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STABILITY CONDITIONS FOR THE UNIQUENESS OF LIMIT CYCLES IN PRE-PREDATOR SYSTEMS OF IVLEV TYPE.

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Abstract

The stability of positive equilibrium of solutions in pre-predator systems of nonlinear autonomous equations is considered. The emphasis is on Lotka-Volterra equations of two-dimensional nonlinear autonomous systems of ordinary differential equations of pre-predator system with Ivlev's functional response. The pre-predator system is linearized with Ivlev's functional response to deduced theorem 2 for the necessary and sufficient conditions for a unique stable limit cycle of the model used. The theorem1 of stability criteria for plane autonomous systems is also used.

***Keywords*:** Nonlinear autonomous equations, Ivlev's functional response, unique stable cycle

1.0 INTRODUCTION

In the study of physical system steady states is most important. However, a steady state has little physical significance unless it is stable. Steady state of a simple physical system corresponds to equilibrium in the phase plane [1]. The study of stability is very important in the pre-predator population, otherwise the population tends to extinction. Therefore, the mathematical descriptions of ecological systems are very important.

The systems $\frac{dx}{dt} = rx(1-x) - (1-e^{-ax})y$ and $\frac{dy}{dt} = y(-D + (1-e^{-ax}))$, -----------(1)

models a pre-predator system of Lotka-Volterra equations, where $r > 0, D > 0, a > 0$ and Ivlev's functional response $(1 - e^{-ax})$ is the capture rate of pre per predator. The existence and uniqueness of limit cycle in (1) was proved by [2] and while the necessary and sufficient condition for the uniqueness of limit cycles of the pre-predator system in (1) was presented by [3].

It has been found that the study of stability of nonlinear autonomous equation is the most interesting part of ecosystem. In the study of physical system, steady state has little physical significance unless it is stable. Since a steady state of a simple physical system corresponds to equilibrium, the study of stability is very important in the pre-predator population, otherwise the population tends to extinction. Therefore, the mathematical descriptions of ecological systems are very important.

The aim is to consider the unique positive equilibrium point of pre-predator system of Lotka-Volterra equations, concentrating on pre-predator system with Ivlev's function response (1), and to deduce the theorem for necessary and sufficient condition for the uniqueness of limit cycles [3]. Here we will be considering nonlinear autonomous equations which have many critical points, but we will concentrate on the unique positive critical point by applying the theorem of stability criteria for plane autonomous equations and linear variation method

2.0 FORMULATIONS

Considering the model with Ivlev's functional response of the form

$$\frac{dx}{dt} = rx(1-x) - (1-e^{-ax})y$$
$$\frac{dy}{dt} = y(-D + (1-e^{-ax})), \qquad (2)$$

where $r > 0, D > 0, a > 0$.

DEFINITION

1. The following definitions will help.
   (i) Suppose that the system (2) can be written in the form
   $$\frac{dx}{dt} = ax + by + G_1(x, y)$$

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$\frac{dy}{dt} = cx + dy + G_2(x, y)$ (3)

where $\left[\frac{G_i(x,y)}{g}\right] \longrightarrow 0\ as\ g \longrightarrow 0, i = 1,2$

The linear systems $\frac{dx}{dt} = ax + by,\ and\ \frac{dy}{dt} = cx + dy$ (4)

is said to be the linearization (or linearized system) of (3) at the origin or equilibrium point if the condition is satisfied.

(ii) If all eigenvalues of matrix A have non-zero real parts, and $f(x)$ is a bounded function which is small and lipschitzian, then the nonlinear system

$\frac{dx}{dt} = Ax + f(x)$ (5)

is topologically equivalent to its linear system

$\frac{dx}{dt} = Ax$ [4] (6)

2. STABILITY

Here is a formal definition of stability of solution function of equation

(i) The equilibrium solution x_0 of (1) is said to be stable if for each number $\varepsilon > 0$, we can find a number $\delta > 0$ (depending on ε) such that if $\psi(t)$ is any solution of (1) having $\|\varphi(t_0) - x_0\| < \delta$, then the solution $\psi(t)$ exists, for $t \geq t_0$ and $\|\psi(t) - x_0\| < \varepsilon$, for $t \geq t_0$.

(ii) The equilibrium solution x_0 of (1) is said to be asymptotically stable if it is stable and if there exist a number $\delta_0 > 0$ such that if $\psi(t)$ is any solution of (1) is such that $\|\varphi(t_0) - x_0\| < \delta_0$, then, $\lim\psi(t) = x_0$ as $t \to \infty$. [5]

3. CRITICAL (EQUILIBRIUM) POINT

A point a in the domain D of $g(x)$ is called a critical point of the system (1) if and only if $g(a) = 0$. Thus, the critical points are those points of D at which $g(x)$ vanish.

THEOREM 1

Stability criteria for plane autonomous system, Let (x_0, y_0) be a critical point of plane autonomous system $\frac{dx}{dt} = f(x, y)\ and\ \frac{dy}{dt} = g(x, y)$ (7)

where $f(x, y)$ and $g(x, y)$ have continuous first-order partial derivatives in a neighbourhood of (x_0, y_0). Let $A = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0. y_0) \end{bmatrix}$

i. If the eigenvalues of A have negative real parts, then (x_0, y_0) is an asymptotically stable critical point of (7)

ii. If A has an eigenvalue with a positive real part, then (x_0, y_0) is an unstable critical point of (7). [6].

THEOREM 2

If $a > -\frac{2D+(1-D)\log(1-D)}{D+(1-D)\log(1-D)}\log(1-D)$ (8)

then system (2) has a unique stable limit cycle, otherwise, system (2) has no limit cycles [3].

