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NUMERICAL SOLUTION OF A TUBERCULOSIS MATHEMATICAL MODEL BY HOMOTOPY ANALYSIS METHOD

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Abstract

In this paper, the homotopy analysis method (HAM) is described and applied to obtain approximate numerical solution to a system of ordinary differential equations for the transmission dynamics of tuberculosis (TB) at the population level. The method yield series solutions that are reasonable and easy to express.

Keywords**:** mathematical model, tuberculosis, series solution, dynamics, homotopy analysis method

1. Introduction

In proffering analytic approximations to nonlinnar problems, perturbation methods are widely applied [1, 2]. These methods use one or more small parameters to convert a nonlinear problem into an infinite number of auxiliary linear sub-problem. Generally, perturbation methods have some restrictions [3, 4]. First, many non- linear problems do not have such small parameter. Second, the validity of perturbation methods is too strongly dependent upon the value of the small parameter. In addition, these methods provides no freedom in choosing initial approximations, base functions and types of governing equations needed to approximate the related auxiliary sub-problems. Finally, perturbation approximations are valid only for nonlinear problems with weak nonlinearity.

Owing to these restrictions, it is therefore worthwhile to develop a new analytic method which can provide us with the freedom and flexibility to apply it and whose approximations is independent of weather the considered nonlinear problems contain small parameter or not. The homotopy analysis method (HAM), which was proposed by Liao [5], is a nonperturbation method and is based on homotopy, which is an important part of topology [6]. It is a general analytic technique to get approximate series solution of different types of nonlinear equations.

The HAM has the advantage that its validity does not depend upon whether or not the nonlinear problem under consideration contain small parameters [7]. Hence, it is valid for most of the nonlinear problem especially those with very strong nonlinearity [8]. Secondly, unlike all previous analytic methods, the HAM provides us with great freedom to express corresponding approximate solution of a given nonlinear problem by means of different base functions [9]. Equal importantly, homotopy analysis method gives great freedom to choose the initial approximation, auxiliary linear operator, auxiliary function and auxiliary parameter. These freedom makes it easier to ensure that the homotopy-series solution is convergent [4]. Moreover, as proved by Liao [8], the three previous nonperturbation methods such as Adomian decomposition method [10], Lyapunov artificial small parameter method [11] and δ -expansion method [12] are special cases of the HAM. It logically contains the three methods and is more general than them. Besides, the homotopy perturbation method (HPM) developed by He [13] is also a special case of the HAM as pointed out by Sajid and Hayat [13] and other researchers. The homotopy analysis method has been widely applied to solve nonlinear problems arising from nonlinear oscillations [15], boundary layer flows [16], heat transfer [17], viscous flow in a porous medium [18] viscous flows of Oldroyd 6- and 8- constant fluids [19,20], MHD flows of non-Newtonian fluids [21], nonlinear water waves [22], Blasius viscous flow [23] nonlinear flows and second grade fluid over a porous plate [24], fourth-order parabolic partial differential equations [25] coupled Van der Pol equations [26] finance problems [27], SEI tuberculosis model [28] and SIR model of tuberculosis [29].

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2. Homotopy Analysis Method

Consider a nonlinear differential equation of the form

$$N[u(t)] = 0 \tag{2.1}$$

where N is a nonlinear operator, u(t) is an unknown function and t is an independent variable. Let $u_0(t)$ denote an initial approximation of the exact solution u(t), L an auxiliary linear operator, $\hbar \neq 0$ and $H(t) \neq 0$ denote an auxiliary parameter and auxiliary function respectively. Using the embedding parameter $r \in [0,1]$, we construct a zero-order deformation equation

$$(1-r)L[\phi(t;r) - u_0(t)] = hr\ H(t)N(\phi(t;r)] \tag{2.2}$$

As pointed out by Liao [9] we have great freedom to choose the initial approximation $u_0(t)$), the auxiliary linear operator L, the non-zero auxiliary (convergent-control) parameter h and the auxiliary function H(t). It is this kind of freedom and flexibility that allows us to control and adjust the convergence region and rate of homotopy solution of the considered nonlinear problem [8]

When r = 0, equation (2.2) becomes

$$\phi(t;0) = u_0(t) \tag{2.3}$$

When r = 1, the zero-order deformation equation (2.2)1s equivalent to

$$\phi(t,1) = u(t) \tag{2.4}$$

Therefore, according to equation (2.3) and (2.4), a the embedding parameter r increases from 0 to 1, $\phi(t;r)$ varies continuously from the initial approximation $u_0(t)$ to the exact solution $u(t)$. In topology, this kind of continuous variation is called deformation.

