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OSCILLATION CRITERIA FOR PERTURBED SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

In this article, we establish sufficient conditions for which the second order nonlinear differential equation of the form $[a(t)x']' + Q(t,x) = P(t,x,x')$ is oscillatory. Examples are given to validate some of the criteria.

***Keywords*:** second order, oscillation, nonlinear differential equations

1. Introduction

Following the classic work of Atkinson [4] in the determination of oscillation criteria for second order nonlinear differential equations, there has arisen a number of literature on this subject. Among these works, we refer specifically to [3, 4, 6 to 15]. In this paper, we are particularly concerned with the oscillatory behaviour of the second order nonlinear differential equations of the form

$$[a(t)x']' + Q(t,x) = P(t,x,x') \tag{1.1}$$

Where $a : [t_0,\infty) \to \mathbb{R}$, $Q : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $P : [t_0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and a(t) > 0.

This paper improves and extends the results of [1] and more particularly, the results of [2] in obtaining oscillatory criteria for (1.1). A solution of (1.1) is said to be oscillatory if it has infinity of zeros in the domain on which it is defined. Then eqn. (1.1) itself is said to be oscillatory if every solution of (1.1) is oscillatory. We shall only be concerned with solutions of the differential equations (1.1) which exist on the interval $[t_0,\infty)$, $t_0 \geq 0$. Also, the uniqueness of solutions of (1.1) is not assumed.

2. Oscillation Theorems

Assume there exist continuous functions $p, q: [t_0,\infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) > 0,\ x \neq 0 \tag{2.1}$$

$$f'(x) \geq k > 0, x \neq 0 \tag{2.2}$$

$$\frac{Q(t,\ x)}{f(x)} \geq q(t) \text{ and } \frac{P(t,\ x,\ x')}{f(x)} \leq p(t) \text{ for } x \neq 0 \tag{2.3}$$

Theorem 1: .Suppose that conditions (2.1), (2.2) and (2.3) hold and let ρ be a positive continuously differentiable function on the interval $[t,\infty)$ such that $\rho' \geq 0$ on $[t_0,\infty)$. Equation (1.1) is oscillatory if

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{\rho(s)}{a(s)} = \infty \tag{2.4}$$

and

$$\int_{t_0}^{\infty} A(s)ds = \infty \tag{2.5}$$

where

$$A(t) = \frac{1}{\rho(t)}[q(t) - p(t)] - \frac{1}{4k}\frac{\rho'^2(t)}{\rho^3(t)}a(t)$$

Proof

Suppose that $x(t)$ is a non-oscillatory solution of (2.1), say $x(t) \neq 0$ on the interval $[t_1,\infty), t \geq t_1 \geq t_0$. We assume that $x(t) > 0$ on $[t_1,\infty)$. The case $x(t) < 0$ can be treated in a similar fashion and is therefore omitted.

But

$$\left[\frac{a(t)x'(t)}{f[x(t)]}\right]' = \frac{P[t,x(t),x'(t)]}{f[x(t)]} - \frac{Q[t,x(t)]}{f[x(t)]} - \frac{a(t)f'[x(t)][x'(t)]^2}{f^2[x(t)]} \tag{2.6}$$

Multiplying (2.6) by $\frac{1}{\rho(t)}$ and integratiog from t_1 to t, we obtain

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$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \le D_{t_1} - \int_{t_1}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds - \int_{t_1}^{t} \frac{\rho'(s)a(s)x'(s)}{\rho^2(s)f[x(s)]}ds - \int_{t_1}^{t} \frac{a(s)[x'(s)]^2 f'[x(s)]}{\rho(s)f^2[x(s)]}ds \quad (2.7)$$

where

$$D_{t_1} = \frac{a(t_1)x'(t_1)}{\rho(t_1)f[x(t_1)]}$$

We set

$$\phi = \frac{a(t)x'(t)}{\rho(t)f[x(t)]}$$

and

$$\Phi = \phi(t)\rho(t) + \frac{\rho'(t)a(t)}{2k\rho(t)}$$

Using the above identities, we have from condition (2.2)

$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \le D_{t_1} - \int_{t_1}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds - \int_{t_1}^{t} \left[\frac{\rho'(s)}{\rho(s)}\phi + k\frac{\rho(s)}{a(s)}\phi^2\right]ds$$

$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \le D_{t_1} - \int_{t_1}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds - \int_{t_1}^{t} \frac{k}{\rho(s)a(s)}\left[\Phi^2(s) - \left(\frac{\rho'(s)a(s)}{2k\rho(s)}\right)^2\right]ds$$

$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \le D_{t_1} - \int_{t_1}^{t} A(s)ds \quad (2.8)$$

