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MATRIX REPRESENTATION OF MULTISET RELATIONS AND ITS COMPOSITIONS

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Abstract

A multiset is a collection in which repetition of elements is significant. In this paper, we present the concept of relations, composition of relations and its matrix representation in multiset context.

KEYWORDS: Multisets, Multiset Relations, Multiset Composition, Matrix Representations.

1.0 INTRODUCTION

The theory of set relations has been studied extensively in mathematics along with its applications in diverse fields especially in solving many real world problems; for example, in solving a complex transportation problem where one needs to see how a city A is related to a city B or problems related to air traffic systems, etc.

In Cantorian set theory, usually called a standard or crisp set theory, a set is considered as "any collection into a whole M of definite and distinguished objects (called the elements of M) of our intuition or our thought" [1]. One unavoidable consequence of Cantor's definition is that no element can occur more than once in a classical set. Cantor's assertion of excluding repeated elements does not go hand in hand with many situations arising in solving real world problems [2&3].. For example, the repeated roots of $x^2 - 2x + 1 = 0$, repeated hydrogen atoms in a water molecule (H_2O), the repeated prime factors of an integer $n > 0$, etc. need to be considered significant. Once we admit repetition of objects, we have multisets.

In view of the recent developments taking place in the study of multisets, a number of important areas of research requiring multiset relations have come to the fore. Especially from a practical point of view, multisets are found useful in providing structures as they arise quite naturally in certain areas of mathematics, computer science, physics, linguistics and philosophy.

Relations are fundamental concepts in discrete mathematics used to define how sets of objects or elements of multiset relate to other sets of objects or other elements of multisets. They provide important modelling tools in almost all areas of scientific developments. In particular, they are used in relational database management systems, task scheduling systems, methods of solving various optimization problems etc,. In this paper, we present multiset relations, operations on multiset relations and their matrix representation.

2.0 PREMINARIES

The term multiset, as noted by [4], was first suggested by N.G.de Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz; list, heap, bunch, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset. Infact, prior to coinage of the term multiset, the term bag was in frequent use. Currently, multiset and bag are being used interchangeably.

The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve mset, is called its multiplicity or characteristic value, denoted by $m_A(x)$ or $c_A(x)$ or simply $A(x)$. In the theory of bags, $x \in\in A$ is used in place of $m_A(x)$. a well formed formula $x \in^n y$ semantically reads that x belong n times to y. The number of distinct elements in an mset (which need not be finite) and their multiplicities jointly determine its cardinality, usually denoted by $\#A$ or $C(A)$ or $|A|$. In other words, the cardinality of an mset is the sum of the multiplicities of all its objects.

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A cardinality-bounded mset space X^n can be defined as the set of all msets whose objects are drawn from a ground set X such that no object in any mset $A \in X^n$ occurs more than n times.
The set of distinct elements of an mset is called its root or support. The cardinality of the root set of an mset is called its dimension.
An mset is called regular or constant if all its objects occur with the same multiplicity and the common multiplicity is called its height. For example, $[a, b]_{3,3}$ is a regular mset of height 3.
An mset is called simple if all its elements are the same. For example,$[a]_3$ is a simple mset.it follows that the root set of a simple mset is a singleton.
A unique mset that does not contain any member is called the empty mset, denoted by $\emptyset$.
Two msets A and B are called equal, denoted by $A = B$ if $M_A(x) = M_B(x)$ for all objects x.
An mset A is called a submultiset (submset,for short) or a multisubset (msubset,for short) of an mset B, denoted by $A \subseteq B$, if $M_A(x) \leq M_B(x)$ for all objects x. An mset is called the parent multiset or overmultiset in relation to its submsets.For example, $[a, b]_{1,2}$ is a submset of $[a, b, c]_{1,3,1}$ and the latter is a parent mset of the former.It follows that $A = B$ if and only $A \subseteq B$ and $B \subseteq A$. Also $A \subset B$ if $A \subseteq B$ and $A \neq B$.
A submset of a given mset is called whole if it contains all multiplicities of the common objects. For example,$[a, b]_{2,3}$ is a whole mset of $[a, b, c]_{2,3,4}$.
A submset of a given mset is called full if it contains all objects of the parent mset. For example,$[a, b, c]_{1,2,3}$ is a full mset of $[a, b, c]_{2,3,4}$.
The powermset of an mset A denoted by $\wp(A)$,is the mset of all submsets of A.