If the initial approximation, auxiliary linear operator, auxiliary parameter h and auxiliary function H(t) are properly chosen, then the homotopy solution $\phi(t;r)$ of the zero-order deformation equation (2.2) exists for all $r \in [0,1]$ and besides its mth-order deformation derivative $\left.\frac{\partial^m \phi(t;r)}{\partial r^m}\right|_{r=0}$ exists for $m \geq 1$.

By Taylor's theorem, we expand the homotopy $\phi(t;r)$ in a power series of the embedding parameter r as follows:

$$\phi(t;r) = \phi(t;0) + \sum_{m=1}^{\infty} u_m(t) r^m$$

Where

$$u_m(t) = \frac{1}{m!}\left.\frac{\partial^m \phi(t;r)}{\partial r^m}\right|_{r=0}$$

Assuming that the auxiliary parameter h, the auxiliary function H(t), the initial approximation $u_0(t)$ and the linear operator L are so properly chosen, so that the solution series (2.6) converges at r = 1. Then, at r = 1, the series (2.6) becomes

$$\phi(t;1) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)$$

Therefore, using equation (2.4), (2.7) can be re-written as

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)$$

which is the approximate solution series of the nonlinear equation (2.1) by homotopy analysis method.

Define the vector

$$\vec{u}_n(t) = \{u_0(t), u_1(t), \ldots, u_n(t)\}$$

Now according to the definition (2.6) the related governing equation of $u_m(t)$ can be derived from the zero-order deformation equation (2 2.). Differentiating the zero-order deformation rn times with respect to r and then dividing by m! and finally setting r = 0, we have the, nth-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \hbar H(t) Q_m(\vec{u}_{m-1}(t))$$

where

$$Q_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;r)}{\partial r^{m-1}}$$

and

$$\chi_m = \begin{cases} 0, & m - 1 \\ 1, & m > 1 \end{cases}$$

At this stage all the solution series $u_1(t), u_2(t), ..., of\ u_m(t)$ can easily be gained by solving the linear high-order deformation equation (2.9) by means of symbolic computation software such as Matlab, Maple and Mathematica. Hence, the mth-order approximation of $u_m(t)$ is given by

$$u(t) \approx \sum_{m=0}^{\infty} u_m(t)$$

It should be noted that the original nonlinear problem (2.1) has been transformed into an infinite number linear sub-problems governed by the high order deformation equation (2.9) and we then use the sum of all the solutions $u_m(t)$ of its first several sub-problems to approximate the exact solution of the considered nonlinear equation.

3. Application of HAM to SEIR Tuberculosis Model

In order to investigate the validity of HAM in solving nonlinear equations. we consider the following nonlinear system of first-order differential equations

$$\frac{dS}{dt} = (1-\gamma)\pi + sI - \beta IS - \mu S \quad (3.1)$$

$$\frac{dE}{dt} = (1-\rho)\beta IS - (\mu + v)E \quad (3.2)$$

$$\frac{dI}{dt} = d\rho\beta IS + dvE + (\mu + \mu r + \varepsilon)S \quad (3.3)$$

$$\frac{dR}{dt} = \varepsilon I - sI - \beta IR - \mu R \quad (3.4)$$

where

S = number of susceptible who do not have the disease yet but could get it

E = number of exposed who have the disease but are yet to show any sign of symptoms

I = number of infectives who have the disease and could transmit it to others.

R = number of recovered or removed who cannot get the disease or transmit it

All other parameters are as defined in Egbetade and Ibrahim [28]

To solve (3.1) by means of HAM, we choose the linear operator

$$L(S(t;r)) = \frac{dS}{dt}(t;r) \quad (3.5)$$

with the property $L(c_1) = 0$ where c_1 is a constant of integration.

The inverse operator L^{-1} is given by

$$L^{-1}(\cdot) = \int_0^t (\cdot)dt \quad (3.6)$$

Define the nonlinear operator

$$N[S(t\ r)] = \frac{dS}{dt}(t;r) - (1-\gamma)\pi - sI(t;r) + \beta I(t;r)S(t;r) + \mu S(t;r) \quad (3.7)$$

Using the above definition and explanation in section 2, we construct the zeroth-order deformation equation

$$(1-r)L[S(t;r)] - s_0(t;r) = r\hbar H(t)N[S(t;r)] \quad (3.8)$$

where $s_0(t)$ is an initial approximation of S(t)

As the embedding parameter r increases from 0 to 1. we have

$$S(t;0) = s_0(t), S(t;1) = S(t) \quad (3.9)$$

Thus, we obtain the mth-order (high order) deformation equation

$$L[S_m(t) - \chi_m S_{m-1}] = \hbar H(t)Q_m(\vec{S}_{m-1}(t)), \quad m \geq 1 \quad (3.10)$$

Where

$$Q_m(S_{m-1}(t)) = \frac{dS_{m-1}(t)}{dt} - (1-\gamma)\pi - sI_{m-1}(t) + \beta I_{m-1}(t)S_{m-1}(t) + \mu S_{m-1}(t), \ m \geq 1 \quad (3.11)$$

