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SOME RESULTS ON 3-PRIME NEAR-RINGS INVOLVING DERIVATIONS

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Abstract

This research work deals some new results on near-rings through derivations and present the commutativity of a 3-prime near-ring satisfying some differential and algebraic identities on nonzero Jordan ideals of 2-torsion free involving derivations by considering two derivations instead of one derivation and also prove some result on special class of near-rings with suitable constraints of its subsets.

Keywords: Derivation, 3-prime near-rings, 2-torsion free, Jordan ideal.

1 Introduction

Throughout the paper a left near-ring $(N, +, \cdot)$ is a triplet, where *N* is a nonempty set, $+$ and $\cdot$ are two binary operations, the structure $(N, +)$ is a group, $(N, \cdot)$ is a semigroup, and $z\cdot(x+y)=z\cdot x+z\cdot y$ for all $x, y, z\in N$. As usual *N* represents $(N, +, \cdot)$. A near-ring *N* is said to be a zero-symmetric if $0x=0$ for all $x\in N$ ($x0-0$ for left distributive law). A near-ring *N* is a prime near-ring if for all $x, y\in N, xNy=(0)$ implies either $x=0$ or $y=0$ and *N* is semi prime near-ring if for $x\in N, xNx=(0)$ implies $x=0$. *N* will be 3-prime near-ring if for all $x, y\in N, xNy=\{0\}$ implies $x=0$ or $y=0$. *N* is called 2-torsion free if $2x=0$ implies $x=0$, for all $x\in N$. In general *N* is called *n*-torsion free if $nx=0$ implies $x=0$ for all $x\in N$. *Z(N)* is known as the multiplicative center of *N*. A near-ring *N* is called derivation on *N* for all $x, y\in N$ satisfies $d(xy)=d(x)y+xd(y)$ or $d(xy)=xd(y)+d(x)y$, $\forall x, y\in N$. The symbols $[x, y]=xy-yx$ and $(x\circ y)=xy+yx$ are both commutator (Lie product) and anticommutator (Jordan product). $[x, xy]=xxy-xyx=x(xy-yx)=x[x, y]$ and $(x\circ xy)=xxy+xyx=x(xy+yx)=x(x\circ y)$ are Jacobi Identities.

The concept of derivation in rings is a fundamental and plays a vital role in many branches of mathematics. The notion of derivation in rings introduced in 1957 by Posner [1] where he established two very striking results on derivations in prime rings. These results give a considerable interest in investigating commutativity of rings, frequently prime ring and semiprime rings that admitting suitable constrained of derivations. Several authors have studies the prime rings and semiprime rings involving derivation in different directions. Bell and Mason [2] have been motivated by the concept of derivation in rings and introduced the notion of derivation in near-rings. Since then, many authors (see for example [3 – 9] for further reference) have investigate 3-prime near-ring and semiprime near-ring with 2-torsion free satisfying certain differential and algebraic identities involving derivations.

2 Some Basic Results

In order to prove main results, we need the following lemmas.

2.1 Lemma [3]

Let *J* be a nonzero Jordan ideal of a 2-torsion free 3-prime near-ring *N*. If $J\subseteq N$, then *N* is a commutative ring.

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2.2 Lemma [2]
Let d be a derivation on a near ring N. Then
$(d(x)y + xd(y))z = d(x)yz + xd(y)z$, $\forall x, y, z \in N$.

3 Some Results on 3-Prime Near-rings with Derivations
Bell and Daif [10], showed that if R is a 2-torsion free prime ring admitting a strong commutativity preserving (in short, SCP) derivation d, i.e., d satisfies $[d(x), d(y)] = [x, y]$ for every $x, y \in R$, then R is commutative. Further, Asma and Huque [3] proved that let J be a nonzero Jordan ideal of a 2-torsion free 3-prime near-ring N. If d_1, d_2 are two nonzero derivations on N such that d_2 is commuting on J and $[d_1(x), d_2(y)] = [x, y]$ for all $y \in J$ and $x \in N$, then either $d_1 = 0$ on J or N is a commutative ring.

Motivated by these observations, it is a natural question to ask if k and i are positive integers instead of $k = 0$ and $i = 0$. In this works we give affirmative answer and extend the result of Asma and Huque [3] for a 3-prime near-ring involving two derivations. In addition, we prove the commutativity of a 3-prime near-ring N with its Jordan ideal.

