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ON WEIBULL-ODD FRÈCHET G FAMILY OF DISTRIBUTIONS: STATISTICAL PROPERTIES AND APPLICATIONS

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Abstract

We proposed a new family of distributions called the Weibull- Odd Frèchet G family of distributions and we derived some of its structural properties. One family of this distribution called Weibull- Odd Frèchet Inverse Exponential distribution is used to fit two data sets using the MLE procedure. A Monte Carlo simulation is used to test the robustness of the parameters of this distribution, in terms of the bias and mean squared error. The results of fitting this new distribution to two different data sets suggest that the new distribution outperforms its competitors.

Keywords: Weibull- odd Frèchet Gfamily.Monte Carlo simulation. Weibull- Odd Frèchet Inverse Exponential distribution.

1. Introduction

Statistical analysis depends heavily on the statistical distribution to address any problem under study. The more different class of distributions available to the researcher, the easier for him to choose an appropriate model for modeling various data of different shape. Survey reveals the existing of abundance statistical distributions in the literature. However, there is still several important dataset that do not follow any existing statistical model; therefore, there is still need to propose several distributions to remedy this problem.

Statistical distributions have received reasonable attention by those working in both theory and application; everyday several models are proposed in order to make it more tractable and flexible. The tractability of a probability distribution makes it easier for the researcher; especially when it comes to simulation of random samples, but the flexibility of probability distributions is of interest because more flexible models give more information than the less flexible models. It is advisable to use the probability distributions that best fit the dataset than to alter the already existing distribution as this may affect the originality of the dataset. As a result of this, several efforts have been made in recent years to ensure that the existing standard distributions are modified; this includes Transmuted Rayleigh distribution [1].

Exponential distribution was first generalized [2] and named it as Exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Some other generators are beta-G [3], Kumaraswamy family [4],Exponentiated generalized class of distributions [5], new technique for Generating Families of probability distribution function as a generator [6], the Lomax Generator of distributions [7],beta Marshall-Olkin family of distributions [8], Kumaraswamy Marshall-Olkin family of distributions [9], Kumaraswamy transmuted-G family of distributions [10], A New Generalized Weibull – Odd Frechet Family of Distributions: Statistical Properties and Applications [11], A New Weibull – Odd Frechet Family of Distributions[12] and many more.

1.1 Weibull Distribution

Weibull distribution [13] is one of the most widely used lifetime distribution and has been identified as a life testing model in reliability and engineering. It is a distribution that can take a form of other types of distributions, based on the value of the shape parameter β. For example, the Weibull distribution reduces to exponential distribution when the shape parameter $\beta=1$. In reliability analysis, the Weibull distribution can be used to find the percentage of items that are expected to fail during the burn-in period.

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A random variable X is said to follow a Weibull distribution, if its CDF and pdf are respectively given by Equation (1) and (2)

$$F(x;\alpha,\beta) = 1 - exp\left[-\alpha x^{\beta}\right], x \geq 0, \alpha > 0, \beta > 0 \tag{1}$$

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1} exp\left[-\alpha x^{\beta}\right], x \geq 0, \alpha > 0, \beta > 0 \tag{2}$$

Where α and β are scale and shape parameters respectively.

1.2 Frèchet Distribution

French mathematician introduced Frèchet distribution [14]. This Frèchet distribution has been described as a distribution for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds. Applications of the Frèchet distribution in various fields given in [15] showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Frèchet distribution can also be called the Inverse Weibull distribution

Several researchers studied Frèchet distribution for a different purpose, some of these are: -estimation the parameter of Frèchet distribution [16], sociological models based on Frèchet random variables [17],applied Frèchet distribution for analyzing the wind speed [18], derivation of best linear unbiased estimators of location and scale parameters of the Frèchet distribution and used MLE to estimate the parameters of the Frèchet distribution [19], Further, studied different estimation methods for Frèchet distribution with known shape [20].

A random variable X is said to follow a Frèchet distribution with one parameter, if its cumulative distribution function (cdf) and probability density function (pdf) are respectively given by equation (3) and (4)

$$F(x;\lambda) = exp\left[-\left(\frac{1}{x}\right)^{\lambda}\right], \qquad x \geq 0, \lambda > 0 \tag{3}$$

and

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}} exp\left[-\left(\frac{1}{x}\right)^{\lambda}\right]; \qquad x \geq 0, \lambda > 0 \tag{4}$$

Where λ is the shape parameter.

