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COEFFICIENT BOUNDS FOR CERTAIN GENERALIZED CLASS OF ANALYTIC FUNCTION INVOLVING BAZILEVIC TYPE FUNCTION AND THE SIGMOID FUNCTION

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Abstract

In this work, some properties of the Fekete-Szego functional for certain generalized class of analytic functions involving logistic sigmoid and Bazilevic type functions are obtained.

Keywords: Analytic function, Univalent function, Bazilevic, Sigmoid, Starlike, Spiralike function.

1.0 INTRODUCTION

Let H denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, z \in U \tag{1.1}$$

which are analytic and univalent in the open disk U = $\{z : |z| < 1\}$ and normalized by
f(z) = f'(0) – 1 = 0. Also, let G denote the subclass of H that are normalized and univalent in U.
For function f(z) of the form (1.1), we can write that

$$f(z)^{\alpha} = z^{\alpha} + \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \alpha > 0, z \in U \tag{1.2}$$

With the aid of Salagean derivative operator [3], we can also write that

$$D^n f(z)^{\alpha} = \alpha^n z^{\alpha} + \sum_{k=j+1}^{\infty} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1}, \alpha > 0, z \in U \tag{1.3}$$

Abdulhalim [4], introduced a generalization of certain family of Bazilevic function satisfying

$$\mathrm{Re}\left\{\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right\} > 0, z \in U, \tag{1.4}$$

where the operator D^n is the Salagean derivative operator defined by
$D^n f(z) = D(D^{n-1} f(z)) = z(D^{n-1} f(z))'$ see [3]. He denoted this class of functions defined in (1.4) by $B_n(\alpha)$.
This generalization includes analytic functions satisfying

$$\mathrm{Re}\left\{\frac{f(z)^{\alpha}}{z^{\alpha}}\right\} > 0, z \in U \tag{1.5}$$

which is non-univalent in the unit disk.
Opoola [5], studied the family of $T_n^{\alpha}(\gamma)$, a more generalized form of (1.4) such that

$$\mathrm{Re}\left\{\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right\} > \gamma, \alpha > 0, z \in U. \tag{1.6}$$

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The theory of special function such as the Sigmoid function has wide applications just as the application of analytic function in many physical problems like heat conduction, electrostatic potential etc.
The most widely used Sigmoid function is the logistic function which has bounds [0,1].
The logistic Sigmoid function is defined as:

$$g(z) = \frac{1}{1+e^{-z}} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 + \ldots \quad (1.7)$$

and has the following properties (cf [1]):
(i) It output real number between 0 and 1
(ii)It maps a very large input domain to a small range of outputs
(iii)It never loses information because it is a one-to-one function
(iv)It increases monotonically.

2.0 LEMMA AND DEFINITION

Here, let the modified Bazilevic function $F_{\alpha,n}(z) \in T_n^{\alpha}$ be defined such that:

$$F_{\alpha,n}(z) = z + \sum_{k=j+1}^{\infty} \alpha_{n,k} a_k(\alpha) z^k, \alpha > 0, j \in N, \quad (2.1)$$

Where,

$$\alpha_{n,k} = \left(\frac{\alpha + k - 1}{\alpha}\right)^n.$$

Using (2.1), we give the following definition.

Definition 2.1

Let $F_{\alpha,n}(z) \in G_\alpha^n(\lambda, \delta, \beta, \theta, j, \gamma)$ then,

$$\text{Re}\left\{(1+\delta^2)\left(\frac{e^{i\theta} z(F'_{\alpha,n}(z))^{\lambda} + (2\delta^2 - \delta)z^2[(F'_{\alpha,n}(z))^{\lambda}]'}{4(\delta - \delta^2)z + (2\delta^2 - \delta)zF'_{\alpha,n}(z) + (2\delta^2 - 3\delta + 1)F_{\alpha,n}(z)} - \gamma\right)\right\} > \quad (2.2)$$

$$\beta\left|(1+\delta^2)\left(\frac{e^{i\theta} z(F'_{\alpha,n}(z))^{\lambda} + (2\delta^2 - \delta)z[(F'_{\alpha,n}(z))^{\lambda}]'}{4(\delta - \delta^2)z + (2\delta^2 - \delta)zF'_{\alpha,n}(z) + (2\delta^2 - 3\delta + 1)F_{\alpha,n}(z)} - 1\right)\right|$$

for $\alpha > 0, \beta \geq 0, 0 \leq \delta \leq 1, \lambda \geq 1, -1 \leq \gamma < 1, n \in N_0 = N \cup \{0\}, n \geq j, 0 \leq \theta < \frac{\pi}{2}$.

