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ON COMBINED EFFECTS OF RADIATION, PERTURBATIONS, OBLATENESS AND DISC ON MOTION AROUND COLLINEAR LAGRANGIAN POINTS OF RESTRICTED THREE-BODY PROBLEM

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Abstract

The present paper divulges combined effects of radiation, perturbations, oblateness and disc on motion around collinear Lagrangian points of the restricted three-body problem when the three bodies are oblate spheroids and surrounded by a disc of dust. The primaries are assumed to be emitting radiation pressure and small perturbations in the Coriolis and centrifugal forces are also assumed to be present. These perturbing forces now act on motion and locations of the infinitesimal mass and the pertinent equations of motion are deduced. The collinear Lagrangian points are also located. It is seen that there exist five such Lagrangian points, two of which exist when simultaneously the mass parameter $\mu \geq 0.15463365$ ***and the density profile parameter*** T ***of the disc is less than*** $\sqrt{2}\mu$***. Invariably, such additional collinear points may not exist in our Solar system. The presence of the perturbed forces due to radiation, oblateness and small perturbation in the centrifugal force do not in any way result in the existence of additional points but define the locations. The stability is discussed and it is seen that the collinear Lagrangian points are unstable. Hence, the presence of the disc in the configuration may not possess the required force sufficient to keep the infinitesimal mass from departing the collinear Lagrangian points.***

Keywords: Restricted three-body problem; collinear Lagrangian points; stability

1. Introduction

Scientists for all time have been engaged in research to produce knowledge and discover results that will be applicable and useful for man. The purpose of research is also about testing theories often generated by pure science and applying them to real situations. While pure mathematics has to do with abstract principles and describing them with theories, applied mathematics by contrast use equations to investigate real life phenomena like mechanics, gravity and several other scientific fields. A branch of theoretical astronomy that deals with the motion of celestial bodies, is called celestial mechanics. Of great importance and interest in celestial mechanics, is the restricted three-body problem (R3BP). This problem illustrates motion of a third body having infinitesimal mass and moving under the mutual gravitational attraction of two main bodies called primaries which move around their center of mass either along circular or elliptical orbits. The R3BP plays an important role in studying the motion of artificial satellites and also used to evaluate the motion of the planets, minor planets and comets. The restricted problem gives an accurate illustration not only regarding the motion of the Moon but also with respect to the motion of other natural satellites. Furthermore, the restricted problem has many applications in physics, mathematics and quantum mechanics, to name a few. In quantum mechanics, a general form of the restricted problem is formed to solve the Schrödinger equation of helium-like ions. Furthermore, in modern solid state physics, the restricted problem can be used to discuss the motion of an infinitesimal mass affected not only by the

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gravitational field but also by light pressure from one (or both) of the primaries. The R3BP is still a stimulating and active research field that has been receiving considerable attention of scientists and astronomers because of its numerous applications.

The solutions of the R3BP have been developed over the centuries; from [1-8]. There exist five specific solutions called the equilibrium or Lagrangian points. Three of these points denoted L_1, L_2 and L_3 are called the collinear equilibrium points and are found on the line joining the primaries while the other two denoted L_4 and L_5 are called triangular equilibrium points because they form a pair of triangles with the primaries.. The collinear equilibrium points are unstable points while the triangular points can be stable [2, 4,8], in that a slight displacement of the infinitesimal mass away from the equilibrium points will not produce unbounded motion but rather an oscillation about the points.

Several modifications of the R3BP have been introduced in order to make it more relevant and applicable to certain systems of Dynamical Astronomy. The characterizations under different permutations have included perturbing forces such as mass variations of the primaries [5,6] radiation pressure and P-R drag of the primaries [9], shapes of the three bodies [8], debris or dust disc around the primaries [7, 10], to whether the bodies are charged masses or magnetic dipoles [11]. In [4] the combined effects of radiation, perturbations and oblateness on locations and stability of the equilibrium points, was discussed. In their study, they assumed that both main masses are radiating bodies and have the shape of an oblate spheroid. Two triangular and three collinear equilibrium points were found and their stability examined. An extension of the formulation in [4] was carried out by [8] by assuming that the infinitesimal mass has the shape of an oblate spheroid, consequently turning the model into the restricted problem of three oblate bodies. They also found two triangular and three collinear points.