This article is structured as follows. Section 2, presents the proposed Weibull odd Frèchet- G (WoFr-G) family of distributions and one of its sub-model. The structural properties of the proposed family including some useful expansions are presented in section 3, section 4 contains some parameter estimation technique and some simulation study in order to assess the robustness of the estimated parameter. Data analysis in the form of fitting the new Weibull odd Frèchet –Inverse Exponential distribution and its competitors to the strength of carbon fibres and the survival time of leukaemiapatients is presented in section 5 while section 6 concludes the paper.

2.0 The Weibull-Odd Frèchet G (WoFr-G) family

This section introduced the new proposed family of distributions.

2.1 Weibull G Family of Distributions

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative functions of the baseline model with parameter vector ξ and consider the Weibull distribution cdf $F(t) = 1 - \exp(-\alpha t^{\beta})$ (for $t > 0$) with positive parameters α and β . Then the cdf of the Weibull-G Family of Distributions [21] is defined by replacing the argument of t by $G(x;\xi)/\bar{G}(x;\xi)$ as follows

$$F(x;\alpha,\beta,\xi) = \alpha\beta\int_0^{\frac{G(x;\xi)}{\bar{G}(x;\xi)}} t^{\beta-1}\exp(-\alpha t^{\beta})dt = 1 - exp\left[-\alpha\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^{\beta}\right]; \quad \alpha,\beta > 0 \tag{5}$$

The corresponding pdf is given by

$$f(x;\alpha,\beta,\xi) = \alpha\beta g(x;\xi)\frac{G(x;\xi)^{\beta-1}}{\bar{G}(x;\xi)^{\beta+1}} exp\left[-\alpha\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)^{\beta}\right]; \quad \alpha,\beta > 0 \tag{6}$$

2.2 The Odd Frèchet-G Family of Probability Distributions

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative distribution functions of the baseline model with parameter vector ξ and consider the Frèchet cdf $F(t,\lambda) = exp\left[-\left(\frac{1}{t}\right)^{\lambda}\right]$ (for $t \geq 0$) with positive parameter $\lambda > 0$, then the cdf of the Odd

Frèchet-G family [22] is defined by replacing the argument of t by $G(x;\xi)/\bar{G}(x;\xi)$, where $\bar{G}(x;\xi) = 1 - G(x;\xi)$ as follow

$$F_{OFG}(x;\lambda,\xi) = \int_0^{\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)} \frac{1}{t^{\lambda+1}} exp\left[-\left(\frac{1}{t}\right)^{\lambda}\right]dx = \exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]; \qquad \lambda > 0 \tag{7}$$

The corresponding pdf is given by

$$f_{OFG}(x;\lambda,\xi) = \frac{\lambda g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{\left[G(x;\xi)\right]^{\lambda+1}} \exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]; \qquad \lambda > 0 \tag{8}$$

2.3 Weibull-odd Frèchet G (WoFr-G) Family of Probability Distributions

We defined the proposed family (Weibull- odd Frèchet-G Family of Probability Distributions) via the cdf and pdf

$$F(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\int_0^{H(x;\lambda,\xi)} t^{\beta-1}\exp(-\alpha t^{\beta})dt$$

$$F_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\int_0^{H(x;\lambda,\xi)} [H(t;\lambda,\xi)]^{\beta-1}\exp(-\alpha[H(t;\lambda,\xi)]^{\beta})dH(t;\lambda,\xi) = 1-\exp\left\{-\alpha[H(x;\lambda,\xi)]^{\beta}\right\} \tag{9}$$

The corresponding pdf is given by:

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\, f_{OFG}(x;\lambda,\xi)\frac{[F_{OFG}(x;\lambda,\xi)]^{\beta-1}}{[1-F_{OFG}(x;\lambda,\xi)]^{\beta+1}}\exp\left\{-\alpha[H(x;\lambda,\xi)]^{\beta}\right\} \tag{10}$$

Where

$$H(t;\lambda,\xi) = \frac{[F_{OFG}(t;\lambda,\xi)]}{[1-F_{OFG}(t;\lambda,\xi)]}$$

After substitution, we have the cdf and corresponding pdf as:

$$F_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = 1-\exp\left[-\alpha\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta}\right] \tag{11}$$

and

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \frac{\alpha\beta\lambda g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{G(x;\xi)^{\lambda+1}}\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta-1}$$
$$\exp\left[-\alpha\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta}\right] \tag{12}$$