It is observed that $TG_\alpha^n(\lambda, \delta, \beta, \theta, j, \gamma) = G_\alpha^n(\lambda, \delta, \beta, \theta, j, \gamma) \cap T,$ where T is the subclass of G consisting of functions of the form

$$F_{\alpha,n}(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, a_k > 0, \forall j+1. \quad (2.3)$$

Remarks:

Several subclasses of analytic functions (well-known and new ones) can be obtained from (2.2) with different values of parameters $\alpha, n, \lambda, \delta, \beta, \theta, \gamma$. For instance,

1. Let $F_{\alpha,n}(z) \in G_1^0(\lambda, 0, \beta, \theta, j, \gamma)$. Then,

$$\text{Re}\left(\frac{e^{i\theta} z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)} - \gamma\right) > \beta\left|\left(\frac{e^{i\theta} z\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}(z)} - 1\right)\right|$$

which is the λ- pseudo-spiralike class of order γ.

2 Let $F_{\alpha,n}(z) \in G_1^0(\lambda, 0, \beta, 0, j, \gamma)$. Then,

$$\text{Re}\left\{\frac{z\,(F'_{1,0}(z))^{\lambda}}{F_{1,0}(z)}-\gamma\right\}>\beta\left|\left(\frac{z\,\left(F'_{1,0}(z)\right)^{\lambda}}{F_{1,0}\,(z)}\right)-1\right|$$

which is the λ- pseudo-starlike class of order γ.

3. Let $F_{\alpha,n}(z)\in G_1^0\,(1,0,\beta,0,j,\gamma)$. Then,

$$\text{Re}\left\{\frac{z\,(F'_{1,0}(z))}{F_{1,0}(z)}-\gamma\right\}>\beta\left|\left(\frac{z\,(F'_{1,0}(z))}{F_{1,0}\,(z)}\right)-1\right|$$

which is the starlike class of order γ.

4. Let $F_{\alpha,n}(z)\in G_1^0\,(2,0,\beta,\theta,j,\gamma)$. Then,

$$\text{Re}\left\{(F'_{1,0}(z)\frac{e^{i\theta}z\left((F'_{1,0}(z)\right)}{F_{1,0}(z)}-\gamma)\right\}>\beta\left|\left(F'_{1,0}(z)\frac{e^{i\theta}z\left((F'_{1,0}(z)\right)}{F_{1,0}(z)}\right)-1\right|$$

which is the product of a combination of geometric expression for bounded turning function and spiralike class of order γ.

5. Let $F_{\alpha,n}(z)\in G_1^0\,(2,0,\beta,0,j,\gamma)$. Then

$$\text{Re}\left\{(F'_{1,0}(z)\frac{z\left((F'_{1,0}(z)\right)}{F_{1,0}(z)}-\gamma)\right\}>\beta\left|\left(F'_{1,0}(z)\frac{z\left((F'_{1,0}(z)\right)}{F_{1,0}(z)}\right)-1\right|$$

which is the product of a combination of geometric expression for bounded turning function and starlike class of order γ (see [1,6,7,8,9] among others).

Lemma (2.4): Let g be sigmoid function of the form (1.1). Also, let $\varphi(z)=2g(z)$ such that

$$\varphi(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right)^k.$$

Then $\varphi\,(z)\in P, |z|<1$ where P is the class of Caratheodory functions and $\varphi(z)$ denotes the celebrated sigmoid functions (see [10]).

Lemma (2.5) Let

$$\varphi(z)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k}\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}z^m\right)^k.$$

Then, $|\varphi(z)|<2$ (see [10]).

