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THE EFFECT OF STOCHASTIC VOLATILITY PROCESS ON THE VALUE OF AN ASSET

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Abstract

In this paper, the effect of stochastic volatility process on the value of an asset was investigated using boundary conditions. The partial differential equation with stochastic volatility term was reduced to financial partial differentiation by making some assumptions in the price of volatility risk and the constant risk-free rate. A set of functions was constructed which transformed the financial problem to a one dimensional heat equation through the exploits of partial differentiation and separation of variables. A function of volatility process and time was derived by setting a parameter α which was a function of the volatility process equal to unity. Analytical solution of the heat equation which was subject to initial conditions was obtained using Fourier transformation. The transformation is a function of spatial variable (stochastic variable) which is independent of time. Uncertainty in the price history of the stock market was dictated as the parameters are varied with respect to the stochastic parameter. Existence of a unique solution was also achieved which represent the volatile behavior of the system. Furthermore, some numerical illustrations of the models which demonstrate the behavior of the system was obtained using the Maple software. The illustrations were examined by using different values of the parameters in the models. The results obtained are comparable to the results of cubic B-spline as found in literature.

Keywords: Financial BVP, Volatility process, Heat equation, cubic B-Spline

Introduction

The measure of the price fluctuation in the market is Market volatility. The fluctuations in prices will be more as the volatility goes higher not only in terms of frequency, but also in terms of difference. A low volatility means a stable and consistent market. This is a rate at which the price of a security increases or decreases for a given set of returns. It is measured by calculating the standard deviation of the annualized returns over a given period of time [1](Will, 2019).

Volatility shows the range to which the price of a security may increase or decrease. It measures the risk of a security. It is used in option pricing formula to gauge the fluctuations in the returns of the underlying assets. Volatility indicates the pricing behavior of the security and helps estimate the fluctuations that may happen in a short period of time. If the prices of a security fluctuate rapidly in a short time span, it is said to have high volatility. If the prices of a security fluctuate slowly in a longer time span, it is said to have low volatility [1](Will, 2019).

This study is aimed at the effect of volatility process on the value of an asset. There are several generalizations of the Black-Scholes model, BSM, that relax various model assumptions. One of these assumptions is that volatility is constant. It has been long known that this is not supported empirically. Two main generalizations to the constant volatility assumption are given by the local volatility and stochastic volatility models. The local volatility models write volatility as a deterministic function of time and stock price, whereas the stochastic volatility models describe the behavior of volatility by another stochastic differential equation. A comprehensive treatment of stochastic volatility models can be found in [2] (Hung et al., 2016).

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A partial differential equation describing the value of any asset $u(s,v,t)$, including accrued payment satisfies the equation of the form

$$\frac{1}{2}vs^2\frac{\partial^2 u}{\partial s^2}+\rho\sigma vs\frac{\partial^2 u}{\partial s\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2}+rs\frac{\partial u}{\partial s}+\{k[\theta-v(t)]-\lambda(s,v,t)\}\frac{\partial u}{\partial v}$$
$$-ru+\frac{\partial u}{\partial t}=0. \quad (1.1)$$

The unspecified term $\lambda(s,v,t)$ represents the price of volatility risk and must be independent of the particular asset. r is the constant risk-free rate, k is the mean reversion speed for the variance, θ is the mean reversion level for the variance while S(t) and v(t) are the price and volatility process respectively at time t. $\rho\in[-1,1]$ is the correlation coefficient. The volatility of volatility is σ. To ensure that zero is an unattainable boundary for the volatility process $v(t)$, then $4k\theta>\sigma^2$. The volatility $v(t)$ process follows the pattern of [3] (Rubinstein, 1985) given by

$$dv(t)=k[\theta-v(t)]dt+2\delta\sqrt{v(t)}dw_2(t), \quad (1.2)$$

where $w_2(t)$ has correlation ρ with $w_1(t)$. The assumption herein is that the spot asset at time t follows the diffusion

$$ds(t)=\mu sdt+\sqrt{v(t)}sdw_1(t), \quad (1.3)$$

where $w_1(t)$ and $w_2(t)$ are Wiener process which takes account of the leverage effect, stock returns and implied volatility which are negatively correlated. Partial deferential equations (PDE.s) are used to model and analyse dynamic systems in fields as diverse as physics, biology, economics, and finance. The linear parabolic ones (LPDE.s) are one class of PDE.s which has received particular attention. The LPDEs make up a large class of PDE.s which is of a succinctly simple structure such that a thorough analysis of them is possible. In [4,5] (Friedman, 1964; Evans, 1998) any interested reader can find an introduction and detailed analysis of their properties.