In this paper, we extend the work of [8] by assuming that the main masses are stars surrounded by a disc of dusts. However, we consider motion only around the collinear equilibrium points. The equilibrium points are very important in exploration and development of space. The Solar and Heliospheric Observatory (SOHO) lunched in 1995 and Microwave Anisotropy Probe (MAP) lunched in 2001 by NASA are currently in operation Sun-Earth L_1 and L_2, respectively. Solar TErrestrial RElations Observatory-Ahead (STEREO-A) made its closest pass to L_5 recently, on its orbit around the Sun. Asteroid 2010 SO16, is currently proximal to L_5 but at a high inclination.

The paper is set up with the introduction given in section 1 while the equations of motion and location of the collinear equilibrium points are presented in sections 2 and 3, respectively. The linear stability analysis of the collinear equilibrium points is done in section 4 while in section 5 and 6, the discussion and conclusions are drawn, respectively.

2. Equations of motion

Let m_1 and m_2 be the masses of the first and second primary, respectively, and let m_3 be the mass of the third body having infinitesimal mass compared to the masses of the primaries. Suppose that the primaries have the shape of an oblate spheroid and are radiation sources such that they emit radiation pressure. Then, the equations of motion in a barycentric rotating coordinate system of an infinitesimal mass in the gravitational field of the primaries, with effective small perturbations in the Coriolis and centrifugal forces, have the form [4]:

$$\ddot{x} - 2\varphi n\dot{y} = U_x$$
$$\ddot{y} + 2\varphi n\dot{x} = U_y \qquad (1)$$

where

$$U = \frac{n^2\psi}{2}\left(x^2 + y^2\right) + \frac{q_1(1-\mu)}{r_1} + \frac{q_2\mu}{r_2} + \frac{A_1q_1(1-\mu)}{2r_1^3} + \frac{A_2q_2\mu}{2r_2^3}$$

where

$$n^2 = 1 + \frac{3}{2}(A_1 + A_2)$$

$$r_1^2 = (x-\mu)^2 + y^2,\; r_2^2 = (x-\mu+1)^2 + y^2$$

r_1 and r_2 are the distances of the body from the first and second primary, respectively;

$\mu = \frac{m_2}{m_1 + m_2}$ is the mass ratio of the primaries and n is their mean motion defined by the oblateness of the primaries $A_i\,(i = 1,2)$.

The parameters φ and ψ represent the small perturbations in Coriolis and the centrifugal forces, respectively and are such that $(\varphi - 1) << 1$ and $(\psi - 1) << 1$.

$q_i\,(i=1,2)$ denote the radiation pressure of the first and second primary, respectively, and are determined by the value of a resultant force F due to the gravitational force F_g and the radiation pressure force F_p acting on infinitesimal mass.

Next, in accordance with [9]; when the shape of the infinitesimal mass is as well assumed to be in the shape of an oblate spheroid, the equations of motion (1) take the form:

$$\ddot{x}-2\varphi n\dot{y}=U_x$$
$$\ddot{y}+2\varphi n\dot{x}=U_y \qquad (2)$$

where

$$U=\frac{n^2\psi}{2}\left(x^2+y^2\right)+\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}+\frac{A_1q_1(1-\mu)}{2r_1^3}+\frac{A_2q_2\mu}{2r_2^3}+\frac{A_3(1-\mu)}{2r_1^3}+\frac{A_3\mu}{2r_2^3}$$

where

$$n^2=1+\frac{3}{2}(A_1+A_2)$$
$$r_1^2=(x-\mu)^2+y^2,\; r_2^2=(x-\mu+1)^2+y^2$$

where A_3 denotes the oblateness of the infinitesimal mass.

Now, working with the modeled equations (2), it is assumed that the configuration has a disc of dusts surrounding it and the extended problem has equations of the type [12]:

$$\ddot{x}-2\varphi n\dot{y}=U_x$$
$$\ddot{y}+2\varphi n\dot{x}=U_y \qquad (3)$$

where

$$U=\frac{n^2\psi}{2}\left(x^2+y^2\right)+\frac{q_1(1-\mu)}{r_1}+\frac{q_2\mu}{r_2}+\frac{A_1q_1(1-\mu)}{2r_1^3}+\frac{A_2q_2\mu}{2r_2^3}+\frac{A_3(1-\mu)}{2r_1^3}+\frac{A_3\mu}{2r_2^3}+\frac{M_d}{\left(r^2+T^2\right)^{1/2}}$$

where

$$r_1^2=(x-\mu)^2+y^2,\; r_2^2=(x-\mu+1)^2+y^2$$

The mean motion in this case is

$$n^2=1+\frac{3}{2}(A_1+A_2)+\frac{2r_cM_d}{\left(r_c^2+T^2\right)^{3/2}} \qquad (4)$$