Lemma (2.6): Let $\varphi\,(z)\in P$ and is starlike, then it is a normalized univalent function of the form (1.2) (see [2]).

Remark: Suppose that k = 1 and also let $\varphi(z)=1+\sum_{m=1}^{\infty}C_m z^m$,

where $C_m=\frac{(-1)^{m+1}}{2m!}$, then $|C_m|\le 2, m=1,2,...$.

This result is the best possible (see[10])

3.0 MAIN RESULT

Theorem (3.1): Let $F_{\alpha,n}(z)\in G_\alpha^n(\lambda,\delta,\beta,\theta,\vartheta,j,\phi)$. Then,

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)\right|\le\frac{1}{2D_2\alpha_{n,j+2}}\left\{\begin{array}{l}\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]-\frac{1}{4D_1^2}Q_1\lambda(\lambda-1)(j+1)^2\\ [e^{i\theta}+(j+1)(2\delta^2-\delta)]+\mu\frac{D_2\alpha_{n,j+2}}{2D_1^2\alpha_{n,j+1}^2},\mu\le\sigma_1\\ \frac{1}{2},\sigma_1\le\mu\le\sigma_2\\ -\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]+\frac{1}{4D_1^2}Q_1\lambda(\lambda-1)(j+1)^2\\ [e^{i\theta}+(j+1)(2\delta^2-\delta)]+\mu\frac{D_2\alpha_{n,j+2}}{2D_1^2\alpha_{n,j+1}^2},\mu\ge\sigma_2.\end{array}\right\}$$

where

$$\sigma_1 = \begin{bmatrix} \frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]- \\ \frac{1}{4D_1^2}Q_2\lambda(\lambda+1)(j+1)^2[e^{i\theta}+(j+1)(2\delta^2-\delta)]-1 \end{bmatrix}\frac{2D_1^2\alpha_{n,j+1}^2}{D_2\alpha_{n,j+2}},$$

$$\sigma_2 = \begin{bmatrix} 1+\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]- \\ \frac{1}{4D_1^2}Q_2\lambda(\lambda+1)(j+1)^2[e^{i\theta}+(j+1)(2\delta^2-\delta)] \end{bmatrix}\frac{2D_1^2\alpha_{n,j+1}^2}{D_2\alpha_{n,j+2}},$$

$$D_1 = Q_1(1-\beta)\lambda(j+1)[e^{i\theta}+j(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1],$$

$$D_2 = Q_1(1-\beta)\lambda(j+2)[e^{i\theta}+(j+1)(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1],$$

$$Q_1 = e^{i\theta}(1-\beta)+(\beta-\vartheta)(1+\delta^2),$$

$$Q_2 = 1-\delta^2.$$

Proof: Let $F_{\alpha,n}(z)\in G_\alpha^n(\lambda,\delta,\beta,\theta,\vartheta,j,\phi)$. Then, by definition there exist $\varphi(z)\in P$ such that

$$\frac{\dfrac{(1+\delta^2)(1-\beta)\left\{e^{i\theta'}z(F'_{\alpha,n}(z))^\lambda+(2\delta^2-\delta)z[(F'_{\alpha,n}(z))^\lambda]'\right\}}{4(\delta-\delta^2)z+(2\delta^2-\delta)zF'_{\alpha,n}(z)+(2\delta^2-3\delta+1)F_{\alpha,n}(z)}-(1+\delta^2)(\vartheta-\beta)}{[e^{i\theta}(1-\beta)+(\beta+\vartheta)(1+\delta^2)]}=p(z) \quad (3.2)$$

