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ON SERIES DEFINITIONS OF THE EXPONENTIAL, COSINE AND SINE FUNCTIONS

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Abstract

There are many methods, in the literature, of defining the exponential, cosine and sine functions. One method is employing series to define them. We here furnish a clear, simple and unflustered proof of convergence of these series.

Keywords: Convergence Ratio Test, Alternating Series Test, the nth Term Test.

1. LANGUAGE AND NOTATION

Our language and notation shall be pretty standard, as found, for example in [1, 2, 3]. From a *real sequence*

$(a_1, a_2, \ldots\ldots)$ …… (RealSeq)

we obtain a real *series*

$a_1 + a_2 + \ldots\ldots\ldots\ldots$ …… (RealSer)

We may, following [2], denote (RealSeq) by $(a_n)_{n=1}^{\infty}$. *Note* : Round brackets ( ). We denote (Real Ser) by $\sum_{n=1,\infty} a_n$ Hence, we shall often write

$(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots\ldots)$

and

$\sum_{n=1,\infty} a_n = a_1 + a_2 + \ldots\ldots\ldots$

The notation ℕ stands for the *positive integers*, while ℝ denotes the *rea1 numbers*. We signify by /// the end or absence of a proof.

In what follows by a *sequence* /*series* we mean a *real* sequence /a *real series*.

As advertised in the abstract, this paper promises a clear simple unclustered *proof of convergence* of the series employed in the definitions of the *exponential*, the *cosine* and the *sine* functions

Checking the correctness of proofs is a delicate matter

– Professor John Horvath of Maryland at College Park, U.S.A in a private communication.

If a student of Analysis *defines* Elementary Real Analysis as the study of the *elementary functions*, it is very unlikely that he/she shall be charged with heresy. And so, this author makes bold to say that *unclustered* proofs of convergence of the series defining the *elementary functions* cannot be overemphasized. The contribution of this paper is its offer of *clarity of idea* in the teaching of *Elementary Real Analysis*.

Next, we assemble, for ease of reference, some elementary *tests* for convergence, or otherwise, of a series.

2 CONVERGENCE TESTS

The nth Term Test 1 The series $a_1 + a_2 + \ldots\ldots\ldots$ converges only if the sequence $(a_n)_{n=1}^{\infty}$ is null. ///

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The Ratio Test 2 Let

$$\sum_{n=1,\infty} a_n \qquad \text{……(Ser)}$$

be a series of strictly positive terms. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$, then (Ser) converges. ///

Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-negative terms, that is $a_n \geq 0$ for all n. The series

$\sum_{n=1,\infty} (-1)^{n+1} a_n = a_1 + (-a_2) + a_3 + (-a_4) + \ldots\ldots\ldots\ldots$

usually written

$a_1 - a_2 + a_3 - a_4 + \ldots\ldots\ldots\ldots$

and the series

$\sum_{n=1,\infty} (-1)^{n} a_n = (-a_1) + a_2 + (-a_3) + a_4 + (-a_5) \ldots\ldots\ldots\ldots$

usually written

$-a_1 + a_2 - a_3 + a_4 - a_5 \ldots\ldots\ldots\ldots$

are each called an *alternating series*. We have

Alternating Series Test 1 3 Suppose the sequence $(a_n)_{n=1}^{\infty}$ is such that

(i) $a_n \geq 0$ for all n,

(ii) $(a_n)_{n=1}^{\infty}$ is a decreasing sequence, and

(iii) $(a_n)_{n=1}^{\infty}$ is null.

Then, the series $\sum_{n=1,\infty} (-1)^{n+1} a_n$ converges. ///

We note

FACT 4 Suppose $\alpha \in \mathbb{R}$ and that the series $\sum_{n=1,\infty} a_n = a_1 + a_2 + \ldots\ldots\ldots$ converges with sum S. Then

(i) The series $\sum_{n=1,\infty} \alpha a_n = \alpha a_1 + \alpha a_1 + \alpha a_3 + \ldots\ldots\ldots$ converges with sum αS, and

(ii) The series $\alpha + \sum_{n=1,\infty} a_n = \alpha + a_1 + a_2 + a_3 + \ldots\ldots\ldots$ converges with sum $\alpha + S$. ///

A sequence

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots\ldots) \qquad \text{..… (Seq)}$$

is called a *decreasing sequence* and said to *decrease* provided $a_{n+1} \leq a_n$ for all n. We shall say that (Seq) *eventually decreases* if there exists $N \in \mathbb{N}$ such that $a_{n+1} \leq a_n$ for all $n \geq N$. With this language we formulate a version of the *Alternating Series Test*.