Where r_c is the radial distance of the particle in the classical R3BP while M_d is the mass of the disc and $T=a+b$ determines the density profile of the disc. Here, a and b are the flatness and core parameters, respectively. The gravitational potential acting on the infinitesimal mass is [13]:

$$V_d(r,T)=-\frac{GM_d}{\sqrt{r^2+T^2}} \qquad (5)$$

where G is the gravitational constant; $r=\sqrt{x^2+y^2}$ is the radial distance of the particle;

3. Collinear equilibrium points

The collinear equilibrium points are the equilibrium points which lie on the line joining the primaries. They are the solutions of the R3BP when the velocity and acceleration are zero. Consequently, we have to equate the R.H.S of equations (4) to zero and solve. That is, we have to solve the equations

$$n^2x\psi-\frac{q_1(1-\mu)(x-\mu)}{r_1^3}-\frac{q_2\mu(x-\mu+1)}{r_2^3}-\frac{3}{2}\frac{A_1q_1(1-\mu)(x-\mu)}{r_1^5}-\frac{3}{2}\frac{A_2q_2\mu(x-\mu+1)}{r_2^5} \qquad (6)$$
$$-\frac{3}{2}\frac{A_3(1-\mu)(x-\mu)}{r_1^5}-\frac{3}{2}\frac{A_3\mu(x-\mu+1)}{r_2^5}-\frac{xM_d}{\left(r^2+T^2\right)^{3/2}}=0$$

and

$$n^2y\psi-\frac{q_1(1-\mu)y}{r_1^3}-\frac{q_2\mu y}{r_2^3}-\frac{3}{2}\frac{A_1q_1(1-\mu)y}{r_1^5}-\frac{3}{2}\frac{A_2q_2\mu y}{r_2^5}-\frac{3}{2}\frac{A_3(1-\mu)y}{r_1^5}-\frac{3}{2}\frac{A_3\mu y}{r_2^5}-\frac{yM_d}{\left(r^2+T^2\right)^{3/2}}=0$$

when $y=0$ ..

For zero velocity and acceleration components in the equations of motion (4) when $y=0$, we have

$$n^2 x\psi - \frac{q_1(1-\mu)(x-\mu)}{r_1^3} - \frac{q_2\mu(x-\mu+1)}{r_2^3} - \frac{3}{2}\frac{A_1q_1(1-\mu)(x-\mu)}{r_1^5} - \frac{3}{2}\frac{A_2q_2\mu(x-\mu+1)}{r_2^5} \tag{7}$$
$$-\frac{3}{2}\frac{A_3(1-\mu)(x-\mu)}{r_1^5} - \frac{3}{2}\frac{A_3\mu(x-\mu+1)}{r_2^5} - \frac{xM_d}{\left(r^2+T^2\right)^{3/2}} = 0$$

where $r_1^2 = (x-\mu)^2$, $r_2^2 = (x-\mu+1)^2$

For zero velocity and acceleration components in the equations of motion (4) when $y = 0$, we obtain equation (7) and suppose we denote it by $f(x)$, so that

$$f(x) = \left[n^2\psi - \frac{M_d}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right]x - \frac{q_1(1-\mu)(x-\mu)}{r_1^3} - \frac{q_2\mu(x-\mu+1)}{r_2^3} - \frac{3(1-\mu)(x-\mu)(A_1q_1+A_3)}{2r_1^5} \tag{8}$$
$$-\frac{3}{2}\frac{\mu(x-\mu+1)(A_2q_2+A_3)}{r_2^5} = 0$$

where $r_1 = |x-\mu|$ and $r_2 = |x-\mu+1|$

The abscissas of the collinear points are the roots of equation (8).