$$\Rightarrow 1+Q_1\left\{(1-\beta)[e^{i\theta}+j(2\delta^2-\delta)]\lambda(j+1)+(\beta-\vartheta)[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]\right\}\alpha_{n,j+1}a_{j+1}(\alpha)z^j$$
$$+Q_1\left\{\begin{matrix}(1-\beta)[e^{i\theta}+(j+2)(2\delta^2-\delta)]\left(\lambda(j+2)\alpha_{n,j+2}a_{j+2}(\alpha)+\frac{\lambda(\lambda-1)}{2}(j+1)^2\alpha_{n,j+1}^2a_{j+1}^2(\alpha)\right)\\+(\beta-\vartheta)\big((j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1\big)\alpha_{n,j+2}a_{j+2}(\alpha)\end{matrix}\right\}z^{j+1}$$
$$+Q_1\left\{\begin{matrix}(1-\beta)[e^{i\theta}+(j+2)(2\delta^2-\delta)]\left(\begin{matrix}\lambda(j+3)a_{j+3}(\alpha)+\lambda(\lambda-1)(j+1)(j+2)\alpha_{n,j+1}\alpha_{n,j+2}a_{j+1}(\alpha)a_{j+2}(\alpha)\\+\frac{\lambda(\lambda-1)(\lambda-2)}{6}(j+3)^3\alpha_{n,j+1}^3a_{n,j+1}^3(\alpha)\end{matrix}\right)\\+(\beta-\vartheta)\big((j+3)(2\delta^2-\delta)+2\delta^2-3\delta+1\big)\alpha_{n,j+3}a_{j+3}(\alpha)\end{matrix}\right\}z^{j+2}=$$
$$1+\left\{p_1+Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+1}a_{n,j+1}(\alpha)\right\}z^j$$
$$+\left\{\begin{matrix}p_2+Q_2p_1[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+1}a_{j+1}(\alpha)\\Q_2[(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+2}a_{j+2}(\alpha)\end{matrix}\right\}z^{j+1}$$
$$+\left\{\begin{matrix}p_3+Q_2p_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+1}a_{j+1}(\alpha)\\+Q_2p_1[(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+1}a_{j+2}(\alpha)\\+Q_2[(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1]\alpha_{n,j+3}a_{j+3}(\alpha)\end{matrix}\right\}z^{j+2}+\ldots \quad (3.3)$$

By comparing coefficients of (3.3) and simplifying, we have

$$a_{j+1}(\alpha)=\frac{p_1}{D_1\alpha_{n,j+1}}, \quad (3.4)$$

and

$$a_{j+2}(\alpha)=\frac{\begin{matrix}2D_1^2p_2+p_1^2\{2D_1Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]-\\Q_1\lambda(\lambda-1)(j+1)^2[e^{i\theta}+(j+1)(2\delta^2-\delta)]\}\end{matrix}}{2D_1^2D_2\alpha_{n,j+2}}, \quad (3.5)$$

where,

$$D_1 = Q_1(1-\beta)\lambda(j+1)[e^{i\theta}+j(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1],$$

$$D_2 = Q_1(1-\beta)\lambda(j+2)[e^{i\theta}+(j+1)(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1].$$

Since, $\varphi(z)$ is univalent and $P\prec\varphi$. The function

$$p_1(z)=\frac{1+\varphi^{-1}(p(z))}{1-\varphi^{-1}(p(z))}=1+q_1z^j+q_2z^{j+1}+\ldots$$

is analytic and has a positive part in U. So, we have that

$$p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \varphi\left(\frac{q_1 z^j + q_2 z^{j+1} + ...}{2 + q_1 z^j + q_2 z^{j+1} + ...}\right)$$

$$\Rightarrow 1 + p_1 z^j + p_2 z^{j+1} + p_3 z^{j+2} = \varphi\left(\tfrac{1}{2} q_1 z^j + \tfrac{1}{2}(q_2 - \tfrac{1}{2} q_1^2) z^{j+1} + ...\right)$$

$$= 1 + \tfrac{1}{4} q_1 z^j + \tfrac{1}{4}\left(q_2 - \tfrac{1}{2} q_1^2\right) z^{j+1}$$

$$\therefore p_1 = \tfrac{1}{4} q_1, \qquad (3.6)$$

$$p_2 = \tfrac{1}{4}\left(q_2 - \tfrac{1}{2} q_1^2\right)$$

Using (3.6) in (3.4) and (3.5), we have

$$a_{j+1}(\alpha) = \frac{q_1}{4D_1 \alpha_{n,j+1}} \qquad (3.7)$$

$$a_{j+2}(\alpha) = \frac{8D_1^2 q_2 + q_1^2 \left\{ \begin{array}{l} 2D_1 Q_2 [(j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1] - \\ Q_1 \lambda(\lambda+1)(j+1)^2 [e^{i\theta} + (j+1)(2\delta^2 - \delta)] - 4D_1^2 \end{array} \right\}}{32 D_1^2 D_2 \alpha_{n,j+2}} \qquad (3.8)$$