2.3.1Survival Function

The survival function (R(x)) is the probability that a patient, device or any objects of interest will survive beyond a specified time, the survival function is also known as the reliability function. The survival function of Weibull-Frèchet-G Family of Probability Distributions is given by equation (13).

$$R(x) = 1-F(x;\alpha,\beta,\lambda,\xi) = \exp\left[-\alpha\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta}\right] \tag{13}$$

2.3.2Hazard Function

Hazard rate, H(x) refers to the rate of death for an item of a given age, and it's also known as the failure rate. It analyses the likelihood that something will survive to an earlier time t. In other words, it is the likelihood that if something survives to one moment; it will also survive to the next. Hazard rate cannot be negative and only applies to those items which cannot be repaired. The hazard function of Weibull-Frèchet-G Family of Probability Distributions is given in equation (14).

$$H(x) = \frac{f(x;\alpha,\beta,\lambda,\xi)}{1-F(x;\alpha,\beta,\lambda,\xi)} = \frac{\alpha\beta\lambda g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{G(x;\xi)^{\lambda+1}}\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta-1} \tag{14}$$

2.3.3 Quantile function

Quantile function also called inverse cumulative distribution function and is associated with probability distribution of a random variable used for simulation study and it's given by:

$$Q(u) = G^{-1}\left[1+\left(\ln\left(1+\left[-\frac{1}{\alpha}\ln(1-u)\right]^{-\frac{1}{\beta}}\right)\right)^{\frac{1}{\lambda}}\right]^{-1} \tag{15}$$

Where $G^{-1}(.)$ is the quantile function of the baseline distribution and u is uniformly distributed on (0,1).

In section (2.3.4), we provide sub-model of the Weibull-Frèchet-G Family of Probability Distributions. The probability density function given in equation (11) will be most useful when the cdf $G(x;\xi)$ and the pdf $g(x;\xi)$ have simple analytic expressions.

2.3.4 NewWeibull- Odd Frèchet–Inverse Exponential (WoFr-IE) Distribution

Suppose that the parent distribution is Inverse-Exponential with cdf and pdf respectively given as $G(x;\theta)=exp\left[-\left(\frac{\theta}{x}\right)\right]$ and $g(x;\theta)=\frac{\theta}{x^2}exp\left[-\left(\frac{\theta}{x}\right)\right]$ for (x> 0), then the Weibull Odd Frèchet- Inverse Exponential distribution has the cdf and corresponding pdf respectively given by equations (16) and (17)

$$F(x;\alpha,\beta,\theta,\lambda)=1-\exp\left[-\alpha\left(\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]-1\right)^{-\beta}\right] \qquad (16)$$

And

$$f(x;\alpha,\beta,\theta,\lambda)=\frac{\alpha\,\beta\,\theta\,\lambda\left(1-exp\left[-\left(\frac{\theta}{x}\right)\right]\right)^{\lambda-1}}{x^2\left(exp\left[-\left(\frac{\theta}{x}\right)\right]\right)^{\lambda}}\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]\left(\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]-1\right)^{-(\beta+1)}$$

$$\times\exp\left[-\alpha\left(\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]-1\right)^{-\beta}\right] \qquad (17)$$

The survival R(x) and hazard h(x) function of a Weibull Frèchet-Inverse Exponential distribution are given in equation (18) and (19) respectively,

$$R(x)=1-F(x;\alpha,\beta,\theta,\lambda)=\exp\left[-\alpha\left(\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]-1\right)^{-\beta}\right] \qquad (18)$$

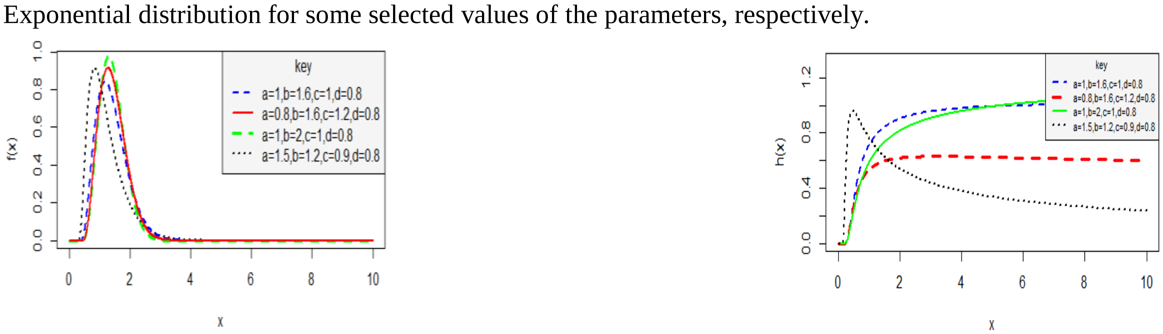
$$\left(h(x)\right)=\frac{\alpha\,\beta\,\theta\,\lambda\left(1-exp\left[-\left(\frac{\theta}{x}\right)\right]\right)^{\lambda-1}}{x^2\left(exp\left[-\left(\frac{\theta}{x}\right)\right]\right)^{\lambda}}\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]\left(\exp\left[\left(exp\left[\left(\frac{\theta}{x}\right)\right]-1\right)^{\lambda}\right]-1\right)^{-(\beta+1)} \qquad (19)$$

