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NUMERICAL APPLICATION OF CHEBYSHEV POLYNOMIAL FOR APPROXIMATION OF SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this study, we developed a new approximate method of solution for various heat equations. We study the numerical accuracy of the method. Detailed numerical results have shown that the method provides better results than the known explicit finite difference method. There is no semi-discretization involved and no reduction of PDE to a system of ODEs in the new approach, but rather a system of algebraic equations directly results.

Keywords: lines; multistep collocation; elliptic; Chebyshev's polynomial

1 Introduction

In this study, we will deal with a single parabolic partial differential equation in one space variable, where t and x are the time and space coordinates respectively, and the quantities h and k are the mesh sizes in the space and time directions.
We consider,

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}, \quad 0 \le x \le b,\ 0 \le t \le T \tag{1.0}$$

Subject to the initial and boundary conditions

$U(x,0)= f(x),\ 0 \le x \le b,$

$U(0,t)= g_1(t),\ \ t \ge 0$

$U(b,t)= g_2(t),\ t \ge 0$

We are interested in the development of numerical techniques for solving heat equations. Of recent, there is a growing interest concerning continuous numerical methods of solution for ODEs [1, 2]. We are interested in the extension of a particular continuous method to solve the heat equation. This is done based on the collocation and interpolation of the PDE directly over multi steps along lines but without reduction to a system of ODEs. We intend to avoid the cost of solving a large system of coupled ODEs often arising from the reduction method by a semi - discretization. The method also, eliminates the usual draw-back of stiffness arising in the conventional reduction method by semi-discretization [3,4].

2 The Solution Method

We subdivide the interval $0 \le x \le b$ into N equal subintervals by the grid points $x_m = mh, \quad m = 0,..., \ N$ where $Nh = b$. On these meshes we seek $l-$ step approximate solution to $U(x,t)$ of the form

$$U(x,t)=\sum_{r=0}^{p-2} a_r Q_r(x,t) \qquad x \in [x_m, x_{m+l}] \tag{2.0}$$

such that $0 = x_0 < ... < x_m < ... < x_N = b$. The basis function $Q_r(x,t), \ \ r = 0,..., p-2$ are assumed known, a_r are constants to be determined and $p \le l+s$, where s is the number of collocation points. The equality holds if the number of interpolation points used is equal to l. There will be flexibility in the choice of the basis function $Q_r(x,t)$ as may be desired for specific application. For this work, we consider the Taylor's polynomial $Q_r(x,t)= x^r t^r$. The interpolation values $U_{m,n},..., U_{m+l-1,n}$ are assumed to have been determined from previous steps, while the method seeks to obtain $U_{m+l,n}$ [5,6,7]. We apply the

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above interpolation conditions on eqn. (2.0) to obtain

$$a_0Q_0\left(x_{m+g},t_n\right)+\ldots+a_{p-2}Q_{p-2}\left(x_{m+g},t_n\right)=U\left(x_{m+g},t_n\right) \qquad g=-\frac{1}{2}\left(\frac{1}{2}\right)l-\frac{5}{2} \tag{2.1}$$

We can write eqn. (2.1) as a simple matrix equation in the augmented form as,

$$\begin{bmatrix} Q_0\left(x_{m-\frac{1}{2}},t_n\right) & \ldots & Q_{p-2}(x_{m-\frac{1}{2}},t_n) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ Q_0\left(x_{m+l-\frac{5}{2}},t_n\right) & \ldots & Q_{p-2}(x_{m+l-\frac{5}{2}},t_n) \end{bmatrix}\begin{bmatrix} a_0 \\ \ldots \\ \ldots \\ \ldots \\ a_{p-2}\end{bmatrix}=\begin{bmatrix} U(x_{m-\frac{1}{2}},t_n) \\ \ldots \\ \ldots \\ \ldots \\ U(x_{m+l-\frac{5}{2}},t_n)\end{bmatrix} \tag{2.2}$$

Using three interpolation points and one collocation point, implies that $s=1$, $p=4$, $l=3$ and $r=0,1,2$.
Substituting for p in eqn. (2.1) we have,

$$a_0Q_0\left(x_{m+g},t_n\right)+a_1Q_1\left(x_{m+g},t_n\right)+a_2Q_2\left(x_{m+g},t_n\right)=U_{m+g,n} \qquad g=-\frac{1}{2},0,\frac{1}{2} \tag{2.3}$$

Putting the values of g in eqn. (2.3) and writing it as matrix in augmented form we have,

$$\begin{bmatrix} Q_0\left(x_{m-\frac{1}{2}},t_n\right) & Q_1\left(x_{m-\frac{1}{2}},t_n\right) & Q_2(x_{m-\frac{1}{2}},t_n) \\ Q_0(x_m,t_n) & Q_1(x_m,t_n) & Q_2(x_m,t_n) \\ Q_0\left(x_{m+\frac{1}{2}},t_n\right) & Q_1\left(x_{m+\frac{1}{2}},t_n\right) & Q_2(x_{m+\frac{1}{2}},t_n) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_3\end{bmatrix}=\begin{bmatrix} U(x_{m-\frac{1}{2}},t_n) \\ U(x_m,t_n) \\ U(x_{m+\frac{1}{2}},t_n)\end{bmatrix} \tag{2.4}$$