From (3.7) and (3.8), we have

$$a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha) = \frac{1}{4D_2 \alpha_{n,j+2}} \left\{ q_2 - \frac{q_1^2}{2} \left[\begin{array}{l} 1 - \frac{1}{2D_1} Q_2 [(j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1] + \\ \frac{1}{4D_1^2} Q_1 \lambda(\lambda-1)(j+1)^2 [e^{i\theta} + (j+1)(2\delta^2 - \delta)] + \mu \frac{D_2 \alpha_{n,j+2}}{2D_1^2 \alpha_{n,j+1}^2} \end{array} \right] \right\}.$$

Hence,

$$\left| a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha) \right| \le \frac{1}{4D_2 \alpha_{n,j+2}} \left\{ q_2 - V q_1^2 \right\} \qquad (3.9)$$

where,

$$V = \frac{1}{2} \left[\begin{array}{l} 1 - \frac{1}{2D_1} Q_2 [(j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1] + \frac{1}{4D_1^2} Q_1 \lambda(\lambda-1)(j+1)^2 \\ [e^{i\theta} + (j+1)(2\delta^2 - \delta)] + \mu \frac{D_2 \alpha_{n,j+2}}{2D_1^2 \alpha_{n,j+1}^2} \end{array} \right] \qquad (3.10)$$

If $\mu \le \sigma_1$, then by lemma (2.4), we have

$$\left| a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha) \right| \le \frac{Q_2 [(j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1]}{4D_1 D_2 \alpha_{n,j+2}} - \frac{Q_1 \lambda(\lambda-1)(j+1)^2 [e^{i\theta} + (j+1)(2\delta^2 - \delta)]}{8D_1^2 D_2 \alpha_{n,j+2}} - \frac{\mu}{4D_1^2 \alpha_{n,j+1}^2} \qquad (3.11)$$

Now, if $\mu \ge \sigma_2$ then by Lemma (2.4), we have

$$\left| a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha) \right| \le -\frac{Q_2 [(j+1)(2\delta^2 - \delta) + 2\delta^2 - 3\delta + 1]}{4D_1 D_2 \alpha_{n,j+2}} + \frac{Q_1 \lambda(\lambda-1)(j+1)^2 [e^{i\theta} + (j+1)(2\delta^2 - \delta)]}{8D_1^2 D_2 \alpha_{n,j+2}} + \frac{\mu}{4D_1^2 \alpha_{n,j+1}^2} \qquad (3.12)$$

Finally, if $\sigma_1 \le \mu \le \sigma_2$

$$\left| a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha) \right| \le \frac{1}{4D_2 \alpha_{n,j+2}} \qquad (3.13)$$

(3.11),(3.12) and (3.13) complete the proof.

Theorem 3.14: Let $\phi(z)=1+\frac{1}{2}z^{j}-\frac{1}{24}z^{j+2}+\frac{1}{240}z^{j+4}-...$. Suppose that $f(z)^{\alpha}=z^{\alpha}+\sum_{k=j+1}^{\infty}a_k(\alpha)z^{\alpha+k-1}$ belong to the class $G_{\alpha}^{n}(\lambda,\delta,\beta,\theta,\vartheta,j,\phi)$,

then for $\alpha>0, n\in N_0, \lambda\geq 1, 0\leq\delta\leq 1, -1\leq\vartheta<1, \beta\geq 0, -\frac{\pi}{2}<\theta\leq\frac{\pi}{2}$ and $j\in N$