The Quantile function of the Weibull Frèchet-Inverse Exponential distribution is given by:

$$Q(u)=\theta\left[-\log\left(\left[1+\left(\log\left(1+\left[-\frac{1}{\alpha}\log(1-u)\right]^{\frac{1}{\beta}}\right)\right)^{\frac{1}{\lambda}}\right]^{-1}\right)\right]^{-1}$$

Where u is a uniformly distributed on the interval (0,1)

Figure 1, Shows some possible shapes of probability density function and hazard function of the Weibull Frèchet-Inverse Exponential distribution for some selected values of the parameters, respectively.



(a) *(b)*

***Figure 1** Plots (a)Pdf of WoFr-IED and (b) hazard function of WoFr-IED*

3. Some Structural Properties of the new family of distribution

This section provides some Statistical Properties of the new family of distributions.

3.1 Useful Expansions of the Proposed Family of Distributions

In this section, we provide a very useful expansion for the proposed Weibull-Frèchet-G Family of Probability Distributions density function.

$$f(x;\alpha,\beta,\lambda,\xi)=\frac{\alpha\beta\lambda g(x;\xi)(1-G(x;\xi))^{\lambda-1}}{G(x;\xi)^{\lambda+1}}\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta-1}$$
$$\exp\left[-\alpha\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta}\right] \quad (20)$$

Consider the power series to the term

$$\exp\left[-\alpha\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-\beta}\right]=\sum_{i=0}^{\infty}\frac{\alpha^{i}(-1)^{i}}{i!}\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-i\beta} \quad (21)$$

Then we

$$f(x;\alpha,\beta,\lambda,\xi)=\sum_{i=0}^{\infty}\frac{\alpha^{i+1}(-1)^{i}}{i!}\frac{\beta\lambda g(x;\xi)(1-G(x;\xi))^{\lambda-1}}{G(x;\xi)^{\lambda+1}}\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-(\beta(i+1)+1)} \quad (22)$$

But

$$\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-(\beta(i+1)+1)}=\frac{\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{-(\beta(i+1)+1)}}{\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{-(\beta(i+1)+1)}} \quad (23)$$

The denominator in equation (23) can be express using the Binomial expansion as:

$$\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{(\beta(i+1)+1)}=\sum_{j=0}^{\infty}(-1)^{j}\binom{(\beta(i+1)+1)}{j}\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{j}$$

Now equation (17) becomes

$$\left(\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right)^{-(\beta(i+1)+1)}=\sum_{j=0}^{\infty}(-1)^{j}\binom{(\beta(i+1)+1)}{j}\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{-(\beta(i+1)+1+j)} \quad (24)$$

Substituting equation (18) in to equation (16), we have

$$f(x;\alpha,\beta,\lambda,\xi)=\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{(\beta(i+1)+1)}{j}\frac{\alpha^{i+1}(-1)^{i+j}}{i!}\frac{g(x;\xi)(1-G(x;\xi))^{\lambda-1}}{G(x;\xi)^{\lambda+1}}\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{-(\beta(i+1)+2-j)}$$

Using power series expansion to the term:

$$\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{-(\beta(i+1)+2-j)}=\sum_{k=0}^{\infty}\frac{\Gamma((\beta(i+1)+2-j)+k)}{k!\Gamma((\beta(i+1)+2-j))}\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda k}$$

