ) $f(x) \neq 0$ for al $x \in J$. ///

The Interval Theorem 4 Let I be an interval and $f : I \to \mathbb{R}$. If $f : I \to \mathbb{R}$ is continuous, then, the range $f(I)$ of f, is an interval. ///
Let I be an interval and $f : I \to \mathbb{R}$. If f is increasing/ decreasing it is called a *monotone* function. Strictly increasing/ strictly decreasing f is called a *strictly monotone* function. Clearly, a strictly monotone function is an injective function.
Let I be an interval and $f : I \to \mathbb{R}$. If f is injective it is also called an *invertible function,* with the *real* function
$f^{-1} : f(I) \to \mathbb{R},\ f(x) \mapsto x,\ x \in I$
called its *inverse*.
We have
Continuity of the Inverse Theorem 5 Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$.
(i) If f is strictly increasing/strictly decreasing and continuous at a, then its inverse, $f^{-1} : f(I) \to \mathbb{R},\ f(x) \mapsto x,\ x \in I$ is also strictly increasing/strictly decreasing and continuous at $f(a)$. ///

2 THE OBSERVATION Let I be an interval and $a \in I$. The *real* function $f : I \to \mathbb{R}$ is said to be *differentiable* at a if the limit $\lim_{x \to a} f^{*a}(x)$ of the function

$$f^{*a} : I - \{a\} \to \mathbb{R},$$
$$x \mapsto \frac{f(x) - f(a)}{x - a}$$

exists. The limit $\lim_{x \to a} f^{*a}(x)$, then denoted $f'(a)$, is called the *derivative* of f at a .
A popular theorem is

THEOREM 1 Let I be an interval, J a subinterval of I, $a \in J$, and the *real* function $f : I \to \mathbb{R}$ differentiable at a. Then, the restriction, $f|J : J \to \mathbb{R},\ x \mapsto f(x),\ x \in J$, of f to J is also differentiable at a with derivative $(f|J)'(a) = f'(a)$. ///
Now to

The Observation 2 Let I be on interval, J a subinterval of I, $a \in J$, and $f : I \to \mathbb{R}$. Then,
(i) a is a left/right endpoint of I and J, and $f|J$ is differentiable at a $\Big\}$ $\Rightarrow f$ is differentiable at a

(ii) a is a left/right endpoint of J, but interior to I and $f|J$ differentiable at a $\Big\}$ may $\not\Rightarrow f$ differentiable at a
[| For an example, the *absolute value* function $|\ |_{\mathbb{R}} : \mathbb{R} \to \mathbb{R},\ x \mapsto |x|,\ x \in \mathbb{R}$ with $a = 0$ |]

(iii) a interior to both J and I and $f|J$ differentiable at a $\Big\}$ $\Rightarrow f$ is differentiable at a

Proof A moment's thought. ///

3 AN APPLICATION We apply the preceding to give proof of the *Inverse Differentiation Theorem* (The IDT). First, we state, for ease of reference Caratheodory's characterization of differentiability.

Caratheodory's Theorem [1] 1

Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$. Then, f is differentiable at a if and only if there exists a function $\varphi : I \to \mathbb{R}$ such that

(i) $f(x) - f(a) = \varphi(x)(x - a)$ for all $x \in I$,

and

(ii) φ is continuous at a.

If this is the case, then $\varphi(a) = f'(a)$. ///

Next, we state and establish

The Inverse Differentiation Theorem 2 Let I be an interval, $f : I \to \mathbb{R}$ strictly monotone and continuous, and

$f^{-1} : f(I) \to \mathbb{R}, f(x) \mapsto x, x \in I$

the strictly monotone and continuous inverse (1.5) of f. Let $a \in I$. Suppose

(i) f is differentiable at a,

and

(ii) $f'(a) \neq 0$.