3.0 REPRESENTATION OF A MULTISET
The use of square brackets to represent an mset has become almost standard. Thus, an mset containing one occurrence of *a*, two occurrences of *b*, and three occurrences of *c* is notationally written as [[*a*, *b*, *b*, *c*, *c*, *c*]] or [*a*, *b* *b*, *c*, *c*, *c*] or$[a, b, c]_{1,2,3}$ or$[a^1b^2c^3]$ or [*a1*, *b*2, *c*3] or $\left[{}^{1}/_{a}, {}^{2}/_{b}, {}^{3}/_{c}\right]$, depending on one's taste and convenience.
In view of the above, an mset A can be represented by a set of pairs as follows:
$A = \{\langle m_A(x_1), x_1\rangle, \ldots, \langle m_A(x_j), x_j\rangle, \ldots\}$
or $A = \{m_A(x_1), x_1, \ldots, m_A(x_j), x_j, \ldots\}$
or $\boldsymbol{A} = \left\{{}^{\mathrm{n}_1}/_{\boldsymbol{x_1}}, \ldots, {}^{\boldsymbol{n_j}}/_{\boldsymbol{x_j}}, \ldots\right\}$ where $m_A(x_j) = n_j$ is the count or the multiplicity of x_j in A
It may be noted that there are other forms of representing an mset [5] and [6]

4.0 MULTISET RELATIONS
4.1 DEFINITIONS
DEFINITION 4.1.1: Multiset Relation
Let X^n be a cardinality-bounded mset space defined on a set X and A be a member of X^n. In consonance with the terminologies used for explicating the notion of relations in set, we say that any submset R of $A \times A$ is called an mset relation on A (or simply, a relation on A if the context is clear), symbolized as $R: A \to A$,where every member of R has a count $C_1(x, y)$ and $C_2(x, y)$.Also ${}^{m}/_{x}$ is R −related to ${}^{n}/_{y}$ is symbolized as $\left({}^{m}/_{x}, {}^{n}/_{y}\right) \in R$ or ${}^{m}/_{x}\, R\, {}^{n}/_{y}$.
Fomally, $R = \{\left({}^{m}/_{x}, {}^{n}/_{y}\right)/mn: \left({}^{m}/_{x}, {}^{n}/_{y}\right) \in^{mn} R\}$.
Note that $\left({}^{m}/_{x}, {}^{n}/_{y}\right) \in^{mn} R$ actually means that ordered pair (x, y) occurs mn times in R.

DEFINITION 4.1.2: Domain and Range of an mset relation
The domain and range of the mset relation R on A is defined as follows:
$DomR = \{x \in^r A: \exists\, y \in^s A \text{ such that } {}^{r}/_{x}\, R\, {}^{s}/_{y}\}$Where
$C_{DomR}(x) = Sup\{C_1(x, y): x \in^r A\}$, which always exist for cardinality bounded msets.
$RanR = \{y \in^s M: \exists\, x \in^r A \text{ such that } {}^{r}/_{x}\, R\, {}^{s}/_{y}\}$Where $C_{ranR}(x) = Sup\{C_2(x, y): y \in^s A\}$, which always exist for cardinality bounded msets.
Note that, for cardinality-bounded msets, the supremum (sup) is simply the maximum (max).
For example: let $A = [{}^{8}/_{x}, {}^{11}/_{y}, {}^{15}/_{z}]$ be an mset and

$R = \{(^2/_x, ^4/_y)/8, (^5/_x, ^3/_x)/15, (^7/_x, ^{11}/_z)/77, (^8/_y, ^6/_x)/48, (^{11}/_y, ^{13}/_z)/143$
$(^7/_z, ^7/_z)/49, (^{12}/_z, ^{10}/_y)/120, (^{14}/_z, ^5/_x)/70\}$
is an mset relation defined on A then $DomR = \{^7/_x, ^{11}/_y, ^{14}/_z\}$ and
$RanR = \{^6/_x, ^{10}/_y, ^{13}/_z\}$

DEFINITION 4.1.3: Inverse of an mset relation

The inverse of an mset relation R denoted by R^{-1} is defined as
$R^{-1} = \{(^n/_y, ^m/_x)/mn: (^m/_x, ^n/_y) \in^{mn} R\}$
Note that,akin to the existence of the inverse of a relation on sets, the inverse of an mset relation always exists as an mset relation.