From eqn. (2.4) we obtain the following values

$$Q_0\left(x_{m-\frac{1}{2}},t_n\right)=1 \qquad Q_1\left(x_{m-\frac{1}{2}},t_n\right)=x_{m-\frac{1}{2}}t_n \qquad Q_2(x_{m-\frac{1}{2}},t_n)=x^2{}_{m-\frac{1}{2}}t^2{}_n$$

$$Q_0\left(x_m,t_n\right)=1 \qquad Q_1\left(x_m,t_n\right)=x_mt_n \qquad Q_2\left(x_m,t_n\right)=x^2{}_mt^2{}_n$$

$$Q_0\left(x_{m+\frac{1}{2}},t_n\right)=1 \qquad Q_1\left(x_{m+\frac{1}{2}},t_n\right)=x_{m+\frac{1}{2}}t_n \qquad Q_2(x_{m+\frac{1}{2}},t_n)=x^2{}_{m+\frac{1}{2}}t_n^2$$

Putting the above values in eqn. (2.4) becomes

$$\begin{bmatrix} 1 & x_{m-\frac{1}{2}}t_n & x^2{}_{m-\frac{1}{2}}t^2{}_n \\ 1 & x_mt_n & x^2{}_mt^2{}_n \\ 1 & x_{m+\frac{1}{2}}t_n & x^2{}_{m+\frac{1}{2}}t^2{}_n \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2\end{bmatrix}=\begin{bmatrix} U_{m-\frac{1}{2},n} \\ U_{m,n} \\ U_{m+\frac{1}{2},n}\end{bmatrix} \tag{2.5}$$

We solve eqn. (2.5) to obtain the value of a_2 to be

$$a_2=\frac{2U_{m+\frac{1}{16},n}+2U_{m-\frac{1}{16},n}-4U_{m,n}}{h^2t^2{}_n},$$

When we substitute $r=0,1,2$ in eqn. (2.0) we obtain

$$U\left(x,t\right)=a_0Q_0+a_1Q_1+a_2Q_2 \tag{2.6}$$

By substitution of Q_0 Q_1 *and* Q_2 in eqn. (2.6) we obtain

$$U(x,t)=a_0+a_1xt+a_2x^2t^2 \tag{2.7}$$

Substituting the value of a_2 in eqn. (2.7) we have

$$U(x,t)=a_0+a_1xt+x^2t^2\left(\frac{2U_{m+\frac{1}{2},n}+2U_{m-\frac{1}{2},n}-4U_{m,n}}{h^2t^2{}_n}\right) \tag{2.8}$$

Taken the first and second derivatives of eqn. (2.8) with respect to x we have

$$U''(x,t) = t^2\left(\frac{2U_{m+\frac{1}{2},n} + 2U_{m-\frac{1}{2},n} - 4U_{m,n}}{h^2 t^2_n}\right), \tag{2.9}$$

we collocate eqn. (2.9) at $t = t_n$ to arrive at

$$U''(x,t) = \frac{2U_{m+\frac{1}{2},n} + 2U_{m-\frac{1}{2},n} - 4U_{m,n}}{h^2} \tag{2.10}$$

Similarly, we reverse the roles of x and t in eqn. (2.0), and we also subdivide the interval $0 \le t \le T$ into y equal subintervals by the grid points $t_n = nk$, $n = 0,..., y$ where $yk = T$. On these meshes we seek $l-$step approximate solution to $U(x,t)$ of the form

$$U(x,t) = \sum_{r=0}^{p-2} a_r Q_r(x,t) \qquad t \in [t_n, t_{n+l}] \tag{2.11}$$

Such that $0 = t_0 < ... < t_n < ... < t_y = T$, the basis function $Q_r(x,t)$, $r = 0,..., p-2$ are assumed known, a_r are constants to be determined and $p \le l + s$, where s is the number of collocation points. The equality holds if the number of interpolation points used is equal to l. There will be flexibility in the choice of the basis function $Q_r(x,t)$ as may be desired for specific application. For this method, we consider the Taylor's polynomial $Q_r(x,t) = x^r t^r$. The interpolation values $U_{m,n},..., U_{m,n+l-1}$ are assumed to have been determined from previous steps, while the method seeks to obtain $U_{m,n+l}$ [8, 9]. We apply the above interpolation conditions on Eqn. (2.11) to obtain

$$a_0 Q_0(x_m, t_{n+f}) + ... + a_{p-2} Q_{p-2}(x_m, t_{n+f}) = U(x_m, t_{n+f}) \quad f = 0\left(\frac{1}{2}\right)l - \frac{5}{2} \tag{2.12}$$