From condition (2.5), we see that

$$\lim_{t\to\infty} \frac{a(t)x'(t)}{\rho(t)f[x(t)]} = -\infty$$

Hence, there exist $t_2 \ge t_1$ such that $x'(t) < 0$ for $t \ge t_2$

Dividing through (1.1) by $f[x(t)]$, we obtain

$$\frac{[a(t)x'(t)]'}{f[x(t)]} = \frac{P[t,x(t),x'(t)]}{f[x(t)]} - \frac{Q[t,x(t)]}{f[x(t)]}$$

or

$$[a(t)x'(t)]' \le -f[x(t)][q(t)-p(t)] \quad (2.9)$$

Multiplying (2.9) by $\frac{1}{\rho(t)}$ and integrating from t_2 to t, we obtain

$$\frac{a(t)x'(t)}{\rho(t)} \le D_{t_2} - \int_{t_2}^{t} \frac{\rho'(s)a(s)x'(s)}{\rho^2(s)}ds - \int_{t_2}^{t} f[x(s)]\frac{1}{\rho(s)}[q(s)-p(s)]ds$$

$$\frac{a(t)x'(t)}{\rho(t)} \le D_{t_2} - f[x(t)]\int_{t_2}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds + \int_{t_2}^{t} x'(s)f'[x(s)]\int_{t_2}^{t} \frac{1}{\rho(u)}[q(u)-p(u)]duds \quad (2.10)$$

where

$$D_{t_2} = \frac{a(t_2)x'(t_2)}{\rho(t_2)} < 0$$

Since $\int_{t_0}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds \to \infty \;\; as\; t\to\infty$ (this is true from (2.5)), we may choose $T \ge t_2$ such that $\int_{t_2}^{T} \frac{1}{\rho(s)}[q(s)-p(s)]ds = 0$ and $\int_{T}^{t} \frac{1}{\rho(s)}[q(s)-p(s)]ds \ge 0 \;\; for\; T \ge t_2$. Now with this choice of T, we get from (2.10)

$$\int_{t_2}^{t} x'(s)ds \le D_{t_2}\int_{t_2}^{t} \frac{\rho(s)}{a(s)}ds$$

We see from condition (2.4) that

$$x(t) \le x(t_2) + D_{t_2}\int_{t_2}^{t} \frac{\rho(s)}{a(s)}ds \to -\infty \; as\; t\to\infty$$

$$\Rightarrow x(t) < 0$$

This contradicts the hypothesis that $x(t) > 0$. This completes the proof.

Example 1 Consider the differential equation

$$[a(t)x']' + [2t^2(3+\cos(t)) + t^2x^3]x = xt^3\sin(t) + \frac{1}{t^5}\frac{x^4\cos(x')}{x^3+1}, \;\; for\; t \ge \frac{\pi}{2}$$

If we chose $f(x) = x, a(t) = t^2$ and $\rho(t) = t$

$$\frac{Q(t,x)}{f(x)} = \frac{[2t^2(3+\cos(t)) \ + t^2x^3\ ]x}{x} \geq 2t^2(3+\cos(t)) = q(t)$$
and
$$\frac{P(t,x,x')}{f(x)} = xt^3\sin(t) + \frac{\frac{1}{t^5}\frac{x^4\cos(x')}{x^3+1}}{x} \leq t^3\sin(t) + \frac{1}{t^5} = p(t)$$
Thus, for every $t \geq t_0 = \frac{\pi}{2}$, we obtain
$$\int_{t_0}^{t} A(s)ds = \int_{t_0}^{t}\left[\frac{1}{s}[2s^2(3+\cos(s)) - s^3\sin(s) - \frac{1}{s^5}] - \frac{1}{4k}\frac{s}{s^2}\right]ds$$
$$= \int_{t_0}^{t}\frac{1}{s}[2s^2(3+\cos(s)) - s^3\sin(s)]ds - \int_{t_0}^{t}\frac{1}{s^6}ds - \frac{1}{4k}\int_{t_0}^{t}\frac{1}{s}ds$$
$$= \int_{t_0}^{t}\frac{d}{ds}[s^2(3+\cos(s))]ds + \frac{1}{5s^5}\Big|_{\frac{\pi}{2}}^{t} - \frac{1}{4k}\log(s)\Big|_{\frac{\pi}{2}}^{t}$$
$$= t^2[3+\cos(t)] - 3\left(\frac{\pi}{2}\right)^2 + \frac{1}{5t^5} - \frac{1}{5}\left(\frac{2}{\pi}\right)^5 - \frac{1}{4k}\log(t) + \frac{1}{4k}\log\left(\frac{\pi}{2}\right)$$
$$\geq t^2 - 3\left(\frac{\pi}{2}\right)^2 - \frac{1}{5}\left(\frac{2}{\pi}\right)^5 - \frac{1}{4k}\log(t)$$
Thus, we see that
$$\int_{t_0}^{\infty} A(s)ds = \ \infty$$
and
$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\rho(s)}{a(s)}ds = \ \int_{t_0}^{t}\frac{s}{s^2}ds = \infty$$
We note that conditions (2.1) – (2.5) are satisfied. Hence, the differential equation is oscillatory.