In finance, for a contingent claim on a single asset, the generic PDE can be written as

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial^2 u}{\partial x^2}+b(x,t)\frac{\partial u}{\partial x}+c(x,t)u=0, \quad (1.4)$$

where t either represents calendar time or time-to-expiry, x represents either the value of the underlying asset or some monotonic function of it (e.g. $\log(s_t)$; log-spot) and u is the value of the claim (as a function of x and t). The terms $a(.)$, $b(.)$ and $c(.)$ are the diffusion, convection and reaction coefficients respectively, and this type of PDE is known as a convection-diffusion PDE.2 This type of PDE can also be written in the form [6] (Richard, 2013) as

$$\frac{\partial u}{\partial t}+a(x,t)\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u}{\partial x}\right)+b(x,t)\frac{\partial}{\partial x}(\beta(x,t)u)+c(x,t)u=0. \quad (1.5)$$

This form occurs in the Fokker-Planck (Kolmogorov forward) equation that describes the evolution of the transition density of a stochastic quantity (e.g. a stock value). This can be put in the form of equation (1.4) if the functions α and β are both once differentiable in x - although it is usually better to directly discretise the form given. A simple application in finance for this PDE can be found in [7] (Friedman, 1975). Through the celebrated Feynman-Kac representation of solutions to PDE.s, LPDEs and discussion models are closely linked [8](Kristensen, 2004). This leads to the representation of derivative prices as solutions to PDE.s in asset pricing theory. To price contingent claim we must make assumption that gives the price of volatility risk [9] (Heston, 1993). Several techniques have been used by many authors to study the existence of solution PDEs with stochastic volatility. In [10](Kanaya and Kristensen, 2016), a two-step estimation method of stochastic volatility models was proposed. In the first step, the non-parametrically (unobserved) instantaneous volatility process is estimated. In the second step, the filtered/estimated volatility process replacing the latent process and the standard estimation methods for fully observed diffusion processes are employed. This method is an extension of the method in [11] (Kristensen, 2008). In [12, 13] (Heston, 1993; Stein and Stein, 1991), a closed form solution was obtained for a European call option that satisfies the PDE (1.1) while [14] Manga et al., (2019) focused on analytical approximations and a study of sensitivities (Greeks) of Asian options with Heston stochastic volatility model parameters.

In this paper, a set of functions is constructed, that transforms the problem of equation (1.1) into a heat equation. Analytical solutions are obtained, and sensitivity analysis given in a concrete setting by the assistance of some boundary conditions.

From equation (1.1), set $S=0$, for the interest in the present (and not the future) value of asset so as to ascertain the effect of the volatility process. Thus, $\lambda(0,v,t)=0$ [15] (Osu et al., 2020). Furthermore, we assume that the stochastic rate, r, changes constantly with volatility process. That is

$$\frac{dr}{dv}=A\Rightarrow r=Av. \quad (1.6)$$

Equation (1.1) reduces to a form equation (1.5) as;

$$\frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2}+k(\theta-v)\frac{\partial u}{\partial v}-Avu+\frac{\partial u}{\partial t}=0. \quad (1.7)$$

In what follows, we construct functions $\alpha(v)$ and $\beta(t)$ so that the set of equations

$$u(v,t)=w(z,\tau,)e^{-\varphi(v,t)}, \quad (1.8)$$

$$z=z(v)=\alpha(v), \quad (1.9)$$

and

$\tau = \tau(t) = \beta(t),$ (1.10)

transforms the financial partial differential of equation (1.7) into a heat equation of the form;

$\frac{\partial^2 w}{\partial z^2} = 2\frac{\partial w}{\partial \tau}\,.$ (1.11)

Materials and Methods

Stochastic Volatility. The word stochastic refers to something that is randomly determined and not be predicted precisely. In the context of stochastic modeling, it refers to successive value of a random variable that are not independent. Stochastic volatility refers to the fact that the volatility of asset price is not constant as assumed in the Black Scholes options pricing model. Stochastic volatility modeling attempts to correct the problem with Black Scholes by allowing volatility to vary over time. It also treats price volatility as a random variable. Allowing the price to vary in the stochastic volatility models improved the accuracy of calculations and forecasts [1] (Will, 2019).