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^{2}(\alpha)\right|\leq\frac{\alpha^{n}}{2D_2(\alpha+j+1)^{n}}\begin{cases}\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]-\frac{1}{4D_1^2}Q_1\lambda(\lambda-1)(j+1)^2\\ [e^{i\theta}+(j+1)(2\delta^2-\delta)]+\mu\frac{D_2\alpha^{n}(\alpha+j+1)^{n}}{2D_1^2(\alpha+j)^{2n}}, \mu\leq\sigma_1\\ 1, \sigma_1\leq\mu\leq\sigma_2\\ -\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]+\frac{1}{4D_1^2}Q_1\lambda(\lambda-1)(j+1)^2\\ [e^{i\theta}+(j+1)(2\delta^2-\delta)]+\mu\frac{D_2\alpha^{n}(\alpha+j+1)^{n}}{2D_1^2(\alpha+j)^{2n}}, \mu\geq\sigma_2.\end{cases}$$

$$\sigma_1=\begin{bmatrix}\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]-\\ \frac{1}{4D_1^2}Q_2\lambda(\lambda+1)(j+1)^2[e^{i\theta}+(j+1)(2\delta^2-\delta)]-1\end{bmatrix}\frac{2D_1^2(\alpha+j)^{2n}}{D_2\alpha^{n}(\alpha+j+1)^{2n}},$$

$$\sigma_2=\begin{bmatrix}1+\frac{1}{2D_1}Q_2[(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1]-\\ \frac{1}{4D_1^2}Q_2\lambda(\lambda+1)(j+1)^2[e^{i\theta}+(j+1)(2\delta^2-\delta)]\end{bmatrix}\frac{2D_1^2(\alpha+1)^{2n}}{D_2\alpha^{n}(\alpha+j+1)^{n}},$$

$$D_1=Q_1(1-\beta)\lambda(j+1)[e^{i\theta}+j(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+1)(2\delta^2-\delta)+2\delta^2-3\delta+1],$$

$$D_2=Q_1(1-\beta)\lambda(j+2)[e^{i\theta}+(j+1)(2\delta^2-\delta)]+[Q_1(\beta-\vartheta)-Q_2][(j+2)(2\delta^2-\delta)+2\delta^2-3\delta+1],$$

$$Q_1=e^{i\theta}(1-\beta)+(\beta-\vartheta)(1+\delta^2),$$

$$Q_2=1-\delta^2.$$

Corollary 3.15: Let $\phi(z)=1+\frac{1}{2}z^{j}-\frac{1}{24}z^{j+2}+\frac{1}{240}z^{j+4}-...$. Suppose that $f(z)^{\alpha}=z^{\alpha}+\sum_{k=j+1}^{\infty}a_k(\alpha)z^{\alpha+k-1}$ belong to the class $G_{\alpha}^{n}(\lambda,0,\beta,\theta,\vartheta,j,\phi)$, then

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^{2}(\alpha)\right|\leq\frac{\alpha^{n}}{2D_2(\alpha+j+1)^{n}}\begin{cases}\frac{1}{2D_1}-\frac{1}{4D_1^2}\lambda(\lambda-1)(j+1)^2[e^{i\theta}][e^{i\theta}(1-\beta)+(\beta-\vartheta)]\\ -\mu\frac{D_2\alpha^{n}(\alpha+j+1)^{n}}{2D_1^2(\alpha+j)^{2n}}, \mu\leq\sigma_1\\ 1, \sigma_1\leq\mu\leq\sigma_2\\ -\frac{1}{2D_1}+\frac{1}{4D_1^2}\lambda(\lambda-1)(j+1)^2[e^{i\theta}][e^{i\theta}(1-\beta)+(\beta-\vartheta)]\\ -\mu\frac{D_2\alpha^{n}(\alpha+j+1)^{n}}{2D_1^2(\alpha+j)^{2n}}, \mu\geq\sigma_2.\end{cases}$$

$$\sigma_1=\left[\frac{1}{2D_1}-\frac{1}{4D_1^2}\lambda(\lambda+1)(j+1)^2e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]-1\right]\frac{2D_1^2(\alpha+j)^{2n}}{D_2\alpha^{n}(\alpha+j+1)^{2n}},$$

$$\sigma_2=\left[1+\frac{1}{2D_1}-\frac{1}{4D_1^2}\lambda(\lambda+1)(j+1)^2e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]\right]\frac{2D_1^2(\alpha+1)^{2n}}{D_2\alpha^{n}(\alpha+j+1)^{n}},$$

$$D_1=\lambda(1-\beta)(j+1)e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]+[(\beta-\vartheta)[e^{i\theta}(1-\beta)+(\beta-\vartheta)]-1],$$

$$D_2=\lambda(1-\beta)(j+2)e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]+[(\beta-\vartheta)[e^{i\theta}(1-\beta)+(\beta-\vartheta)]-1],$$