Then we have:

$$f(x;\alpha,\beta,\lambda,\xi)=\beta\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma((\beta(i+1)+2-j)+k)}{k!\Gamma((\beta(i+1)+2-j))}\binom{(\beta(i+1)+1)}{j}\frac{\alpha^{i+1}(-1)^{i+j}}{i!}\frac{g(x;\xi)(1-G(x;\xi))^{\lambda(k+1)-1}}{G(x;\xi)^{\lambda(k+1)+1}} \quad (25)$$

But

$$(1-G(x;\xi))^{\lambda(k+1)-1}=\sum_{l=0}^{\infty}(-1)^{l}\binom{\lambda(k+1)-1}{l}G(x;\xi)^{l} \quad (26)$$

Substituting equation (26) in to equation (25), we have

$$f(x;\alpha,\beta,\lambda,\xi)=\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma((\beta(i+1)+2-j)+k)}{k!\Gamma((\beta(i+1)+2-j))}\binom{(\beta(i+1)+1)}{j}\binom{\lambda(k+1)-1}{l}\frac{\alpha^{i+1}(-1)^{i+j+l}}{i!}g(x;\xi)G(x;\xi)^{l-(\lambda(k+1)+1)}$$

This can be rewrite as

$$f(x;\alpha,\beta,\lambda,\xi)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}h_{(l-(\lambda(k+1)+2))}(x) \quad (27)$$

Where

$$h_{(l-(\lambda(k+1)+2))}(x)=(l-(\lambda(k+1)+2))g(x;\xi)G(x;\xi)^{l-(\lambda(k+1)+1)}$$

And

$$\Omega_{k,l}=\beta\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma((\beta(i+1)+2-j)+k)}{k!(l-(\lambda(k+1)+2))\Gamma((\beta(i+1)+2-j))}\binom{\beta(i+1)+1}{j}\binom{\lambda(k+1)-1}{l}\frac{\alpha^{i+1}(-1)^{i+j+l}}{i!} \quad (28)$$

And $G(x;\xi)$ is a baseline cdf, which depends on a parameter vector ξ .

3.2 Mathematical Properties of the Proposed Family

In this section, we provide some structural properties of the Weibull Odd Frèchet Generalized (WoFrG) family of distributions.

3.2.1Moments

The r^{th} moments of the proposed family can be obtained from equation (27) as

$$E(X^r)=\sum_{k=o}^{\infty}\sum_{l=o}^{\infty}\Omega_{k,l}E(Z_{k,l}^r) \quad (29)$$

Where $Z_{k,l}$ denotes the exponential-G distribution with power parameter $(l-(\theta(k+1)+1))$. Since the inner quantities in pdf (21) are absolutely integrable, the incomplete moments of *X* can be written as

$$I_X(y)=\int_{-\infty}^{y}x^r f(x)dx=\sum_{k=o}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}I_{k,l}(y) \quad (30)$$

Where

$$I_{k,l}(y)=\int_{-\infty}^{y}x^r h_{(l-(\theta(k+1)+2))}(x;\xi)dx$$

3.2.2Moment Generating Function

The moments generating function of the proposed family is defined as;

$$M_x(t)=\sum_{k=o}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}E(\mathrm{e}^{tz_{k,l}}) \quad (31)$$

Where

$$E(\mathrm{e}^{tz_{k,l}})=\int_{0}^{\infty}e^{tz_{k,l}}h_{(l-(\theta(k+1)+2))}(z;\xi)dz$$

3.2.3 Entropies

The Renyi entropy of a random variable *X* represents a measure of variation of the uncertainty. The Renyi entropy is defined by:

$$I_\theta(x)=\frac{1}{1-\theta}\log\int_{-\infty}^{\infty}f(x)^\theta dx,\theta>0 \text{ and } \theta\neq 1$$

Using the pdf of our proposed class of distributions (27), we can write

$$f(x)^\theta=\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}h_{(l-(\theta(k+1)+2))}(x)\right)^\theta$$

Then, the Renyi entropy of a random variable *X* having the Weibull-Frèchet Generalized family of distributions is given by

$$I_\theta(x)=\frac{1}{1-\theta}\log\left\{\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}\right)^\theta\int_{-\infty}^{\infty}\left(h_{(l-(\theta(k+1)+2))}(x)\right)^\theta dx\right\};\theta>0 \text{ and } \theta\neq 1 \quad (32)$$

The q-entropy, say $H_q(x)$, is given by

$$H_q(x)=\frac{1}{q-1}\log\left\{1-\int_{-\infty}^{\infty}f(x)^q dx\right\};q>0 \text{ and } q\neq 1$$