Theorem 2

Suppose conditions (2.1) – (2.4) hold and let ρ be a positive continuously differentiable function on the interval $[t,\infty)$ such that $\rho' \geq 0$ on $[t_0,\infty)$.If
$$\int_{t_0}^{\infty}\frac{1}{\rho(s)}[q(s)-p(s)]ds < \ \infty \qquad (2.11)$$
$$\lim_{t\to\infty} inf\left[\int_{T}^{t}A(s)ds\right] \ \geq 0 \ \ for\ all\ large\ T \qquad (2.12)$$
$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{\rho(s)}{a(s)}\int_{s}^{\infty}A(u)duds = \infty \qquad (2.13)$$
and
$$\int_{\epsilon}^{\infty}\frac{dy}{f(y)} < \ \infty \ \ and \ \ \int_{-\epsilon}^{-\infty}\frac{dy}{f(y)} < \ \infty \ \ for\ every\ \ \epsilon > 0 \qquad (2.14)$$
where
$$A(t) = \ \frac{1}{\rho(t)}[q(t) \ - p(t)] - \frac{1}{4k}\frac{\rho'^2(t)}{\rho^3(t)}a(t)$$
then all solutions of (1.1) are oscillatory.

Proof

Let $x(t)$ be a non-oscillatory solution of (1.1), i.e. $x(t) \neq 0$. We assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in theorem 1, we have from (2.8)
$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \leq D_{t_1} - \int_{t_1}^{t}A(s)ds$$
We now consider the following three cases for the behaviour of $x'(t)$.

Case 1: $x'(t) > 0$ for $t \geq b$ for some $b \geq t_1$

1f $x'(t) > 0$, then by condition (2.11), we have
$$0 \leq \frac{a(b)x'(b)}{\rho(b)f[x(b)]} - \int_{b}^{\infty}A(s)ds \quad for\ \ t \geq b$$

or

$$\int_b^\infty A(s)ds \leq \frac{a(b)x'(b)}{\rho(b)f[x(b)]}$$

Hence, for all $t \geq b$, it follows that

$$\int_b^\infty A(s)ds \leq \frac{a(t)x'(t)}{\rho(t)f[x(t)]}$$

Now as $t \to \infty$ and invoking condition (2.14),

$$\lim_{t\to\infty} \int_b^t \frac{\rho(s)}{a(s)} \int_b^\infty A(u)duds \leq \lim_{t\to\infty} \int_{x(b)}^{x(t)} \frac{dy}{f(y)} < \infty$$

This however is a contradiction since the integral on the left diverges by condition (2.13).

Case 2: $x'(t) = 0$ for some $t \geq t_1 \geq t_0$

$x'(t) = 0$ implies that there exist a sequence $(t_n: n = 1,2,3 \ldots)$ on the interval $[t_0,\infty)$, $[t_0 \geq 0$ such that : $x'(t) \leq 0$.

We now choose N sufficiently large such that (2.12) holds. Thus from (2.8),

$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]} \leq D_{t_N} - \int_{t_N}^t A(s)ds \qquad (2.15)$$

Taking the limit superior of (2.15), we obtain

$$\lim_{t\to\infty} sup \frac{a(t)x'(t)}{\rho(t)f[x(t)]} \leq \lim_{t\to\infty} sup\, D_{t_N} + \lim_{t\to\infty} sup \left[-\int_{t_N}^t A(s)ds\right]$$

$$\lim_{t\to\infty} sup \frac{a(t)x'(t)}{\rho(t)f[x(t)]} \leq D_{t_N} + \lim_{t\to\infty} sup \left[-\int_{t_N}^t A(s)ds\right]$$

$$\lim_{t\to\infty} sup \frac{a(t)x'(t)}{\rho(t)f[x(t)]} \leq D_{t_N} - \lim_{t\to\infty} inf \left[\int_{t_N}^t A(s)ds\right]$$

$$< 0$$

This is a contradiction since $x'(t)$ oscillates.