Now, from equation (8), if $x = \pm\infty$, we have $\frac{df(x)}{dx} = n^2\psi$; when $x = \mu-1$ and $x = \mu$, both cases gives $\frac{df(x)}{dx} = \infty$. When $x = \mu-2$, we observed that $f(\mu-2) < 0$, for $\psi \approx 1$, $M_d << 1$, $A_i << 1$ $(i = 1,2,3)$ and $q_i \approx 1$. When $x = 0$, we observe that $f(0) > 0$, while for $x = \mu+1$, $f(\mu+1) > 0$. Therefore, the real roots of the equation (8) will lie in the opens interval $(\mu-2, \mu-1)$, $(\mu-1, 0)$ and $(\mu, \mu+1)$. These roots correspond to the collinear points and we shall denote them appropriately when they are found.

Now, we express equation (8) as

$$f(x) = P(x) + Q(x) \tag{9}$$

where

$$P(x) = n^2\psi x - \frac{(1-\mu)(x-\mu)}{|x-\mu|^3}\left\{q_1 + \frac{3}{2}\frac{(A_1+A_2)}{|x-\mu|^2}\right\} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3}\left\{q_2 + \frac{3}{2}\frac{(A_2+A_2)}{|x-\mu+1|^2}\right\} \tag{10}$$
$$Q(x) = -\frac{M_d x}{\left(x^2+T^2\right)^{3/2}}$$

Next, observe that the first equation in system (10) can have different forms depending on positions of the primaries. Hence, in order to investigate the position of collinear equilibrium points, we divide the orbital plane *Oxy* into three parts with respect to the primaries: $x < \mu-1$, $\mu-1 < x < \mu$ and $\mu < x$. The first case $x < \mu-1$, implies that $x-\mu+1 < 0$ and if this happens, then $x-\mu < 0$. In the second case, when $\mu-1 < x < \mu$, we have $x-\mu < 0$ and $x-\mu+1 > 0$. The third case when $\mu < x$, we have $x-\mu > 0$ and so $x-\mu+1 > 0$. We follow the pattern used in [10] and we found that an equilibrium point L_1 exists in the interval $(-\infty, \mu-1)$ while a second point L_2 lies in the interval $(\mu-1, 0)$. When $T < \sqrt{2}\mu$, the third and fourth equilibrium points denoted by L_{21} and L_{22} exist in the open interval $(0, \mu)$, each lying in the interval $\left(0, \frac{T}{\sqrt{2}}\right)$ and $\left(\frac{T}{\sqrt{2}}, \mu\right)$, respectively. A fifth collinear equilibrium point denoted L_3 lies in the open interval (μ, ∞). Hence, there exist five collinear equilibrium points when there is a disc around the system.

Following [10], we explore equation (7) numerically and our result reveals that when the mass parameter $\mu < 0.15463365$ for any system, there exist only three collinear equilibrium points. When simultaneously $\mu \geq 0.15463365$ and $M_d > 0$, only then do the additional collinear equilibrium points L_{21} and L_{22} exists, provided $T < \sqrt{2}\mu$. Invariably, such equilibrium points may not exist in our Solar system. The presence of the perturbed forces coming from radiation, oblateness, small perturbation in the centrifugal force do not result in the existence of additional collinear points but they affect the locations. These additional collinear points now referred to as the Jiang and Yeh points [14] do not appear in the classical R3BP because the effects arising due to the gravitational potential from the disc is not present.

4 Stability of collinear equilibrium points

In order to study the stability of any of the equilibrium points $L_i\left(1=1,2,3\right)$ and L_{21}, L_{22}, we displace the infinitesimal mass a little from an equilibrium point, by applying a small velocity. Next, introduce $x = x_0 + \xi$ and $y = y_0 + \eta$, where (ξ, η) is a small displacement and substitute it in the equations of motion (4). On expanding the equations of motion into first-order terms with respect to ξ and η, we obtain the variational equations:

$$\ddot{\xi} - 2n\varphi\dot{\eta} = U^0_{xx}\xi + U^0_{xy}\eta$$

$$\ddot{\eta} + 2n\varphi\dot{\xi} = U^0_{xy}\xi + U^0_{yy}\eta \tag{11}$$

where the superscript O indicates that the derivatives are to be calculated at the collinear equilibrium points.
The associated characteristic equation is

$$\lambda^4 - \left(U^0_{xx} + U^0_{yy} - 4\varphi^2 n^2\right)\lambda^2 + U^0_{xx}U^0_{yy} - \left(U^0_{xy}\right)^2 = 0 \tag{12}$$