Example: $let\ A = [^4/_x, ^7/_y, ^9/_z]$ and
$R = \left\{(^2/_x, ^3/_y)/6, (^3/_x, ^5/_z)/15, (^3/_y, ^4/_z)/12, (^4/_z, ^5/_y)/20, (^3/_z, ^3/_x)/9\right\}$
be an mset relation on$A$ then
$R^{-1} = \left\{(^3/_y, ^2/_x)/6, (^5/_z, ^3/_x)/15, (^4/_z, ^3/_y)/12, (^5/_y, ^4/_z)/20, (^3/_x, ^3/_z)/9\right\}$.
Is also an mset relation on$A$

4.2 OPERATIONS ON MULTISET RELATIONS

Relations on msets are simply msets. That is, subsets of ordered pairs of the Cartesian product of an mset. It therefore make sense to use the usual multiset operations (intersection $\cap$, union $\cup$, sum $\uplus$ and difference $-$ or $\ominus$) to combine relations to create new relations. Hence, if R_1 and R_2 are two mset relations on $A \in X^n$, then $R_1 \cup R_2$, $R_1 \uplus R_2$, $R_1 \cap R_2$, $R_1 - R_2$ the compliment of R_1 or R_2 , $R_1 \Delta R_2$ e.t.c are also mset relations on A. For example, $R_1 \cup R_2$ can be defined as the mset consisting of all distinct pairs of R_1 and R_2 and the common pairs appearing only once or equivalently, appearing with its maximum multiplicity.

For illustration, let R_1 and R_2 be two msets relations defined on $A \in X^n$ given
$A = [^4/_x, ^7/_y, ^9/_z]$
$R_1 = \left\{(^2/_x, ^3/_y)/6, (^3/_x, ^5/_z)/15, (^3/_y, ^4/_z)/12, (^4/_z, ^5/_y)/20, (^3/_z, ^3/_x)/9\right\}$ and $R_2 =$
$\left\{(^2/_x, ^3/_y)/6, (^4/_x, ^4/_z)/16, (^4/_z, ^5/_y)/20, (^6/_z, ^3/_x)/18, (^3/_z, ^4/_z)/12\right\}$ then
$R_1 \cup R_2 = \{(^2/_x, ^3/_y)/6, (^4/_x, ^5/_z)/20, (^5/_y, ^4/_z)/20, (^3/_y, ^4/_y)/12$
$(^4/_z, ^5/_y)/20, (^6/_z, ^3/_x)/18, (^3/_z, ^4/_z)/12\}$ and
$R_1 \cap R_2 = \{(^2/_x, ^3/_y)/6, (^3/_x, ^4/_z)/12, (^3/_z, ^3/_x)/9\}$

4.3 COMPOSITION OF MULTISET RELATIONS

It is a known fact that transitive closures of relations have applications in certain areas such as networks, syntactic analysis, fault detection and diagnosis in switching circuits [7 & 8].which require computing compositions of relations. Below shows some details for constructing a composition of mset relations.

Let A, B, C be three msets and R be an mset relation from A to B and S be an mset relation from B to C. The composition of R and S denoted by $S \circ R$ (or $R \circ S$, depending on the convention one follows) is an mset relation from A to C and is defined as follows:

If $^m/_x$ is in A and $^k/_z$ in C then $^m/_x\,(S \circ R)\,^k/_z$ if and only if there is some $^m/_x\, R\, ^n/_y$ and $^n/_y\, S\, ^k/_z$ such that $C_1(x,z) = \text{Max}\{Min\{C_1(x,y), C_1(y,z)\}\}$ and$C_2(x,z) = \text{Max}\left\{Min\{C_2(x,y), C_2(y,z)\}\right\}$ meaning that
$m = \max\{\min\{m,\text{n}\}\}$ and $k = \max\{\min\{n,\text{k}\}\}$

For example

Let $A = [^4/_x, ^5/_y, ^3/_z]$
$B = \left[^5/_a, ^3/_b, ^4/_c\right]$
$C = [^4/_p, ^2/_r, ^3/_t]$ be given msets

Let R be an mset relation from A to B and S be an mset relation from B to C defined as
$R = \{(^3/_x, ^2/_a)/6, (^2/_x, ^2/_b)/4, (^3/_y, ^3/_c)/9\,(^2/_z, ^1/_b)/2, (^1/_z, ^3/_a)/3\}$
$S = \{(^3/_a, ^1/_r)/3, (^2/_a, ^2/_t)/2, (^2/_b, ^3/_p)/6\,(^2/_c, ^2/_p)/4, 2\}$
Then
$SoR = \{(^2/_x, ^2/_p)/4, (^3/_x, ^1/_r)/3, (^2/_x, ^2/_t)/4\,(^2/_y, ^2/_p)/4, (^1/_y, ^2/_c)/2$
$(^2/_z, ^1/_p)/2, (^1/_z, ^1/_r)/1\,(^1/_z, ^2/_t)/2,\}$