The problem in mathematical modelling of ecological system has been that of determining conditions for the stability of limit cycles in pre-predator models.

In pre-predator system, the existence and stability of limit cycle is related to the existence and stability of equilibrium. If the equilibrium is asymptotically stable, there may exist limit cycles, the innermost of which must be unstable from the inside and the outermost of which must be stable from the outside [7]

3.0 RESULTS

Considering the stability of the unique positive equilibrium point of the system (1)

The linear approximation of is $\frac{dx}{dt} = rx$ and $\frac{dy}{dt} = -Dy$ (9)

and the jacobian matrix of (9) at the origin is $A = \begin{bmatrix} r & 0 \\ 0 & -D \end{bmatrix}$ (10)

Hence we have

$|A - \lambda I| = \begin{vmatrix} r - \lambda & 0 \\ 0 & -D - \lambda \end{vmatrix} = 0,\ \Rightarrow\ \lambda_1 = r,\ \lambda_2 = -D$ (11)

By theorem 1 the eigenvalues $\lambda_1 = r,\ \lambda_2 = -D$ have different sign and hence the system is unstable.

The jacobian matrix of (2) is $A = \begin{bmatrix} r - 2rx - aye^{-ax} & -(1 - e^{-ax}) \\ aye^{-ax} & (1 - e^{-ax} - D) \end{bmatrix}$ (12)

The critical points of (2) are the solutions of

$rx(1-x)-(1-e^{-ax})y=0$ *and* $y(1-e^{-ax}-D)=0$ (13)

we have $y=0,\ e^{-ax}=1-D \implies x=-\frac{1}{a}\log(1-D)$ (14)

For $x=-\frac{1}{a}\log(1-D), y=-\frac{r}{Da}\log(1-D)\,[1+\frac{1}{a}\log(1-D)]$ (15)

Let $\rho=-\frac{1}{a}\log(1-D)$, (16)

then we have $x=\rho,\ y=\frac{r\rho}{D}(1-\rho)$ as the critical point.

The jacobian matrix of (2) at the point $[\rho,\ \frac{r\rho}{D}(1-\rho)]$ is

$$A=\begin{bmatrix} r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D) & -D \\ \frac{ar\rho}{D}(1-\rho)(1-D) & 0 \end{bmatrix} \quad (17)$$

Hence $$|A-\lambda I|=\begin{vmatrix} r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)-\lambda & -D \\ \frac{ar\rho}{D}(1-\rho)(1-D) & -\lambda \end{vmatrix} \quad (18)$$

$$\lambda^2-\left[r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)\right]\lambda+ar\rho(1-\rho)(1-D)=0 \quad (19)$$

$$\lambda=\frac{r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)\pm\sqrt{\left(r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)\right)^2-4ar\rho(1-\rho)(1-D)}}{2} \quad (20)$$

Let $\tau=r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)$ and $\Delta=ar\rho(1-\rho)(1-D)$ (21)

then, we have $\lambda=\frac{\tau\pm\sqrt{\tau^2-4\Delta}}{2} \implies \lambda_1=\frac{\tau+\sqrt{\tau^2-4\Delta}}{2},\quad \lambda_2=\frac{\tau-\sqrt{\tau^2-4\Delta}}{2}$ (22)

If $\lambda_1<\lambda_2<0$, then this type of critical point corresponds to the case when both eigenvalues are negative, and the critical point is asymptotically stable. It is called a stable improper node.

If $\lambda_1,\ \lambda_2$ are complex conjugates, say $\lambda_1=\alpha+i\beta$ and $\lambda_2=\alpha-i\beta$ with $\beta\neq 0$, then the critical point is called a focus and it is asymptotically stable, for $\alpha<0$.

To establish theorem 2, we assumed that $r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)>0$ in (19) holds.

If $r-2r\rho-\frac{ar\rho}{D}(1-\rho)(1-D)>0$, (23)

with $\rho=-\frac{1}{a}\log(1-D)$,

$r+\frac{2r}{a}\log(1-D)+\frac{r}{D}\log(1-D)\,[1+\frac{1}{a}\log(1-D)](1-D)>0$ (24)

$1+\frac{2}{a}\log(1-D)+\frac{1}{D}\log(1-D)\left[1+\frac{1}{a}\log(1-D)\right](1-D)>0$ (25)

$a[D+(1-D)\log(1-D)]>-[2D+(1-D)\log(1-D)]\log(1-D)$ (26)

$a>-\frac{2D+(1-D)\log(1-D)}{D+(1-D)\log(1-D)}\log(1-D)$ (27)

The inequality (27) is the necessary and sufficient condition for a system to have a unique stable limit cycle (Theorem 2).

CONCLUSION

Theorem 2 was established by transforming (2) into a Lienard system and applying the properties of $f(x)$ and $g(x)$ of the Lienard system of the form

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0 \quad (28)$$

In the result obtained (27) which is the same with (8), the system (2) is linearized and the eigenvalues of the jacobian matrix for the system are utilised to deduce the result of [3], which is inequality (27).

If $a>-\frac{2D+(1-D)\log(1-D)}{D+(1-D)\log(1-D)}\log(1-D)$ then system (2) has a unique stable limit cycle; otherwise, system (2) has no limit cycles. If $D<1-e^{-a}$, the system (2) has critical points and limit cycles exist otherwise system (2) has no critical points and therefore no limit cycles exist [3]. The asymptotic stability of unique positive equilibrium point of nonlinear autonomous system using linear variation method was considered.

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