where

$$U^0_{xx} = n^2\psi + \frac{2q_1(1-\mu)}{\left|x_0-\mu\right|^3} + \frac{2q_2\mu}{\left|x_0+1-\mu\right|^3} + \frac{6(1-\mu)(A_1+A_2)}{\left|x_0-\mu\right|^5} + \frac{6\mu(A_2+A_3)}{\left|x_0+1-\mu\right|^5} + \frac{3x_0^2M_d}{\left(x_0^2+T^2\right)^{\frac{5}{2}}} - \frac{M_d}{\left(x_0^2+T^2\right)^{\frac{3}{2}}}$$

$$U^0_{yy} = n^2\psi - \frac{2(1-\mu)}{\left|x_0-\mu\right|^3}\left\{q_1 + \frac{3(A_1+A_2)}{\left|x_0-\mu\right|^2}\right\} - \frac{2\mu}{\left|x_0+1-\mu\right|^3}\left\{q_2 + \frac{3(A_2+A_2)}{\left|x_0+1-\mu\right|^2}\right\} - \frac{M_d}{\left(x_0^2+T^2\right)^{\frac{3}{2}}} \tag{13}$$

$$U^0_{xy} = U^0_{yx} = 0 \tag{14}$$

Equations (13) and (14) are the second order partial derivatives estimated at the collinear point x_0.
Inspecting the reduced characteristic equation when (14) is substituted in (12), we see that
$U^0_{xx} + U^0_{yy} - 4\varphi^2 n^2 < 0$, $U^0_{xx}U^0_{yy} < 0$ since $U^0_{xx} > 0$ and $U^0_{yy} < 0$, $0 < \mu \leq \frac{1}{2}$, $M_d << 1$, $\psi \approx 1$, $q_i \approx 1\left(i=1,2\right)$ and $A_i << 1\ \left(i=1,2,3\right)$.
Hence, the roots of (12) contain at least a positive root in accordance with Descartes rule and this root induces instability at the collinear equilibrium points. Hence, the collinear points are linearly unstable.

5. Discussion

The equations of motion of an infinitesimal body has been derived under the assumption that the three bodies involved in the model of the R3BP are surrounded by a disc and all have the shape of an oblate spheroid with further assumptions that both primaries are radiating and small perturbation in the Coriolis and centrifugal forces are considered to be effective. These equations are affected by radiation pressure, oblateness, mass of the disc and the perturbations in the Coriolis and centrifugal forces of the primaries. These equations are similar but contain more parameters than other previous studies of [4,7,8,10,15,16,17]. Therefore, the five collinear obtained only have their positions shifted due to the combine effects of these parameters in the governing equations. However, it is important to state that the additional two collinear equilibrium points which have been mentioned in the works of [7,15,16,17], which exist when density profile parameter T is less that $\sqrt{2\mu}$ only holds in specific range of the mass parameter. Our numerical effort reveals that when the mass parameter $\mu < 0.15463365$, $T < \sqrt{2\mu}$, there exist only three collinear points, but when simultaneously $\mu \geq 0.15463365$ and $T < \sqrt{2\mu}$, two additional collinear equilibrium points L_{21} and L_{22} exists. These additional collinear points do not appear in the classical R3BP because the effects arising due to the gravitational potential from the disc is ignored. The study of the stability of the collinear equilibrium points did not yield anything different from the already established fact that the collinear equilibrium points are unstable.

6. Conclusion

This paper divulges the dynamical behavior of motion of an infinitesimal mass around collinear equilibrium points in the R3BP under the influence of small perturbations in the Coriolis and centrifugal forces, radiation pressure and oblateness of the primaries which are both enclosed by a disc. The infinitesimal mass is also assumed to have the shape of an oblate spheroid. The equations of motion have been presented and the collinear equilibrium points reassessed. There are five collinear equilibrium points, three are the usual classical locations while two points exist particularly when simultaneously

$\mu \geq 0.15463365$ and $T < \sqrt{2}\mu$. The presence of the perturbed forces due to radiation of the primaries, oblateness of the three bodies in the set up and small perturbations do not in any way result in the existence of additional collinear points but they affect the positions of these points. These points are all unstable due to the presence of a positive root of the governing characteristic equation. Thus, the presence of the disc in the configuration may not hold bound the infinitesimal mass in orbit around the collinear points.

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