Case 3: $x'(t) < 0$ for some $t \geq t_2 \geq t_1(\geq t_0)$

We note that condition (2.11) implies that for every $T_0 \geq t_0$, there exist $T_1 \geq T_0$ such that $\int_{T_1}^t \frac{1}{\rho(s)}[q(s) - p(s)]ds \geq 0 \quad for\ all\ \ t \geq T_1$.

Multiplying (2.9) by $\frac{1}{\rho(t)}$ and integrating from t_1 to t, we obtain

$$\frac{a(t)x'(t)}{\rho(t)} \leq D_{t_1} - \left[\int_{t_1}^t f[x(s)]\frac{1}{\rho(s)}[q(s) - p(s)]ds - \int_{t_1}^t x'(s)f'[x(s)]\int_{t_1}^t \frac{1}{\rho(u)}[q(u) - p(u)]duds\right]$$

$$\frac{a(t)x'(t)}{\rho(t)} \leq D_{t_1} - f[x(t)]\int_{t_1}^t \frac{1}{\rho(s)}[q(s) - p(s)]ds + \int_{t_1}^t x'(s)f'[x(s)]\int_{t_2}^t \frac{1}{\rho(u)}[q(u) - p(u)]duds$$

where

$$D_{t_1} = \frac{a(t_1)x'(t_1)}{\rho(t_1)} < 0$$

Thus

$$x'(t) \leq D_{t_1}\frac{\rho(t)}{a(t)} \qquad (2.16)$$

Integrating (2.16) from T to t, we obtain

$$x(t) - x(T) \leq D_{t_1}\int_T^t \frac{\rho(s)}{a(s)}ds$$

$\Rightarrow x(t) \to -\infty\ as\ t \to \infty$. This contradicts the hypothesis that $x(t) > 0$. This completes the proof.

Theorem 3: Suppose conditions (2.1) – (2.3) hold and assume there is a constant $c > 0$ such that

$$\frac{\rho(t)}{a(t)} \leq c \qquad (2.17)$$

$$if \quad \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^t \int_{t_0}^s \frac{1}{\rho(u)}[q(u) - p(u)]duds = \infty \qquad (2.18)$$

then all solutions of (1.1) are oscillatory.

Proof

Let $x(t)$ be a non-oscillatory solution of (1.1) with $x(t) \neq 0$ for $t \geq t_1 \geq t_0$. Then from (1.1)

$$\frac{a(t)x'(t)}{\rho(t)f[x(t)]}+\int_{t_1}^{t}\frac{1}{\rho(s)}[q(s)-p(s)]ds+\int_{t_1}^{t}\frac{\rho'(s)a(s)x'(s)}{\rho^2(s)f[x(s)]}ds+\int_{t_1}^{t}\frac{a(s)[x'(s)]^2f'[x(s)]}{\rho(s)f^2[x(s)]}ds\le D_{t_1} \quad (2.19)$$

where

$$D_{t_1}=\frac{a(t_1)x'(t_1)}{\rho(t_1)f[x(t_1)]}$$

Integrating (2.19) a second time from $t_1\ to\ t$ yields

$$\int_{t_1}^{t}\frac{a(s)x'(s)}{\rho(s)f[x(s)]}ds+\int_{t_1}^{t}\int_{t_1}^{s}\frac{1}{\rho(u)}[q(u)-p(u)]duds+\int_{t_1}^{t}\int_{t_1}^{s}\frac{\rho'(u)a(u)x'(u)}{\rho^2(u)f[x(u)]}duds+\int_{t_1}^{t}\int_{t_1}^{t}\frac{a(s)[x'(s)]^2f'[x(s)]}{\rho(s)f^2[x(s)]}duds \le D_{t_1}(t-t_1)$$

or

$$\int_{t_1}^{t}\frac{a(s)x'(s)}{\rho(s)f[x(s)]}ds+\int_{t_1}^{t}\int_{t_1}^{s}\frac{1}{\rho(u)}[q(u)-p(u)]duds+\int_{t_1}^{t}\int_{t_1}^{s}\frac{\rho'(u)a(u)x'(u)}{\rho^2(u)f[x(u)]}duds$$
$$+\int_{t_1}^{t}\int_{t_1}^{s}\frac{a(u)[x'(u)]^2f'[x(u)]}{\rho(u)f^2[x(u)]}duds\le D_{t_1}t \quad (2.20)$$