Then,

(Σ_1) f^{-1} is differentiable at $f(a)$, and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)},$$

and

(Σ_2) If $y \in f(I)$ and f is differentiable at $f^{-1}(y)$ with $f'(f^{-1}(y)) \neq 0$, then f^{-1} is differentiable at y with derivative

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))},$$

Proof Clearly, (Σ_2) is simply a restatement of (Σ_1). So, we establish (Σ_1). Assume f strictly increasing. By Caratheodory's Theo- rem 1, there exists $\varphi : I \to \mathbb{R}$ such that

(α) $f(x) - f(a) = \varphi(x)(x - a)$ for all $x \in I$,

(β) φ is continuous at a,

and

(γ) $\varphi(a) = f'(a)$.

By the hypothesis (ii), therefore $\varphi(a) \neq 0$, and so by the INTERP. 1.2 &1.3, there exists a subinterval J of I such that

(ρ) $a \in J$, and $\varphi(x) \neq 0$, for all $x \in J$.

Clearly, from (α) :

If $x \in J$, then

$f(x) - f(a) = f(f^{-1}(f(x))) - f(f^{-1}(f(a))) = \varphi(f^{-1}(f(x)))[f^{-1}(f(x)) - f^{-1}(f(a))]$.

That is, if $x \in J$, then

$$f(x) - f(a) = \varphi(f^{-1}(f(x)))[f^{-1}(f(x)) - f^{-1}(f(a))] \qquad \ldots\ldots(\nabla)$$

Of course by 1.5, $f|J$ is strictly increasing, continuous and has the inverse $(f|J)^{-1}$ with domain the *interval* (1.4) $f(J) \ni f(a)$. So, in (∇), we are moving forward and backward between J and $f(J)$; and f and f^{-1} there are actually $f|J$ and $(f|J)^{-1}$.

Now, from (ρ), we have

$\varphi(f^{-1}(f(x))) \neq 0$

for all $x \in J$.

Therefore, from (∇), we obtain:

For $x \in J$,

$$\left.\begin{aligned} &f^{-1}(f(x)) - f^{-1}(f(a)) = \frac{1}{\varphi(f^{-1}(f(x)))}(f(x) - f(a)) \\ &= \left(\frac{1}{\varphi \circ f^{-1}}\right)(f(x))\cdot(f(x) - f(a) \end{aligned}\right\} \quad \ldots\ldots(\nabla\nabla)$$

As noted earlier, $f^{-1}[| = (f \mid J)^{-1}|]$ is a continuous function and so, the composition

$\varphi o f^{-1} : f(J) \to \mathbb{R}$

is continuous at $f(a)$. Hence,

$\frac{1}{\varphi \circ f^{-1}}$ is also continuous at $f(a)$) ……($\nabla\nabla\nabla$)

By ($\nabla\nabla$), ($\nabla\nabla\nabla$) and Caratheodory's Theorem 1, therefore;

$(f \mid J)^{-1}$ is differentiable at $f(a)$ with derivative

$$\frac{1}{\varphi \circ (f \mid J)^{-1}}(f(a)) = \frac{1}{\varphi(a)}$$

$$= \frac{1}{f'(a)}$$

That is,

$$[|(f \mid J)^{-1}|]'(f(a)) = \frac{1}{f'(a)} \quad \ldots\ldots(\prod)$$

We know that the functions

$f^{-1} : f(I) \to \mathbb{R}$

and

$(f \mid J)^{-1} : f(J) \to \mathbb{R}$

are such that

(ξ_1) both are strictly increasing,

(ξ_2) $f(J)$ is a subinterval of $f(I)$,

(ξ_3) $f(a) \in f(J) \subseteq f(I)$,

and

(ξ_4) by ($\prod$), $(f \mid J)^{-1}$ is differentiable with derivative $\frac{1}{f'(a)}$.It is clearly, immediate from (ξ_1), (ξ_2), (ξ_3), (ξ_4), 1.2, 1.3 and The Observation 2.2 that f^{-1} is also differentiable at $f(a)$ with derivative $\frac{1}{f'(a)}$

REFERENCES

[1] R.G. Bartle and Donald R. Sherbert, *Introduction to Real Analysis* 3rd Edition, John Wiley, New York, 2000.