And χ_m is as defined in equation (3.11)

By the concept of $\hbar$ curves we simply need to replace the values of $\hbar$ while setting H(t) = 1 to obtain solutions of the mth-order deformation equations for various values of $\hbar$. If we choose $\hbar$ = -1 then we have the solution of the mth-order deformation equation (3.10) as

$$S_m(t) = \chi_m S_{m-1}(t) - \int_0^t \left(\frac{d}{dt} S_{m-1}(t) - (1-\gamma)\pi - sI_{m-1}(t) + \beta I_{m-1}(t) S_{m-1}(t) + \mu S_{m-1}(t) \right), \ m \geq 1 \quad (3.12)$$

By observing all other steps in (3.5) - (3.12), the solutions of the mth-order deformation equations of $E_m(t), I_m(t),$ *and* $R_m(t)$ *for* $\hbar = -1$ are respectively

$$E_m(t) = \chi_m E_{m-1}(t) - \int_0^t \left(\frac{d}{dt} E_{m-1}(t) - (1-\rho)\beta I_{m-1}(t) S_{m-1}(t) + (\mu + v) E_{m-1}(t) \right) dt, \ m \geq 1 \quad (3.13)$$

$$I_m(t) = \chi_m I_{m-1}(t) - \int_0^t \left(\frac{d}{dt} I_{m-1}(t) - d\rho\beta I_{m-1}(t) S_{m-1}(t) - dvE_{m-1}(t) - (\mu + \mu_T + \varepsilon) I_{m-1}(t) + SI_{m-1}(t) dt, \ m \geq 1 \quad (3.14)$$

$$R_m(t) = \chi_m R_{m-1}(t) - \int_0^t \left(\frac{d}{dt} R_{m-1}(t) - \varepsilon I_{m-1}(t) S_{m-1}(t) + \beta I_{m-1}(t) R_{m-1}(t) + \mu R_{m-1}(t) \right) dt, \ m \geq 1 \quad (3.15)$$

4.0 Numerical Results

The following values (Table 1) are assumed for variables and parameters of the model:

Table 1: Variable and Parameter values for series solution

Variable	Assigned values
S	30
E	10
I	15
R	20
β	0.002
γ	0.14
s	0.15
μ	0.1
μ_T	0.02
v	0.03
ε	0.13
ρ	0.01
d	0.24
π	0.1

Using Maple 18 computation software [30] the 1st - 5th terms series approximtion for S(t), E(t), I(t) and R(t) are calculated for $\hbar = (-1, -2)$

Case 1: $\hbar = -1$

1st term approximations

$$S_1(t) = \sum_{m=0}^{1} S_m(t) = 30 - 27.664\,t$$

$$E(t) = \sum_{m=0}^{1} E_m(t) = 10 + 26.23t$$

$$I_1(t) = \sum_{m=0}^{1} I_m(t) = 15 + 1.6368\,t$$

$$R_1(t) = \sum_{m=0}^{1} R_m(t) = 20 - 18.7t$$

2nd term approximations

$$S_2(t)=\sum_{m=0}^{2} S_m(t)=60-27.664t+12.48164\ t^2,$$

$$E_2(t)=\sum_{m=0}^{2} E_m(t)=20+26.23t-11.5216732\ t^2,$$

$$I_2(t)=\sum_{m=0}^{2} I_m(t)=30+1.6368\ t+0.149926368\ t^2,$$

$$R_2(t)=\sum_{m=0}^{2} R_m(t)=40-18.7t+7.603552\ t^2,$$

3rd term approximations

$$S_3(t)=\sum_{m=0}^{3} S_m(t)=90-109.414t+23.45732\ t^2-7.171745268\ t^3,$$

$$E_3(t)=\sum_{m=0}^{3} E_m(t)=30+109.92t-22.3875964\ t^2+6.887681955\ t^3,$$

$$I_3(t)=\sum_{m=0}^{3} I_m(t)=45+3.4032\ t+0.123584736\ t^2-0.0064255814\ 3t^3,$$

$$R_3(t)=\sum_{m=0}^{3} R_m(t)=60-73.4t+15.036472\ t^2-4.12159895\ t^3,$$

4th term approximations

$$S_4(t)=\sum_{m=0}^{4} S_m(t)=120-166.414t+174.0967\ t^2+256.1046086\ t^3-79.64519315\ t^4,$$

$$E_4(t)=\sum_{m=0}^{4} E_m(t)=40+239.07t-139.7830574\ t^2+14.12114378\ t^3-3.902590858\ t^4,$$

$$I_4(t)=\sum_{m=0}^{4} I_m(t)=60+5.5152\ t-0.426528624\ t^2+0.0735355504\ 7t^3-0.0168315513\ 3t^4,$$

$$R_4(t)=\sum_{m=0}^{4} R_m(t)=80-164.1t+93.664208\ t^2-8.786295927\ t^3+2.176984693\ t^4,$$