Boundary Condition. This is a set of conditions specified for the behavior of the solution to a set of differential or partial differential equations at the boundary of its domain. A system with boundary conditions is known as the boundary value problem.

Solution of a Boundary Value Problem. This is a function u that satisfies the differential equation on an open region D, continuous on $D \cup \partial D$ and satisfies the specified boundary condition on ∂D.

Heat Equation. Heat equation is a partial differential equation that describes how the distribution of some quantity such as heat evolves over time in a solid medium as it spontaneously flow from higher places to lower places. It is a special case of the diffusion equation [16](Rozier, 1984)**.**

Separation of Variables. Variable separation is a method of solving ordinary and partial differential equations in which algebra allows one to rewrite an equation so that each of two variables occurs on a different sides of the equation [17] (Andrei, 2001).

Fourier Transform. This is a tool that breaks a waveform (a function or signal) into an alternate representation characterized by sine and cosines. Fourier transform is also a mathematical technique that transform a function $x(t)$ into a function of frequency $X(w)$. It is a special case of the Fourier series when the period T ends to infinity.

Results

Given equation (1.7) implies that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(we^{-\varphi(v,t)}\right) = \frac{\partial w}{\partial t}e^{-\varphi(v,t)} + w\frac{\partial}{\partial t}e^{-\varphi(v,t)}$$
$$= \frac{\partial w}{\partial t}e^{-\varphi(s,v,t)} - we^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial t}\,.$$

By chain rule for $w = w(z,\tau,\theta)$ w.r.t gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial t} = \beta'\frac{\partial w}{\partial t}.$$

Therefore

$$\frac{\partial u}{\partial t} = \beta'(t)\frac{\partial w}{\partial t}e^{-\varphi(s,v,t)} - we^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial t}. \quad (1.12)$$

On the other hand

$$\frac{\partial u}{\partial v} = \frac{\partial}{\partial v}\left(we^{-\varphi(v,t)}\right) = \frac{\partial w}{\partial v}e^{-\varphi(v,t)} + w\frac{\partial}{\partial v}\left(e^{-\varphi(v,t)}\right)$$
$$= \left(\frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial v}\right)e^{-\varphi(v,t)} - we^{-\varphi(v,t)}\frac{\partial \varphi}{\partial v}.$$

But

$$\frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial \tau}\frac{\partial \tau}{\partial v} = \alpha'(v)\frac{\partial w}{\partial \tau},$$

Therefore

$$\frac{\partial u}{\partial v} = \alpha'(v)\frac{\partial w}{\partial \tau}e^{-\varphi(s,v,t)} - we^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial v}\,, \quad (1.13)$$

Again

$$\frac{\partial^2 u}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial v}e^{-\varphi(v,t)} - we^{-\varphi(v,t)}\frac{\partial \varphi}{\partial v}\right)$$
$$= \frac{\partial}{\partial v}\left(\alpha'(v)\frac{\partial w}{\partial \tau}e^{-\varphi(v,t)}\right) - \frac{\partial}{\partial v}\left(we^{-\varphi(v,t)}\frac{\partial \varphi}{\partial v}\right)$$