By using the pdf of our proposed class of distributions, we can write

$$f(x)^q=\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}h_{(l-(\theta(k+1)+2))}(x)\right)^q$$

Then the q- entropy of a random variable *X* having the Weibull-Frèchet Generalized family of distributions is given by

$$H_q(x) = \frac{1}{q-1}\log\left\{1-\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Omega_{k,l}\right)^q \int_{-\infty}^{\infty}\left(h_{(l-(\theta(k+1)+2))}(x)\right)^q dx\right\}; q>0 \text{ and } q \neq 1 \quad (33)$$

3.2.4 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from our proposed class of distributions and $X_{1:n} \le X_{2:n} \le ...X_{n:n}$ denote the corresponding order statistics. $f_{i:n}(x)$ denote, the pdf of the ith order statistics $X_{i:n}(x)$,then the order statistics for our proposed class of distribution is obtained as :

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x;\theta,\lambda,\xi)F(x;\theta,\lambda,\xi)^{i-1}\left[1-F(x;\theta,\lambda,\xi)\right]^{n-i}$$

But $\left[1-F(x;\theta,\lambda,\xi)\right]^{n-i} = \sum_{j=0}^{i-1}(-1)^j\binom{n-i}{j}F(x;\theta,\lambda,\xi)^j$

Then we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{i-1}(-1)^j\binom{n-i}{j}f(x;\theta,\lambda,\xi)F(x;\theta,\lambda,\xi)^{i+j-1} \quad (34)$$

After substitution and applying useful expansion, we have the order statistics given in equation (35)

$$f_{i:n}(x) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Psi_{m,r}h_{(r-\theta(m+1)+2)}(x) \quad (35)$$

Where

$$h_{(r-\theta(m+1)+2)}(x) = (r-\theta(m+1)+2)g(x;\xi)G(x;\xi)^{(r-\theta(m+1)+1)}$$

And

$$\Psi_{m,r} = \varphi\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{i+j-1}{k}\binom{\theta(m+1)-1}{r}\frac{(-1)^{k+j+r}\alpha^l\Gamma(\beta(1+l)-1+n)(i+j+k-1)^l\Gamma(\beta(1+l)+n+m)}{l!m!n!\Gamma(\beta(1+l)-1)\Gamma(\beta(1+l)+n)}$$

4. Parameter Estimation

4.1 maximum likelihood method.

The maximum likelihood method is the most common method employed in most of the research for parameter estimate, so we consider the estimation of the unknown parameters of our proposed class of distributions from complete samples only by maximum likelihood estimates. Let $x_1, x_2, ..., x_n$ be the observed values from the Weibull Odd Frèchet Generalized family of distributions with parameters α, β, λ and ξ . Let $\eta = (\alpha,\beta,\lambda,\xi)^T$ be the $P\times 1$ parameter vector. Then the log-likelihood function of η is given by

$$l(\eta) = n\log\alpha + n\log\beta + n\log\lambda + \sum_{i=1}^{n}\log g(x_i;\xi) - (\lambda+1)\sum_{i=1}^{n}\log G(x_i;\xi) + (\lambda-1)\sum_{i=1}^{n}\log\bar{G}(x_i;\xi) + \sum_{i=1}^{n}(H_i(x_i;\xi))^{\lambda}$$
$$-(\beta+1)\sum_{i=1}^{n}\left(\left[\left(H_i(x_i;\xi)\right)^{\lambda}\right]-1\right) - \alpha\sum_{i=0}^{n}\left(\exp\left[\left(H_i(x_i;\xi)\right)^{\lambda}\right]-1\right)^{-\beta}$$

Where

$H_i(x;\xi) = \frac{\bar{G}(x;\xi)}{G(x;\xi)}$. The components of the score function

$U(\eta) = \frac{\partial l}{\partial\eta} = \left(\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial\beta},\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial\xi}\right)^T = \left(U_\alpha, U_\beta, U_\lambda, U_{\xi_r}\right)^T$, are given by