Alternating Series Test 2 5 Suppose the sequence $(a_n)_{n=1}^{\infty}$ is such that

(i) $a_n \geq 0$ for all n,

(ii) $(a_n)_{n=1}^{\infty}$ eventually decreases and

(iii) $(a_n)_{n=1}^{\infty}$ is null.

Conclusion The alternating series $\sum_{n=1,\infty} (-1)^{n+1} a_n$ converges. ///

With FACT 4(i), we have from the above

The Alternating Series Test 3 6 Suppose the sequence $(a_n)_{n=1}^{\infty}$ is such that

(i) $a_n \geq 0$ for all n,

(ii) $(a_n)_{n=1}^{\infty}$ eventually decreases

and

(iii) $(a_n)_{n=1}^{\infty}$ is null.

Conclusion The alternating series $\sum_{n=1,\infty}(-1)^n\, a_n$ converges. ///

We now proceed to the task of this paper of furnishing clear, unclustered proofs of convergence of the series defining the elementary functions – the *exponential*, the *cosine* and the *sine*.

3 THE EXPONENTIAL SERIES The series

$$\sum_{n=0,\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots.., x \in \mathbb{R}, \qquad \ldots\ldots\text{(ExpSer)}$$

is called the *exponential series*. We examine (ExpSer) for convergence.

Case x = 0 Clearly, if $x = 0$ (ExpSer) trivially converges.

Case x > 0 For $x > 0$,

$$\frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n} = \frac{x}{n+1}.$$

And so, clearly,

$$\lim_{n\to\infty}\frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty}\frac{x}{n+1} = 0.$$

For $x > 0$, the terms of (ExpSer) are non-negative and

$$\lim_{n\to\infty}\frac{x^{n+1}/(n+1)!}{x^n/n!} = 0 < 1.$$

By the Ratio Test 2(2.2),therefore (ExpSer) converges. And this con- cludes the proof of the Case $x > 0$.

Still suppose $x > 0$. By The nth Term Test(2.1), and the preceding,

(i) the sequence $\left(\frac{x^n}{n!}\right)_{n=1}^{\infty}$ is null,

(ii) $\frac{x^n}{n!} \geq 0$ for all n,

and

(iii) $\frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}$ is eventually less than 1.

By the Alternating Series Test 2(2.5), the series

$$\sum_{n=1,\infty}(-1)^{n+1}\frac{x^n}{n!} = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots\ldots\ldots$$

converges. We record this as

FACT 1 For $x > 0$, the alternating series

$$\sum_{n=1,\infty}(-1)^{n+1}\frac{x^n}{n!} = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots\ldots \qquad \ldots\text{.(AltExpSer, } x > 0)$$

converges. ///

We now move to

Case x < 0 Now suppose $y < 0$, and so, $-y > 0$. By FACT 1 above, therefore, the series

$$\sum_{n=1,\infty}(-1)^{n+1}\frac{(-y)^n}{n!}$$

converges. That is, the series

$$\sum_{n=1,\infty}(-1)^{n+1}\ \frac{((-1)y)^n}{n!}$$

converges. That is, the series

$$\sum_{n=1,\infty}(-1)^{n+1}\ \frac{(-1)^n y^n}{n!}$$

converges. That is, the series

$$\sum_{n=1,\infty}(-1)^{n+1+n}\ \frac{y^n}{n!}$$

converges. That is, the series

$$\sum_{n=1,\infty}(-1)\ \frac{y^n}{n!}$$

By 2.4(i), it follows from this that

$$\sum_{n=1,\infty}(-1)\ \frac{y^n}{n!} = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\ldots\ldots$$

converges. By 2.4(ii), therefore,

$$1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\ldots\ldots\ldots$$

converges. By hypothesis, $y < 0$, and we have thus concluded the proof of the ***Case x < 0***.