5th term approximations

$$S_5(t)=\sum_{m=0}^{5} S_m(t)=150-434.914t+288.42482\ t^2+234.777354\ t^3+233.6884026\ t^4+41.2673192\ t^5,$$

$$E_5(t)=\sum_{m=0}^{5} E_m(t)=50+318.76t-186.2103684\ t^2+17.48882594\ t^3-3.854153975\ t^4+-1.142696178\ t^5,$$

$$I_5(t)=\sum_{m=0}^{5} I_m(t)=75+7.3248\ t+0.465154848\ t^2-0.8976069707\ t^3-0.5487489555\ t^4-0.1011761705\ t^5,$$

$$R_5(t)=\sum_{m=0}^{5} R_m(t)=100-218.5t+214.974288\ t^2-73.95147233\ t^3+1.505253779\ t^4-0.9769096572\ t^5,$$

Case 2: $\hbar=-2$

1st term approximations

$$S_1(t)=\sum_{m=0}^{1} S_m(t)=30-55.328t$$

$$E(t)=\sum_{m=0}^{1} E_m(t)=10+52.46t$$

$$I_1(t)=\sum_{m=0}^{1} I_m(t)=15+3.2736\ t$$

$$R_1(t)=\sum_{m=0}^{1} R_m(t)=20-37.4t$$

2nd term approximations

$$S_2(t)=\sum_{m=0}^{2} S_m(t)=60-27.664t+24.96328\ t^2$$

$$E_2(t)=\sum_{m=0}^{2} E_m(t)=20+26.23t-23.0433464\ t^2$$

$$I_2(t)=\sum_{m=0}^{2} I_m(t)=30+1.6368\,t+0.299852736\quad t^2$$

$$R_2(t)=\sum_{m=0}^{2} R_m(t)=40-18.7t+15.207104\;\, t^2$$

3rd term approximations

$$S_3(t)=\sum_{m=0}^{3} S_m(t)=90-191.164\,t+34.433\,t^2-15.24929928\;\, t^3$$

$$E_3(t)=\sum_{m=0}^{3} E_m(t)=30+185.61t-33.2535196\;\, t^2+14.67191652\;\, t^3$$

$$I_3(t)=\sum_{m=0}^{3} I_m(t)=45+5.1696\,t+0.097243104\quad t^2-0.0106717019\quad 7t^3$$

$$R_3(t)=\sum_{m=0}^{3} R_m(t)=60-128.1t+22.469392\;\, t^2-8.8553611\;\, t^3$$

4th term approximations

$$S_4(t)=\sum_{m=0}^{4} S_m(t)=120-223.414\,t+371.65072\;\, t^2+519.3810625\;\, t^3-158.617861\;\, t^4,$$

$$E_4(t)=\sum_{m=0}^{4} E_m(t)=40+372.22t-257.1785184\;\, t^2+21.3546056\;\, t^3-8.459694168\;\, t^4,$$

$$I_4(t)=\sum_{m=0}^{4} I_m(t)=60+7.6272\,t-0.97664198\;\, t^2+0.1534936824\;\, t^3-0.0335545000\quad 2t^4,$$

$$R_4(t)=\sum_{m=0}^{4} R_m(t)=80-254.8t+172.291944\;\, t^2-13.4509979\;\, t^3+4.775816761\;\, t^4$$

5th term approximations

$$S_5(t)=\sum_{m=0}^{5} S_m(t)=150-703.414\,t+750.94634\;\, t^2+212.0508623\;\, t^3-367.731612\;\, t^4+58.50547248\;\, t^5,$$

$$E_5(t)=\sum_{m=0}^{5} E_m(t)=50+398.45t-232.6376794\;\, t^2+20.8565081\;\, t^3-3.805717092\;\, t^4-1.672029308\;\, t^5,$$

$$I_5(t)=\sum_{m=0}^{5} I_m(t)=75+9.1344\,t+1.35683832\;\, t^2-1.868749492\;\, t^3+1.114329463\;\, t^4-0.200271635\;\, t^5,$$

$$R_5(t)=\sum_{m=0}^{5} R_m(t)=100-272.9t+336.284368\;\, t^2-139.1166487\;\, t^3+0.833522864\;\, t^4-2.32374607\;\, t^5$$

5. Discussion of Results and Conclusion

The results of HAM implementation for SEIR tuberculosis model considered here agree well with those of Awawdeh et al [31] and Vahdati et.al. [32] for SIR epidemic model in a homogeneous population. For various order of approximations the method was shown to be computationally efficient as it yields solutions that are reasonable and easy to express. These results further confirm the validity and potential of HAM in handling many functional equations arising in the field of science

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