$Q_1 = e^{i\theta}(1-\beta)+(\beta-\vartheta),$
$Q_2 = 1.$

Corollary 3.16: Let $\phi(z)=1+\frac{1}{2}z^j-\frac{1}{24}z^{j+2}+\frac{1}{240}z^{j+4}-\dots$. Suppose that $f(z)^\alpha = z^\alpha+\sum_{k=j+1}^{\infty}a_k(\alpha)z^{\alpha+k-1}$ belong to the class $G_\alpha^n(\lambda,1,\beta,\theta,\vartheta,j,\phi)$, then

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)\right| \le \frac{\alpha^n}{2D_2(\alpha+j+1)^n}\begin{cases} -\frac{1}{4D_1^2}\lambda(\lambda-1)(j+1)^2(e^{i\theta}+j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)] \\ -\mu\frac{D_2(\alpha+j+1)^n}{2D_1^2\alpha^n(\alpha+j)^{2n}}, \mu\le\sigma_1 \\ 1, \sigma_1\le\mu\le\sigma_2 \\ \frac{1}{4D_1^2}\lambda(\lambda-1)(j+1)^2(e^{i\theta}+j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)] \\ +\mu\frac{D_2(\alpha+j+1)^n}{2D_1^2\alpha^n(\alpha+j)^{2n}}, \mu\ge\sigma_2. \end{cases},$$

$$\sigma_1=\left[-\frac{1}{4D_1^2}\lambda(\lambda+1)(j+1)^2(e^{i\theta}+j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)]-1\right]\frac{2D_1^2(\alpha+j)^{2n}}{D_2\alpha^n(\alpha+j+1)^{2n}},$$

$$\sigma_2=\left[1-\frac{1}{4D_1^2}\lambda(\lambda+1)(j+1)^2(e^{i\theta}+j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)]\right]\frac{2D_1^2(\alpha+1)^{2n}}{D_2\alpha^n(\alpha+j+1)^n},$$

$$D_1=\lambda(1-\beta)(j+1)(e^{i\theta}+j)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)]+(\beta-\vartheta)(j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)],$$

$$D_2=\lambda(1-\beta)(j+2)(e^{i\theta}+j+1)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)]+(\beta-\vartheta)(j+2)[e^{i\theta}(1-\beta)+2(\beta-\vartheta)],$$

$Q_1=e^{i\theta}(1-\beta)+2(\beta-\vartheta),\ Q_2=0.$

Corollary 3.17: Let $\phi(z)=1+\frac{1}{2}z^j-\frac{1}{24}z^{j+2}+\frac{1}{240}z^{j+4}-\dots$. Suppose that $f(z)^\alpha = z^\alpha+\sum_{k=j+1}^{\infty}a_k(\alpha)z^{\alpha+k-1}$ belong to the class $G_\alpha^n(1,0,\beta,\theta,\vartheta,j,\phi)$, then

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)\right| \le \frac{\alpha^n}{2D_2(\alpha+j+1)^n}\begin{cases} \frac{1}{2D_1}-\mu\frac{D_2(\alpha+j+1)^n}{2D_1^2\alpha^n(\alpha+j)^{2n}}, \mu\le\sigma_1 \\ 1, \sigma_1\le\mu\le\sigma_2 \\ -\frac{1}{2D_1}-\mu\frac{D_2(\alpha+j+1)^n}{2D_1^2\alpha^n(\alpha+j)^{2n}}, \mu\ge\sigma_2. \end{cases},$$

$$\sigma_1=\left[\frac{1}{2D_1}-1\right]\frac{2D_1^2(\alpha+j)^{2n}}{D_2\alpha^n(\alpha+j+1)^{2n}},$$

$$\sigma_2=\left[\frac{1}{2D_1}+1\right]\frac{2D_1^2(\alpha+1)^{2n}}{D_2\alpha^n(\alpha+j+1)^n},$$

$$D_1=(1-\beta)(j+1)e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]+[(\beta-\vartheta)[e^{i\theta}(1-\beta)+(\beta-\vartheta)]-1],$$

$$D_2=(1-\beta)(j+2)e^{i\theta}[e^{i\theta}(1-\beta)+(\beta-\vartheta)]+[(\beta-\vartheta)[e^{i\theta}(1-\beta)+(\beta-\vartheta)]-1].$$