So, we have

THEOREM 2 The *exponential series*

$$\sum_{n=0,\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \ldots\ldots\ldots$$

Converges for all x. ///

4 THE COSINE SERIES Consider the series

$$\sum_{n=1,\infty}(-1)^n\ \frac{x^{2n}}{(2n)!} = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ \ldots\ldots\ldots \qquad \ldots..\text{(Ser cos)}$$

Clearly, for $x = 0$, (Ser cos) converges. Suppose $x \neq 0$. Clearly then

(i) $\dfrac{x^{2n}}{(2n)!} \geq 0$ for all n,

(ii) the sequence $\left(\dfrac{x^{2n}}{(2n)!}\right)_{n=1}^{\infty}$ eventually decreases, since

$$\frac{x^{2(n+1)}/[2(n+1]!}{x^{2n}/(2n)!} = \frac{x^2}{(2n+1)(2n+2)}$$

is eventually les than 1
and

(iii) the sequence $\left(\dfrac{x^{2n}}{(2n)!}\right)_{n=1}^{\infty}$ is null. For, it is a subsequence of $\left(\dfrac{x^{n}}{n!}\right)_{n=1}^{\infty}$ which is null by The nth Term Test(2.1) and 3.2.

Therefore, by 2.6, (Ser cos) converges, for $x \neq 0$. Hence by 2.4(ii), the cosine series 1 + (Ser cos) converges for all x. That is,

$$\sum_{n=1,\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots\ldots$$

converges. We record this as

THEOREM 1 The *cosine* series

$$\sum_{n=0,\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots\ldots$$

converges for all x. ///

5 THE SINE SERIES We show that the *sine series*

$$\sum_{n=1,\infty}(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots \qquad \text{…..(Sin Ser)}$$

converges for all x.

Case x = 0 If $x = 0$, (Sin Ser) clearly converges.

Case x > 0 Let $x > 0$. Then, by now familiar arguments, the sequence

$$\left(\frac{x^{2n-1}}{(2n-1)!}\right)_{n=1}^{\infty} \qquad \text{……. (Seq Sin)}$$

(i) is a sequence of non-negative term,

(ii) eventually decreases,

and

(iii) is null.

Hence, by 2.6, the series $\sum_{n=1,\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!}$ converges. And so, by 2.4(i),

the series

$$\sum_{n=1,\infty}(-1)(-1)^n \frac{x^{2n-1}}{(2n-1)!} \qquad \text{….}(\sigma^1)$$

converges. By a Property of ℝ [| For a, $b \in \mathbb{R}$, $b \neq 0$, $\frac{a}{-b} = -\frac{a}{b}$ |],

$$\frac{1}{-1} = -\frac{1}{1} = -1,$$

and so,

$$(-1)^{n-1} = \frac{(-1)^n}{-1} = (-1)^n \cdot \frac{1}{-1} = (-1)^n(-1)$$

for all n. That is,

$(-1)^{n-1} = (-1)^n(-1) = (-1)(-1)^n$ for all n ….(σ^2)

Clearly, from (σ^1) and (σ^2) follows that the series

$$\sum_{n=1,\infty}(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \qquad \text{…...(SinSer)}$$

converges.

Case x < 0 By now every familiar arguments!!

Thus, we have

THEOREM 1 The *sine* series

$$\sum_{n=1,\infty}(-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$$

converges for all x. ///

6 SERIES DEFINITIONS We now give the series definitions of the elementary functions advertised in the *Abstract*.

(i) The *exponential*(the *exponential function*) is the function

$$E : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n\,!},\ \ x \in \mathbb{R}.$$

(ii) The *cosine* (the *cosine function*) is the function

$$\cos : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{n=0}^{\infty} (-1)^n\ \frac{x^{2n}}{(2n)\,!},\ \ x \in \mathbb{R}.$$

(iii) The *sine* (the *sine function*)is the function

$$\sin : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{n=1}^{\infty} (-1)^{n-1}\ \frac{x^{2n-1}}{(2n-1)\,!},\ \ x \in \mathbb{R}.$$

REFERENCES

[1] Adegoke Olubummo, *Introduction to Real Analysis* , Heinemann, Nig. Ltd, Ibadan, 1978

[2] Karl Stromberg, *INTRODUCTION TO CLASSICAL REAL ANALYSIS*, Wadsworth Inc., Bellmot, California, 1981.

[3] R.G. Bartle and Donald R. Sherbert, *INTRODUCTION TO REAL ANALYSIS*, 3rd Edition, John Wiley & Sons, Inc., New York, 2000.