3.1 Theorem
Let J be a nonzero Jordan ideal of a 2-torsion free 3-prime near ring N. If d_1, d_2 are two nonzero derivations on N such that d_2 is commuting on J and satisfying one of the following conditions:
(1) $[d_1(y), d_2(x)] = x^k[x, y]x^i$, for all $x \in J$ and $y \in N$.
(2) $[d_1(y), d_2(x)] = x^k(x \circ y)x^i$, for all $x \in J$ and $y \in N$;
(3) $(d_1(y) \circ d_2(x)) = x^k[x, y]x^i$, for all $x \in J$ and $y \in N$;
(4) $(d_1(y) \circ d_2(x)) = x^k(x \circ y)x^i$, for all $x \in J$ and $y \in N$

where both k and i are integers greater than zero ($k > 0$, and $i > 0$). Then either $d_1 = 0$ on J or N is a commutative ring.

Proof
(1) By hypothesis,
$$[d_1(y), d_2(x)] = x^k[x, y]x^i, \text{ for all } x \in J \text{ and } y \in N. \quad (1)$$
Replacing y by xy in equation (1) we have
$$[d_1(xy), d_2(x)] = x^k[x, xy]x^i.$$
$$[d_1(xy), d_2(x)] = x(x^k[x, xy]x^i). \quad (2)$$
Substitute equation (1) in equation (2) to obtain
$$[d_1(xy), d_2(x)] = x[d_1(y), d_2(x)].$$
$$d_1(xy)d_2(x) - d_2(x)d_1(xy) = x(d_1(y)d_2(x) - d_2(x)d_1(y)).$$
Applying definition of derivation on d_1 we get
$$(xd_1(y) - d_1(x)y)d_2(x) - d_2(x)(xd_1(y) - d_1(x)y) = x(d_1(y)d_2(x) - d_2(x)d_1(y)).$$
Using Lemma 2, the last equation yield the expression below, since d_2 is commuting on J
$$d_1(x)yd_2(x) = d_1(x)d_2(x)y \quad \forall x \in J \text{ and } y \in N. \quad (3)$$
Substituting yz as y in equation (3), we find that
$$d_1(x)yzd_2(x) = d_1(x)d_2(x)zy.$$
$$d_1(x)N[d_2(x), z] = \{0\}, \quad \forall x \in J \text{ and } y \in N.$$
By 3-primeness of N, we obtain
$$d_1(x) = 0 \text{ or } - \ \forall x \in J. \quad (4)$$
If $d_2(x) \in Z(N)$ for all $x \in J$, then our hypothesis becomes $[x, y] = 0$ for all $x \in J$ and $y \in N$ which means that $x \in Z(N)$ for all $x \in J$. Therefore, (4) becomes
$$d_1(x) = 0 \text{ or } x \in Z(N). \ \forall x \in J. \quad (5)$$

Hence, by Lemma 1, we obtain the result.
(2) By hypothesis,

$$[d_1(y), d_2(x)] = x^k(x \circ y)x^i \text{, for all } x \in J \text{ and } y \in N. \qquad (6)$$

Replacing *y* by *xy* in equation (6) we have

$$[d_1(xy), d_2(x)] = x^k(x \circ xy)x^i.$$

$$[d_1(xy), d_2(x)] = x(x^k(x \circ xy)x^i). \qquad (7)$$

Substitute equation (6) in equation (7) to obtain

$$[d_1(xy), d_2(x)] = x[d_1(y), d_2(x)].$$

$$d_1(xy)d_2(x) - d_2(x)d_1(xy) = x(d_1(x)d_2(x) - d_2(x)d_1(y)).$$

Applying definition of derivation on d_1 we get

$$(xd_1(y) + d_1(x)y)d_2(x) - d_2(x)(xd_1(y) + d_1(x)y) = x(d_1(x)d_2(x) - d_2(x)d_1(y)).$$

Using Lemma 2, the last equation yield the expression below, since d_2 is commuting on *J*

$$d_1(x)yd_2(x) = d_1(x)d_2(x)y \quad \forall x \in J \text{ and } y \in N. \qquad (8)$$

Substituting *yz* as *y* in equation (8), we find that

$$d_1(x)yzd_2(x) = d_1(x)d_2(x)zy.$$

$$d_1(x)N(d_2(x) \circ z) = \{0\}, \quad \forall x \in J \text{ and } z \in N.$$

By 3-primeness of *N*, we obtain

$$d_1(x) = 0 \text{ or } x \in Z(N), \forall x \in J, \qquad (9)$$

If $d_2(x) \in Z(N)$ for all $x \in J$, then our hypothesis becomes $(x \circ y) = 0$ for all $x \in J$ and $y \in N$ which means that $x \in Z(N)$ for all $x \in J$. Therefore, (9) becomes

$$d_1(x) = 0 \text{ or } x \in Z(N), \ \forall x \in J. \qquad (10)$$

Hence, by Lemma 1, we conclude that *N* is a commutative ring.
(3) By hypothesis,

$$(d_1(y) \circ d_2(x)) = x^k[x, y]x^i \text{, for all } x \in J \text{ and } y \in N. \qquad (11)$$