It is easy to verify that the stipulations made above in order that composite relation is defined hold for the example described above. In particular, $Dom(SoR) \subseteq DomR$ and
$Ran(SoR) \subseteq RanS$. on the same line, for a given relation R on an mset $A, RoR, RoRoR, \ldots$ can be composed. In particular, RoR, denoted by R^2, can be defined and, in turn , RoR^2, denoted by R^3, can be defined and so on. This eventually leads to obtaining transitive closure of R as it is done in the case of relation on set viz., $R^+ = R \cup R^2 \cup R^3$ ...
On a final note, the composition of mset relations is not commutative. For example, let $A = [^4/_a, ^3/_b, ^5/_c]$ be a given mset.
Let $R = \{(^3/_a, ^2/_a)/6, (^2/_a, ^3/_c)/6, (^2/_b, ^2/_c)/2, (^3/_c, ^3/_c)/9\}$ and
$S = \{(^2/_a, ^2/_b)/4, (^2/_b, ^3/_a)/4, (^3/_c, ^2/_b)/6\}$ be a relation on A
Then $SoR = (^2/_a, ^2/_b)/4, (^2/_b, ^2/_b)/4, (^3/_c, ^2/_b)/6\}$ and
$RoS = \{(^2/_b, ^3/_c)/6, (^2/_b, ^2/_a)/4, (^2/_c, ^2/_c)/4\}$ proving that $SoR \neq RoS$
It can also be easily proved that $(SoR)^{-1} = R^{-1}oS^{-1}$

4.4 SOME PROPERTIES OF MULTISET RELATIONS
Similar to defining some particular properties of a relation on a set, we can define them on an mset as well [9 & 10].. Let R be an mset relation on A
(i). *Void mset Relation*: as any relation on an mset A is a subset of $A \times A$ and the empty mset $\emptyset$ is a unique submset of $A \times A$, it is called the Void mset relation. However, the way an mset relation is defined in 4.1.2, the emergence of empty mset relation is excluded.
(ii). *Identity mset relation*: the identity on an mset A denoted by I_A, is the mset of all pairs having equal coordinates in $A \times A$. Such a relation keeps every element of A fixed.
(iii). *Universal mset relation*: The universal mset relation on an mset A is simply the multiset $A \times A$ itself. Follows that I_A is the diagonal of $A \times A$.
(iv). *Reflexive mset relation*: R is called reflexive if $(^m/_x, ^m/_x) \in R$ for every $^m/_x$ in A. Note that I_A is contained in any R on A which is reflexive. Follows that if R is reflexive, then $R \cap R^{-1} \neq \emptyset$.
(v).Symmetric mset relation: R is called symmetric if $(^m/_x, ^n/_y) \in R$, then $(^n/_y, ^m/_x) \in R$ for all $^m/_x, ^n/_y \in A$. Follows that R is symmetric if and only if $R \cap R^{-1} \neq \emptyset$
(vi). *Transitive mset relation*: R is called transitive on A if $(^m/_x, ^n/_y) \in R$ and $(^n/_y, ^p/_z) \in R$, then $(^m/_x, ^p/_z) \in R$, for all $^m/_x, ^n/_y, ^p/_z \in A$
(vii). *Equivalence mset relation*: R is called an equivalence relation on A if it is reflexive, symmetric and transitive on A.
For example, let $A = [^4/_x, ^5/_y, ^6/_z]$ and
$R = \{(^4/_x, ^4/_x)/16, (^4/_x, ^6/_z)/24, (^6/_z, ^4/_x)/24, (^5/_y, ^5/_y)/25, (^6/_z, ^6/_z)/36,\}$,
then R is an equivalence relation on A. Blizard (1989) and Girish and Sunil (2009)

4.5 MATRIX REPRESENTATION OF MULTISET RELATIONS AND ITS COMPOSITION
Similar to the representation of a relation on a set by a matrix or graph, an mset relation can also be represented.
Let X^n be a cardinality-bounded mset space generated by the elements of a given generic set X, and let R be an mset relation on $A \in X^n$. If we assume that the elements of X appear in a certain order, then R can be represented by a

matrix whose elements are mn if $\left({}^{m}/_{x}, {}^{n}/_{y}\right) \in R$ and $0s$ if $\left({}^{m}/_{x}, {}^{n}/_{y}\right) \notin R$. Note that a relation R from mset A having p distinct objects to mset B having q distinct objects will have p rows and q columns; mn in row i and column j of the matrix means that the relation R holds for the ith element ${}^{m}/_{x}$ of A and jth element ${}^{n}/_{y}$ of B; "0" means the relation does not hold.