Corollary 3.18: Let $\phi(z)=1+\frac{1}{2}z^j-\frac{1}{24}z^{j+2}+\frac{1}{240}z^{j+4}-\dots$. Suppose that $f(z)^\alpha = z^\alpha+\sum_{k=j+1}^{\infty}a_k(\alpha)z^{\alpha+k-1}$ belong to the class $G_\alpha^n(1,1,0,0,-1,j,\phi)$, then

$$\left|a_{j+2}(\alpha)-\mu a_{j+1}^2(\alpha)\right| \le \frac{\alpha^n}{6(j+2)(j+3)(\alpha+j+1)^n}\begin{cases} -\frac{\mu(j+3)(\alpha+j+1)^n}{6(j+1)^2(j+2)\alpha^n(\alpha+j)^{2n}}, \mu\le\sigma_1 \\ 1, \sigma_1\le\mu\le\sigma_2 \\ \frac{\mu(j+3)(\alpha+j+1)^n}{6(j+1)^2(j+2)\alpha^n(\alpha+j)^{2n}}, \mu\ge\sigma_2. \end{cases},$$

$$\sigma_1 = -\frac{6(j+1)^2(j+2)(\alpha+j)^{2n}}{(j+3)(\alpha+j+1)^n},$$

$$\sigma_2 = \frac{6(j+1)^2(j+2)(\alpha+j)^{2n}}{(j+3)(\alpha+j+1)^n}.$$

Corollary 3.19: Let $\phi(z) = 1 + \frac{1}{2}z^j - \frac{1}{24}z^{j+2} + \frac{1}{240}z^{j+4} - \ldots$. Suppose that $f(z)^\alpha = z^\alpha + \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1}$ belong to the class $G_\alpha^n(1,0,0,0,-1,j,\phi)$, then

$$\left|a_{j+2}(\alpha) - \mu a_{j+1}^2(\alpha)\right| \le \frac{\alpha^n}{2(2j+5)(\alpha+j+1)^n}\begin{Bmatrix} \frac{1}{2(2j+3)} - \mu\frac{(2j+5)(\alpha+j+1)^n}{2(2j+3)^2\alpha^n(\alpha+j)^{2n}}, \mu \le \sigma_1 \\ 1, \sigma_1 \le \mu \le \sigma_2 \\ -\frac{1}{2(2j+3)} - \mu\frac{(2j+5)(\alpha+j+1)^n}{2(2j+3)^2\alpha^n(\alpha+j)^{2n}}, \mu \ge \sigma_2. \end{Bmatrix},$$

$$\sigma_1 = -\frac{(5+4j)(3+2j)(\alpha+j)^{2n}}{\alpha^n(5+2j)(\alpha+j+1)^n},$$

$$\sigma_2 = \frac{(7+4j)(3+2j)(\alpha+j)^{2n}}{\alpha^n(5+2j)(\alpha+j+1)^n},$$

Corollary 3.20: Let $\phi(z) = 1 + \frac{1}{2}z^j - \frac{1}{24}z^{j+2} + \frac{1}{240}z^{j+4} - \ldots$. Suppose that $f(z)^\alpha = z^\alpha + \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1}$ belong to the class $G_1^0(1,1,0,0,-1,1,\phi)$, then

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{72}\begin{Bmatrix} -\frac{\mu}{18}, \mu \le 18 \\ 1, -18 \le \mu \le 18 \\ \frac{\mu}{18}, \mu \ge 18. \end{Bmatrix}.$$

Remark:
For recent articles on Fekete-Szego problem, interested readers can refer to [11], [12], [13] among others.

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