$$= \alpha''(v)\frac{\partial w}{\partial \tau}e^{-\varphi(v,t)} + \alpha'(v)e^{-\varphi(v,t)}\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial \tau}\right) + \alpha'(v)\frac{\partial w}{\partial \tau}\frac{\partial}{\partial v}e^{-\varphi(v,t)}$$
$$-\frac{\partial w}{\partial v}e^{-\varphi(v,t)}\frac{\partial \varphi}{\partial \varphi} + w\frac{\partial}{\partial v}e^{-\varphi(v,t)}\frac{\partial \varphi}{\partial v} + we^{-\varphi(v,t)}\frac{\partial^2 \varphi}{\partial v^2}$$
$$= \alpha''(v)\frac{\partial w}{\partial \tau}e^{-\varphi(v,t)} + \alpha'(v)e^{-\varphi(v,t)}[\frac{\partial}{\partial \tau}\left(\frac{\partial w}{\partial \tau}\right)\frac{\partial \tau}{\partial v} + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial \tau}\right)\frac{\partial z}{\partial v}]$$
$$+\alpha'(v)\frac{\partial w}{\partial \tau}\left(-e^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial v}\right) - \alpha'(v)\frac{\partial w}{\partial \tau}e^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial v} + we^{-\varphi(s,v,t)}\frac{\partial^2 \varphi}{\partial v^2} - we^{-\varphi(s,v,t)}\frac{\partial^2 \varphi}{\partial v^2}$$
$$= \alpha''(v)\frac{\partial w}{\partial \tau}e^{-\varphi(v,t)} + \alpha'(v)e^{-\varphi(v,t)}\left[\frac{\partial^2 w}{\partial \tau^2}\alpha'(v)\right] - \alpha'(v)e^{-\varphi(v,t)}\frac{\partial w}{\partial \tau}\frac{\partial \varphi}{\partial v}$$
$$-\alpha'(v)e^{-\varphi(v,t)}\frac{\partial w}{\partial \tau}\frac{\partial \varphi}{\partial v} + we^{-\varphi(v,t)}\left(\frac{\partial \varphi}{\partial v}\right)^2 - we^{-\varphi(v,t)}\frac{\partial^2 \varphi}{\partial v^2}.$$

Hence

$$\frac{\partial^2 u}{\partial v^2} = \alpha''(v)\frac{\partial w}{\partial \tau}e^{-\varphi(v,t)} + \left(\alpha'(v)\right)^2 e^{-\varphi(v,t)}\frac{\partial^2 w}{\partial \tau^2} - \alpha'(v)e^{-\varphi(s,t)}\frac{\partial w}{\partial z}\frac{\partial \varphi}{\partial v}$$
$$+we^{-\varphi(v,t)}\left(\frac{\partial \varphi}{\partial v}\right)^2 + we^{-\varphi(s,v,t)}\frac{\partial^2 \varphi}{\partial v^2} - we^{-\varphi(v,t)}\frac{\partial^2 \varphi}{\partial v^2}, \qquad (1.14)$$

Substituting for equations (1.12), (1.13) and (1.14) into equation (1.7) gives

$$\frac{1}{2}v\sigma^2\{\alpha''(v)\frac{\partial w}{\partial z}e^{-\varphi(v,t)} + \alpha'(v)e^{-\varphi(v,t)}\left[\frac{\partial^2 w}{\partial z^2}\alpha'(v)\right] - \alpha'(v)e^{-\varphi(v,t)}\frac{\partial w}{\partial z}\frac{\partial \varphi}{\partial v}$$
$$-\alpha'(v)e^{-\varphi(v,t)}\frac{\partial w}{\partial z}\frac{\partial \varphi}{\partial v} + we^{-\varphi(v,t)}\left(\frac{\partial \varphi}{\partial v}\right)^2 - we^{-\varphi(v,t)}\frac{\partial^2 \varphi}{\partial v^2}\}$$
$$+k(\theta - v)\left(\alpha'(v)\frac{\partial w}{\partial \tau}e^{-\varphi(s,v,t)} - we^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial v}\right) - Avw + \beta'(t)\frac{\partial w}{\partial t}e^{-\varphi(s,v,t)} - we^{-\varphi(s,v,t)}\frac{\partial \varphi}{\partial t} = 0.$$

That is

$$\beta'(t)\frac{\partial w}{\partial t} + \frac{1}{2}v\sigma^2\left(\alpha'(v)\right)^2\frac{\partial^2 w}{\partial z^2} + \left(\frac{1}{2}v\sigma^2\alpha''(v) - v\sigma^2\frac{\partial \varphi}{\partial v}\alpha'(v) + k(\theta - v)\alpha'(v)\right)\frac{\partial w}{\partial \tau}$$
$$-\left[Av + \frac{1}{2}v\sigma^2\left(\frac{\partial \varphi}{\partial v}\right)^2 - \frac{1}{2}v\sigma^2\frac{\partial^2 \varphi}{\partial v^2} + k(\theta - v)\frac{\partial \varphi}{\partial v} + \frac{\partial \varphi}{\partial t}\right]w = 0. \qquad (1.15)$$