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=0}^{n}\left(\exp s_i\right)^{-\beta}$$

$$U_\beta = \frac{n}{\beta} - \sum_{i=1}^{n}s_i + \alpha\beta\sum_{i=0}^{n}\left(z_i\right)^{-\beta-1}$$

$$U_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n}\log G(x_i;\xi) - \sum_{i=1}^{n}\log\bar{G}(x_i;\xi) + \lambda\sum_{i=1}^{n}(H_i(x_i;\xi))^{\lambda-1} - (\beta+1)\sum_{i=1}^{n}\left(s_i^{(\lambda)}\right) - \alpha\sum_{i=0}^{n}\left(z_i^{-\beta}\right)^{(\lambda)}$$

and

$$U_{\xi_r} = \sum_{i=0}^{n} \frac{\partial g(x;\xi)/\partial \xi_k}{g(x;\xi)} - (\lambda+1)\sum_{i=0}^{n} \frac{\partial G(x;\xi)/\partial \xi_k}{G(x;\xi)} + (\lambda-1)\sum_{i=0}^{n} \frac{\partial \overline{G}(x;\xi)/\partial \xi_k}{\overline{G}(x;\xi)} - \sum_{i=0}^{n} \partial H(x;\xi)^{\lambda}/\partial \xi_k$$

$$-(\beta+1)\sum_{i=0}^{n} \partial(s_i)/\partial \xi_k - \alpha \sum_{i=0}^{n} \partial z_i^{-\beta}/\partial \xi_k$$

Where

$$s_i = \left(\left[(H_i(x_i;\xi))^{\lambda}\right] - 1\right)$$

And

$$z_i = \left(\exp\left[(H_i(x_i;\xi))^{\lambda}\right] - 1\right)$$

Equating $U_{\alpha}, U_{\beta}, U_{\lambda}, U_{\xi_k}$ (for k=1…, p) to zero and solving the equations simultaneously yields the MLE $\hat{\eta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\Phi}_k)^T$ of $\eta = (\alpha, \beta, \lambda, \Phi_k)^T$ .To solve these nonlinear equations, it is better to use statistical software and solve numerically using the nonlinear optimization methods such as quasi-Newton algorithm.

4.2Simulation Study

To assess the performance of WoFr-Inv-Exp. distribution, the simulation study was conducted using Monte Carlo Simulation on the basis of parameter estimate $\hat{\eta}$, bias($\hat{\eta}$), variance($\hat{\eta}$) and mean square error($\hat{\eta}$) from maximum likelihood estimation method. The Simulation is performed using random samples of sizes n= 50, 100, 150 drawn from the WoFr- Inverse exponential distribution. Each sample is replicated 1000 times. Each sample is drawn from the WoFr-Inverse exponential distribution with parameter values $(\alpha, \beta, \lambda, \theta) = (0.1000,\ 0.5000,\ 0.3100,\ 0.0125)$

Table 1: Monte Carlo Simulation

Sample sizes (n)	Parameters	Estimates	Bias	Variance	MSE
50	$\hat{\alpha}$	0.0718852	-0.0281149	0.0016526	0.0024430
	$\hat{\beta}$	0.5485885	0.0485885	0.0127350	0.0150959
	$\hat{\lambda}$	0.3651294	0.0551294	0.0131844	0.0162237
	$\hat{\theta}$	0.0059385	-0.0065615	0.0025523	0.0025953
100	$\hat{\alpha}$	0.0718901	-0.0281199	0.0014618	0.0022520
	$\hat{\beta}$	0.5354652	0.0354652	0.0039707	0.0052285
	$\hat{\lambda}$	0.3480368	0.0380368	0.0051606	0.0066074
	$\hat{\theta}$	0.0007905	-0.0117095	2.9224e-07	0.0001374
150	$\hat{\alpha}$	0.0763255	-0.0236746	0.0013389	0.0018994
	$\hat{\beta}$	0.5287699	0.0287699	0.0033482	0.0041759
	$\hat{\lambda}$	0.3401882	0.0301882	0.0042789	0.0051902
	$\hat{\theta}$	0.0008603	-0.0116397	1.4868e-07	0.0001358

Table 1 represents the results obtained from the Monte Carlo Simulation study. These results show that the bias, variance and mean square error decreases toward zero with an increase in sample size.

5. Applications

In this section, we demonstrate the significance of Weibull Odd Frèchet –Inverse Exponential Distribution using two real dataset. The ML estimates, as well as goodness-of-fit measures, are computed and compared with other competing models:

Data 1: Relates to the strength of carbon fibers tested under tension at gauge lengths of 10 mm.