Replacing *y* by *xy* in equation (11) we have

$$(d_1(xy) \circ d_2(x)) = x^k[x, xy]x^i.$$

$$(d_1(xy) \circ d_2(x)) = x(x^k[x, y]x^i). \qquad (12)$$

Substitute equation (11) in equation (12) to obtain

$$(d_1(xy) \circ d_2(x)) = x(d_1(y) \circ d_2(x)).$$

$$d_1(xy)d_2(x) + d_2(x)d\ (xy) = x(d_1(y)d_2(x) + d_2(x)d_1(y)).$$

Applying definition of derivation on d_1 we get

$$(xd_1(y) + d_1(x)y)d_2(x) + d_2(x)(xd_1(y) + d_1(x)y) = x(d_1(y)d_2(x) + d_2(x)d_1(y)).$$

Using Lemma 2, the last equation yield the expression below, since d_2 is commuting on *J*

$$d_1(x)yd_2(x) = d_1(x)d_2(x)y, \quad \forall x \in J \text{ and } y \in N. \qquad (13)$$

Substituting *yz* as *y* in equation (13), we find that

$$d_1(x)yzd_2(x) = d_1(x)d_2(x)zy.$$

$$d_1(x)N[d_2(x), z] = \{0\}, \quad \forall x \in J, \text{ and } z \in N.$$

By 3-primeness of *N*, we obtain

$$d_1(x) = 0 \text{ or } x \in Z(N), \forall x \in J, \qquad (14)$$

If $d_2(x)\in Z(N)$ for all $x\in J$, then our hypothesis becomes $[x,\ y]=0$ for all $x\in J$ and $y\in N$ which means that $x\in Z(N)$ for all $x\in J$. Therefore, (14) becomes

$$d_1(x)=0 \text{ or } x\in Z(N),\ \forall x\in J. \tag{15}$$

Hence, by Lemma 1, we obtain the result.

(4) By hypothesis,

$$(d_1(y)\circ d_2(x))=x^k(x\circ y)x^i \text{, for all } x\in J \text{ and } y\in N. \tag{16}$$

Replacing *y* by *xy* in equation (16) we have

$$(d_1(xy)\circ d_2(x))=x^k(x\circ xy)x^i.$$

$$(d_1(xy)\circ d_2(x))=x(x^k(x\circ y)x^i). \tag{17}$$

Substitute equation (16) in equation (17) to obtain

$$(d_1(xy)\circ d_2(x))=x(d_1(y)\circ d_2(x)).$$

$$d_1(xy)d_2(x)+d_2(x)d_1(xy)=x(d_1(y)d_2(x)+d_2(x)d_1(y)).$$

Applying definition of derivation on d_1 we get

$$(xd_1(y)+d_1(x)y)d_2(x)+d_2(x)(xd_1(y)+d_1(x)y)=x(d_1(y)d_2(x)+d_2(x)d_1(y)).$$

Using Lemma 2, the last equation yield the expression below, since d_2 is commuting on *J*

$$d_1(x)yd_2(x)=-(d_1(x)d_2(x)y),\ \forall x\in J \text{ and } y\in N. \tag{18}$$

Substituting *yz* as *y* in equation (18), we find that

$$d_1(x)yzd_2(x)=-(d_1(x)d_2(x)zy).$$

$$d_1(x)N(d_2(x)\circ z)=\{0\},\ \forall x\in J \text{ and } y\in N.$$

By 3-primeness of *N*, we obtain

$$d_1(x)=0 \text{ or } x\in Z(N),\ \forall x\in J, \tag{19}$$

If $d_2(x)\in Z(N)$ for all $x\in J$, then our hypothesis becomes $(x\circ y)=0$ for all $x\in J$ and $y\in N$ which means that $x\in Z(N)$ for all $x\in J$. Therefore, (19) becomes

$$d_1(x)=0 \text{ or } x\in Z(N),\ \forall x\in J. \tag{20}$$

Hence, by Lemma 1, we obtain *N* is a commutativity ring.

4 Conclusion

This paper has provided a useful result on near-rings through derivations and presents their fundamentals. Firstly, it is shown that some commutativity results for 3-prime near-rings with algebraic identities of Lie and Jordan products involving derivations. Secondly, the certain results presented in this paper are extension of previously obtained results and also prove some result on special class of near-rings with suitable constraints of its subsets via derivations. Thirdly, it is prove that the commutativity of prime near-ring satisfying the differential identities on Jordan ideals involving derivations. Finally, we improve and extend some recent results on 3-prime near-rings.

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