In equation (1.15) set

$$\frac{1}{2}v\sigma^2\alpha''(v) - v\sigma^2\frac{\partial \varphi}{\partial v}\alpha'(v) + k(\theta - v)\alpha'(v) = 0 \qquad (1.16)$$

and

$$Av + \frac{1}{2}v\sigma^2\left(\frac{\partial \varphi}{\partial v}\right)^2 - \frac{1}{2}v\sigma^2\frac{\partial^2 \varphi}{\partial v^2} + k(\theta - v)\frac{\partial \varphi}{\partial v} + \frac{\partial \varphi}{\partial t} = 0 \qquad (1.17)$$

to get

$$\beta'(t)\frac{\partial w}{\partial t} = -\frac{1}{2}v\sigma^2\left(\alpha'(v)\right)^2\frac{\partial^2 w}{\partial z^2}. \qquad (1.18)$$

In (1.18) set

$$\beta'(t) = -v\sigma^2\left(\alpha'(v)\right)^2. \qquad (1.19)$$

The form (1.19) means that

$$\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}, \qquad (1.20)$$

is a heat equation.

Now equation (1.19) must be a function of t only. Thus set $v\left(\alpha'(v)\right)^2 = 1$ to get

$$\alpha'(v) = \pm\frac{1}{\sqrt{v}} = \pm v^{-\frac{1}{2}}.$$

Thus

$$\alpha(v) = \pm 2\sqrt{v} + B. \qquad (1.21)$$

Hence $\beta'(t) = -\sigma^2$, so that

$$\beta(t) = -\sigma^2 t + C. \qquad (1.22)$$

To derive the formula for $\varphi(v,t)$, when $\alpha(v) = 2\sqrt{v} + B$, we note that for this case, $\alpha' = v^{-\frac{1}{2}}$ and $\alpha'' = -\frac{1}{2}v^{-\frac{3}{2}}$, so that equation

$$\frac{1}{2}v\sigma^2\left(-\frac{1}{2}v^{-\frac{3}{2}}\right) - v\sigma^2\frac{\partial \varphi}{\partial v}v^{-\frac{1}{2}} + k(\theta - v)v^{-\frac{1}{2}} = 0.$$

That is

$$-\frac{1}{4}\sigma^2 - v\sigma^2\frac{\partial\varphi}{\partial v} + k(\theta - v) = 0. \quad (1.23)$$

Without loss of generality, set

$$\frac{1}{4}\sigma^2 = k\theta, \quad (1.24)$$

then one gets from (1.23)

$$\frac{\partial\varphi}{\partial v} = -\frac{k}{\sigma^2} = -\frac{1}{4\theta}. \quad (1.25)$$

Consequently,

$$\frac{\partial^2\varphi}{\partial v^2} = 0 \text{ and } \left(\frac{\partial\varphi}{\partial v}\right)^2 = \frac{1}{16\theta^2}. \quad (1.26)$$

From equation (1.25),

$$\varphi(v,t) = -\frac{1}{4\theta}v + \emptyset(t), \quad (1.27)$$

implies that $\emptyset$ is independent of v, that is $\frac{\partial\emptyset}{\partial v} = 0$. Using equations (1.26), (1.27) and (1.16), one gets;

$$\emptyset'(t) = \frac{\partial\varphi}{\partial t} = -Av - \frac{1}{2}v\sigma^2\left(\frac{1}{4\theta}\right)^2 + \frac{1}{2}v\sigma^2(0) + k(\theta - v)\frac{1}{4\theta}$$

$$= -Av - \frac{kv}{8\theta} + \frac{k}{4} - \frac{kv}{4\theta}$$

$$= -\left(A + \frac{3k}{8\theta}\right)v + \frac{k}{4}.$$

Therefore

$$\emptyset(t) = -\left(A + \frac{3k}{8\theta}\right)vt + \frac{k}{4}t.$$

Since $\frac{\partial\emptyset}{\partial v} = 0$, we must have $A = -\frac{3k}{8\theta}$, so that

$$\emptyset(t) = \frac{k}{4}t.$$

Accordingly

$$\varphi(v,t) = \frac{k}{4}t - \frac{1}{4\theta}v = \frac{1}{4}\left(kt - \frac{v}{\theta}\right). \quad (1.28)$$