For example let $A = [{}^{4}/_{x}, {}^{5}/_{y}, {}^{6}/_{z}]$ and R an mset relation defined on A by $R = \left\{({}^{4}/_{x}, {}^{4}/_{x})/16, ({}^{4}/_{x}, {}^{6}/_{z})/24, ({}^{6}/_{z}, {}^{4}/_{x})/24, \left({}^{5}/_{y}, {}^{5}/_{y}\right)/25, ({}^{6}/_{z}, {}^{6}/_{z})/36\right\}$ the matrix representation of R is as follows

$$\begin{bmatrix} 16 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & 36 \end{bmatrix}$$

It may be noted that if A contains 3 objects, then the relation matrix is 3×3.It can also be represented in a tabular form as follows:

	x	y	z
x	16	0	24
y	0	25	0
z	24	0	36

As in the case of matrix representation of a relation on a set A it is possible to draw some of the properties of an mset relation from its matrix. For example, if a relation R on an mset A is reflexive, then all the diagonal entries in its matrix must be non zero squared numbers. However, this is only a necessary condition as can be checked by following the definition. On the other hand, if R is irreflexive, then all the diagonal entries must be zero. R may be neither reflexive nor irreflexive. Similarly, if a relation is symmetric, then both $\left({}^{m}/_{x}, {}^{n}/_{y}\right)$ and $\left({}^{n}/_{y}, {}^{m}/_{x}\right)$ must appear. For R being antisymmetric, no such dual pairs appear.However, as in the case of a relation in a set, it is not straightforward to determine from the matrix of an mset relation R, whether it is transitive.It may also be noted that if the ground set X is large and R defined on $A \in X^{n}$ is required to possess many properties, both the graphical and matrix representations of R become unwieldy. However, the matrix representation in contrast to its graph can be obtained on a computer, simply because computers are not good in looking at pictures.

It is straightforward to see that aforesaid matrix representation of an mset relation in an mset A can as well be extended to a composition of two relations on A. For an illustration, let us consider the example discussed at the end of the section 4.3. We have as follows:

$$M_R = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 0 & 6 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$M_{SoR} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 6 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

Similarly

$$M_{RoS} = \begin{bmatrix} 0 & 0 & 6 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} \odot \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that, akin to the technique used for constructing the matrix of a composite set relatrions, Boolean product of the matrices of the two mset relations is computed. For example, the entry in the second row and third column of $M_{R \circ S}$ is given by $\{(6$ and $6)$ or $(4$ or $4)$ or $(0$ and $9)\} = \{6$ or 4 or $0\} = 6$.

Note that the motivation behind representing the occurrence of $\left({}^{m}/_{x}, {}^{n}/_{y}\right)$ by the number mn in the aforesaid matrix representation is just to indicate that an mset relation are in consideration ,otherwise the same $0 - 1$ matrix used for representing relations in sets does work.

REFERENCE

[1] W. D. Blizard (1989), Multiset Theory, Notre Dame J. Formal Logic, 30, 36-66

[2] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh (2008), A Systemization of Fundamentals of Multisets, Lecturas Mathematics, 29, 33-48

[3] W. D. Blizard (1991), The Development of Multiset Theory, Modern Logic, 1, 319-352.

[4] D. E. Knuth (1981), The Art of Computer Programming, Vol. 2 (Seminumerical Algorithms), Second Edition, Addison-Wesley.

[5] A. Syropoulos (2001), Mathematics of Multisets, In Multiset Processings (Claud et al eds), LNCS 2235, Springer, 347-358

[6] K. P. Girish, S. J. John (2009), Relations and functions in Multiset context, Information Sciences, 179, 758-768

[7] N. J. Wildberger (2003). A New Look at Multiset. School of Mathematics, UNSW Sydney 2052, Australia.

[8] R. R. Yager (1986), On the Theory of Bags, International Journal Of General System 13, 23-37.

[9] D. Singh and J. N. Singh (2008), A Note on the Definition of Multisubset, Bulletin of the Calcutta Mathematical Society, 31, 19-20

[10] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh (2007), An Overview of The Application of Multisets, Novi Sad Journal Of Mathematics, No. 2, 37:73-92.