The first data set represents the strength of carbon fibres tested under tension at gauge lengths of 10 mm. The observations are as follows: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

For these data, we fit and compared the performances of Weibull Odd Frechet- Inverse Exponential (WoFr-IED) with the other competitive models i.e. Weibull-inverse Exponential (W-IED), Frechet-inverse Exponential(Fr-IED) and Burr III-inverse Exponential (BURRIII-IED)distributions by finding the value of Akaike Information Criterion(AIC), Bayesian information criterion(BIC) and Hannan-Quinn information criterion (HQIC) statistic as a goodness of fit measures computed using the Adequacy model package in R.

Table 2: parameters Estimates and Goodness of fit measures for first data set

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	**AIC**	**BIC**	**HQIC**
WoFr-IED	0.624969	1.635836	1.880452	1.790351	122.8915	131.4641	126.2632
W-IED	0.693926	1.967006	-	1.884405	164.2749	170.7043	166.8036
Fr-IED	-	-	1.920383	1.855323	165.1739	169.4602	166.8597
BIII-IED	1.664407	1.664305	1.933496	1.064385	195.8606	204.4331	199.2322

Table 2 provides the parameters estimate and goodness of fit measures for the Weibull- odd Frèchet- Inverse Exponential distribution (WFr-IED)and other competitor distributions.

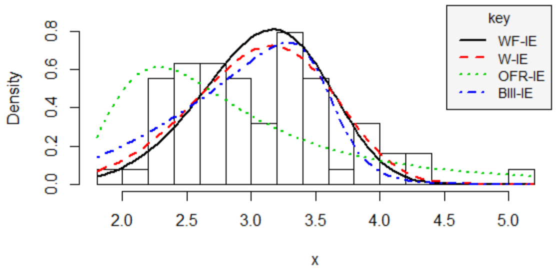


Figure 2*: Strength of carbon fibres (histogram) and the fitted distributions.*

From Table 2 and Figure 2 it is clear that the new Weibull- odd Frèchet- Inverse Exponential distribution (WoFr-IED) performs better than itS competitors.

Data 2: Data set represents the survival times, in weeks of 33 patients.

The second data set represents the survival times, in weeks of 33 patients suffering from acutemyelogeneous leukaemia.

65,156,100,134,16,108,121,4,39,143,56,26,22,1,1,5,65,56,65,17,7,16,22,3,4,2,3,8,4,3,30,4,43

For the second data, we adopted the same competitive distributions as well as criterion for evaluating the performances of Weibull Frèchet- Inverse Exponential (WFr-IED).

Table 3: parameters Estimates and Goodness of fit measures for second data set

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	**AIC**	**BIC**	**HQIC**
WoFr-IED	0.171395	1.098713	0.568999	1.834774	312.0375	317.0235	314.0516
W-IED	0.175445	0.571708	-	1.565443	313.3527	317.8422	314.8633
Fr-IED	-	-	1.885914	0.110329	323.3333	326.3263	324.3404
BIII-IED	1.756828	1.793026	0.648492	1.833535	320.6559	326.6420	322.6701

Table 3 provides the parameters estimate and goodness of fit measures for the Weibull- odd Frèchet- Inverse Exponential distribution (WoFr-IED) and other competitor distributions.

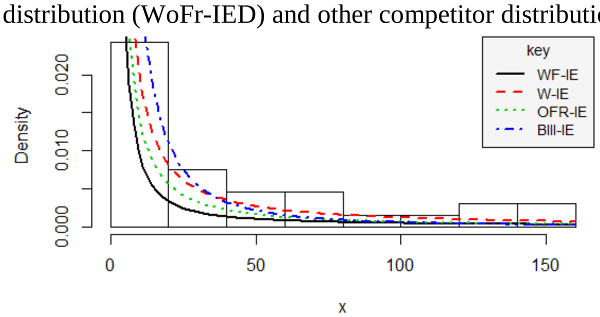


Figure 3*: Survival times of patients and the fitted distributions*

From Table 3 and Figure 3 it is clear that the new Weibull- odd Frèchet- Inverse Exponential distribution (WoFr-IED) performs better than the other distributions.

6. Conclusion

Generalizing a standard distribution provides more flexibility in modeling real data. We proposed a new family called Weibull- Odd Frèchet G family (WoFr-G) family of distribution in order to provide great flexibility to any continuous

distribution by adding three extra parameters. We provide some structural properties of the new family including shapes, moments and incomplete moments, moment generating functions, order statistic and Renyi entropies. The model parameters are estimated by maximum likelihood. The special model of the proposed family was fitted to two real dataset to illustrate flexibility of the proposed family. The result indicates that the special model provides better fit than the other competitive distributions for describing this data based on Akaike information criterion (AIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC).

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