In the sequel we state;

Theorem 2.1: The set of functions

$$\left.\begin{aligned} u(v,t) &= w(z,\tau)e^{-\varphi(v,t)} \\ z &= 2\sqrt{v} \\ \tau &= -4k\theta t \\ \varphi(v,t) &= \frac{1}{4}\left(kt - \frac{v}{\theta}\right) \end{aligned}\right\} \quad (1.29)$$

transforms the linear pde

$$\frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2} + k[\theta - v(t)]\frac{\partial u}{\partial v} - Avu + \frac{\partial u}{\partial t} = 0$$

into a heat equation of the form

$$\frac{\partial w}{\partial\tau} = \frac{1}{2}\frac{\partial^2 w}{\partial z^2}.$$

It follows that a European call option with strike K and maturity at time T satisfies the PDE (1.1) subject to the boundary conditions;

$$\left.\begin{aligned} u(s,v,T) &= \max(v,1), \\ u(s,0,t) &= 0, \\ \frac{\partial u}{\partial v}(s.\infty,t) &= 1, \\ \frac{1}{2}v\sigma^2\frac{\partial^2 u}{\partial v^2} + \{k[\theta - v(t)]\}\frac{\partial u}{\partial v} - Avu + \frac{\partial u}{\partial t} &= 0, \\ u(\infty,v,t) &= 0. \end{aligned}\right\} \quad (1.30)$$

Exponential (Analytical) solution

For the exponential solution of the heat equation (2.9), we consider Fourier transform of the Initial-value problem for the heat equation of the form;

$$\begin{cases} w_\tau - \Delta w = 0 \; in \; \mathbb{R}^n \times (0,\infty) \\ w = g \; on \; \mathbb{R}^n \times (\tau = 0) \end{cases}. \quad (1.31)$$

A method for solving (1.20) by using (1.31) and computing $\widehat{w}$, is therefore the Fourier transform of w in the spatial variable v only. Thus

$$\begin{cases} \hat{w}_\tau + |z|^2\hat{w} = 0 \quad for\ t > 0 \\ \quad w = g \quad for\ \tau = 0; \end{cases}$$

whence

$\hat{w} = e^{-|z|^2 t}\hat{g}.$

Consequently, $w = \left(e^{-|z|^2 t}\hat{g}\right)^{\check{}}$, so that

$$w = \frac{g * F}{(4\pi)^{n/2}}, \qquad (1.32)$$

where $\hat{F} = e^{-|z|^2\tau}$. But

$$F = \left(e^{-|z|^2\tau}\right)^{\check{}} = \frac{1}{(4\pi)^{n/2}}\int_{R^n} e^{iv.z-|z|^2\tau}dz = \frac{1}{(4\pi)^{n/2}}e^{-\frac{|z|^2}{4\tau}}.$$

By (1.32), we get

$$w(v,\tau) = \frac{1}{(8\pi\tau)^{n/2}}\int_{R^n} e^{-\frac{|v-z|^2}{4t}}\ g(z)dz \quad v \in \mathbb{R}^n, \tau > 0. \qquad (1.33)$$

This implies that by invoking (1.28), one gets

$$u(v,\tau) = \frac{e^{-\frac{1}{4}(k\tau)}}{(4\pi\tau)^{n/2}}\int_{R^n} e^{-\left(\frac{|v-2\sqrt{v}|^2}{4\tau}+\frac{v}{4\theta}\right)}\ g(2\sqrt{v})d\sqrt{v}, \qquad (1.34)$$

and

$$u(v,t) = \frac{e^{-k^2\theta t}}{(16k\theta\pi t)^{n/2}}\int_{R^n} e^{-\left(\frac{|v-2\sqrt{v}|^2}{16k\theta t}+\frac{v}{4\theta}\right)}\ g(2\sqrt{v})d\sqrt{v}. \qquad (1.35)$$

Numerical Results

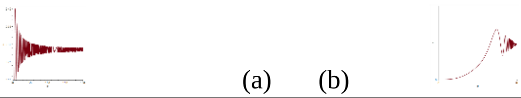
In this section, some numerical illustrations of our results in the above sections in a specific example is presented using the Maple software. This is to enable us to observe the behaviour of the volatility as against the expected final surplus of the worth of investment(see figures 1-6).The values of parameters that considered in this paper, are the following; t=2.5 .5 v= 3.7 theta= 3.5 k=10, L=4 H=6, L=150, H=3, theta= 3.5, t=60, k=10 are applied in equation (1.35).