Multiplying through (2.20) by $\frac{1}{t}$ and invoking condition (2.2), we have

$$\frac{1}{t}\int_{t_1}^{t}\frac{a(s)x'(s)}{\rho(s)f[x(s)]}ds+\frac{1}{t}\int_{t_1}^{t}\int_{t_1}^{s}\frac{1}{\rho(u)}[q(u)-p(u)]duds+\frac{1}{t}\int_{t_1}^{t}\int_{t_1}^{s}\frac{\rho'(u)a(u)x'(u)}{\rho^2(u)f[x(u)]}duds+\frac{k}{t}\int_{t_1}^{t}\int_{t_1}^{s}\frac{a(u)[x'(u)]^2}{\rho(u)f^2[x(u)]}duds\le D_{t_1}$$

From condition (2.18), we see that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}\frac{a(s)x'(s)}{\rho(s)f[x(s)]}ds=-\infty$$

Now define

$$R(t)=\left|\int_{t_1}^{t}\frac{a(s)x'(s)}{\rho(s)f[x(s)]}ds\right| \quad (2.21)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$R^2(t)\le t\int_{t_1}^{t}\left[\frac{a(s)x'(s)}{\rho(s)f[x(s)]}\right]^2ds$$

Thus

$$-\frac{R(t)}{t}+\frac{k}{t}\int_{t_1}^{t}\frac{\rho(s)R^2(s)}{a(s)s}ds\le 0$$

or

$$-\frac{R(t)}{t}+\frac{kc}{t}\int_{t_1}^{t}\frac{R^2(s)}{s}ds\le 0$$

Now we set

$$\varphi(t)=\int_{t_1}^{t}\frac{R^2(s)}{s}ds$$

then

$$\frac{k^2c^2}{t}\le\frac{\varphi'(t)}{\varphi^2(t)} \quad (2.22)$$

Integrating (2.22) from t_2 to t where $t_2\ge t_1$, we obtain

$$k^2c^2log\left[\frac{t}{t_2}\right]\le\frac{1}{\varphi(t_2)}-\frac{1}{\varphi(t)}\le\frac{1}{\varphi(t_2)}$$

This is a contradiction as the left hand side of the inequality diverges as $t\to\infty$. Hence the proof.

Example 2. Consider the differential equation

$$[a(t)x']'+\left[\frac{1}{2}t^{\frac{1}{2}}(2+\cos(t))+t^2e^3\right]x=xt^{\frac{3}{2}}\sin(t)+\frac{1}{t^4}\frac{x^4cos^2(x')}{x^2+1},\ \ for\ t\ge\frac{\pi}{2}$$

If we choose $f(x)=x, a(t)=t^{\frac{3}{2}}$ and $\rho(t)=t$,

$$\frac{Q(t,x)}{f(x)}\ge\frac{1}{2}t^{\frac{1}{2}}(2+\cos(t))=q(t)$$

and

$$\frac{P(t,x,x')}{f(x)} \le t^{\frac{3}{2}}\sin(t) + \frac{1}{t^4} = p(t)$$

Thus, for every $t \ge t_0 = \frac{\pi}{2}$, we have

$$\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\frac{1}{u}\left[\frac{1}{2}u^{\frac{1}{2}}(2+\cos(u)) - u^{\frac{3}{2}}\sin(u) - \frac{1}{u^4}\right]duds$$

But

$$\int_{t_0}^{t}\frac{1}{s}\left[\frac{1}{2}s^{\frac{1}{2}}(2+\cos(s)) - s^{\frac{3}{2}}\sin(s) - \frac{1}{s^4}\right]ds$$

$$= \int_{t_0}^{t}\frac{d}{ds}s^{\frac{1}{2}}(2+\cos(s))\,ds - \int_{t_0}^{t}\frac{1}{s^5}ds$$

$$\ge t^{\frac{1}{2}} - 2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} - \frac{1}{4}\left(\frac{2}{\pi}\right)^4$$

Now

$$\frac{1}{t}\int_{t_0}^{t}\left\{s^{\frac{1}{2}} - \left[2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \frac{1}{4}\left(\frac{2}{\pi}\right)^4\right]\right\}ds$$

$$\ge \frac{2}{3}t^{\frac{1}{2}} - \left[2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \frac{1}{4}\left(\frac{2}{\pi}\right)^4\right] - \frac{2}{3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{1}{t} + \frac{\pi}{2}\left[2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \frac{1}{4}\left(\frac{2}{\pi}\right)^4\right]\frac{1}{t}$$

$$\ge \frac{2}{3}t^{\frac{1}{2}} - \frac{2}{3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{1}{t} - \left[2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \frac{1}{4}\left(\frac{2}{\pi}\right)^4\right]$$

We clearly see that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\frac{1}{\rho(u)}[q(u) - p(u)]duds = \infty$$

and

$$\frac{\rho(t)}{a(t)} = \frac{1}{\sqrt{t}} \le c, c > 0$$

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