(a) (b)

Figure 1: Profiles of Eq. (1.35), Substituting different Values of the Parameter v, t, σ, θ for 2D Graph (m=n).

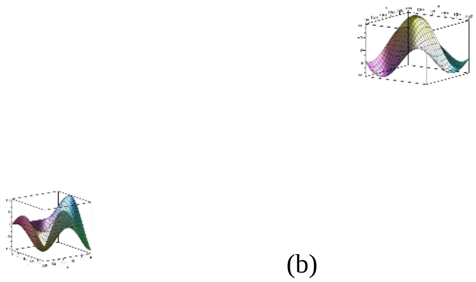
(a) (d) (e)

Figure 2: Profiles of Eq. (1.35) , Substituting different Values of the Parameters v, t, σ, θ for 2D Graph.



(a) (b)

Figure 3: Profiles of Eq. (1.35) , Substituting different Values of the Parameters v, t, σ, θ for 2D Graph



(b) (a)

Figure 4: Profiles of Eq. (1.35), Substituting different Values of the Parameters v, t, σ, θ for the 3D Graphs.

(a) (b)

Figure 5: Profiles of Eq. (1.35), Substituting different Values of the Parameters v, t, σ, θ for the 3D Graphs.

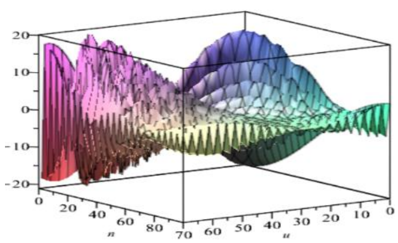


Figure 6: Profiles of Eq. (1.35) , Substituting different Values of the Parameters v, t, σ, θ for the 3D Graphs.

Discussion

This study is supported by efficient numerical simulations showing the behaviour of the system using the values of the parameters.

Figure 1 is the numerical illustrations of the results for different values of the parameters. In (a) there are existence of incomplete cone shape and unequal amplitude with respect to the parameters showing discontinuity in the buying and selling of asset. This invariably causes distortion in stock prices. This agrees with the numerical solution for $\alpha = 1, h = 0.1$, $\Delta t = 0.01$ as in [18] (Demir and Bildik, 2012). Unequal amplitude in (b) shows sensational variations in terms of prices which is highly periodic in nature. The downwall turn dictates helding of assets which in turns leads to recovery in the market.

In figure 2, 2 dimensional graph were obtained for different values of the parameters. The trajectory is shown using frequency dependent. Jump continuities which arise as a result of volatility parameter was obtained. The jump continuties continued over time as the frequency is nearly uniform convergence.

In figure 4, positve trajectories were obtained with high amplitude. This is caused by the decrease in the volatility in relation to income price. Frequency convergence in term of price was obtained which shows positve synergist profit. The wavelike nature is positive due to high convergence rate.
In figure 5, 3 dimensional plot of the stochastic parameters were obtained. The wavelike nature was obtained but restricted to quadratic nature. This quadratic nature shows uncertainty, inequality in the values of assets that is held long or protected against a rise in the value of assets held short. In this case, the price may be inconsistent with the value of the option as predicted by the PDE with stochastic volatility. This agrees with the geometrical Interpretation of Heat equation at different time level $t = 0.1$ and $0.2\ at\ 0 < v < 5$ in [6] using cubic B-spline.
In figure 6, 3 dimensional graph for different parameters were obtained. In (a) to (b) the wavelike form is cubic in nature showing that the PDE is nonlinear. This nonlinearity causes speculations in the market which depend on the volatility rate.

Conclusion

In this study, the Fourier transformation was successfully utilized for the existence of unique solution of the partial differential equation with stochastic volatility. The transform equation was expressed in terms of the volatility parameters which have effects on the system. These effects were critically examined using numerical simulation. The numerical simulation which describes the behaviour of the system gave different trajectories for different values of the parameters. The different trajectories obtained dictate uncertainty in the price history of the stock market which is determined by the stochastic parameter (v). This invariably leads to instability in the stock price.

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