We can write (2.12) as a simple matrix equation in the augmented form as

$$\begin{bmatrix} Q_0(x_m,t_n) & ... & Q_{p-2}(x_m,t_n) \\ ... & ... & ... \\ ... & ... & ... \\ ... & ... & ... \\ Q_0(x_m,t_{n+l-\frac{5}{2}}) & ... & Q_{p-2}(x_m,t_{n+l-\frac{5}{2}}) \end{bmatrix} \begin{bmatrix} a_0 \\ ... \\ ... \\ ... \\ a_{p-2} \end{bmatrix} = \begin{bmatrix} U(x_m,t_n) \\ ... \\ ... \\ ... \\ U(x_m,t_{n+l-\frac{5}{2}}) \end{bmatrix} \tag{2.13}$$

Using two interpolation points and one collocation point in eqn. (2.13) implies that $p = 3, r = 0,1$ $l = 2$ and $f = 0, \frac{1}{2}$, and by substitution eqn.(2.13) becomes

$$\begin{bmatrix} Q_0(x_m,t_n) & Q_1(x_m,t_n) \\ Q_0\left(x_m,t_{n+\frac{1}{2}}\right) & Q_1\left(x_m,t_{n+\frac{1}{2}}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} U(x_m,t_n) \\ U(x_m,t_{n+\frac{1}{2}}) \end{bmatrix} \tag{2.14}$$

From eqn. (2.14) we obtain the following values:

$$\left.\begin{array}{ll} Q_0(x_m,t_n) = 1 & Q_1(x_m,t_n) = x_m t_n \\ Q_0(x_m,t_{n+\frac{1}{2}}) = 1 & Q_1(x_m,t_{n+\frac{1}{2}}) = x_m t_{n+\frac{1}{2}} \end{array}\right\} \tag{2.15}$$

Substituting the values of eqn. (2.15) into eqn. (2.14), we have this matrix below

$$\begin{bmatrix} 1 & x_m t_n \\ 1 & x_m t_{n+\frac{1}{2}} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} U_{m,n} \\ U_{m,n+\frac{1}{2}} \end{bmatrix} \tag{2.16}$$

Solving eqn. (2.16) for value of a_1 we obtain

$$a_1 = 16\,\frac{\left(U_{m,n+\frac{1}{2}} - U_{m,n}\right)}{kx_m}$$

we substitute $r = 0,1,$ into eqn.(2.11), we obtain

$$U(x,t) = a_0 Q_0 + a_1 Q_1 \tag{2.17}$$

By substituting the values of a_1, Q_0, Q_1 in equation (2.17) we have

$$U(x,t) = a_0 + 2xt\,\frac{\left(U_{m,n+\frac{1}{2}} - U_{m,n}\right)}{kx_m} \tag{2.18}$$

Taken the first derivatives of equation (2.18) with respect to t we obtain

$$U'(x,t) = 2x\left(\frac{U_{m,n+\frac{1}{2}} - U_{m,n}}{kx_m}\right) \tag{2.19}$$

We collocate eqn. (2.19) at $x = x_m$ yields

$$U'(x,t) = 2\,\frac{\left(U_{m,n+\frac{1}{2}} - U_{m,n}\right)}{k} \tag{2.20}$$

But from eqn. (1.0) we discovered that eqn. (2.20) is equal to eqn. (2.10), which implies that

$\dfrac{2U_{m,n+\frac{1}{2}} - 2U_{m,n}}{k} = \dfrac{2U_{m+\frac{1}{2},n} + 2U_{m-\frac{1}{2},n} - 4U_{m,n}}{h^2}$, manipulating mathematically and putting $r = \dfrac{k}{h^2}$, we obtain

$$U_{m,n+\frac{1}{2}} = (1-2r)U_{m,n} + 2r\left(U_{m+\frac{1}{2},n} + U_{m-\frac{1}{2},n}\right) \tag{2.21}$$

Eqn. (2.21) is a new scheme for solving the heat equation.
To illustrate this method, we use it to solve problems (5.1) and (5.2) respectively.

3. Advantages of the method

1) We intend to avoid the cost of solving a large system of coupled ODEs often arising from the reduction methods.
2) We also intend to eliminate the usual draw-back of stiffness arising in the conventional reduction method by semi-discretization.

4. Specific Problem

Example 4.1

Use the scheme to approximate the solution to the heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < t$$

$$U(0,t) = U(\pi,t) = 0,\ t > 0$$

$$U(x,0) = \sin x, \quad 0 \le x \le \pi$$

Table 1: Result of action of Eqn. (2.21) on problem 4.1

x	New Method $U(x,t)$	Schmidt Method $U(x,t)$	Exact Solution $U(x,t)$	Errors	
				New Method	Schmidt Method
0	0	0	0	0	0
$\frac{\pi}{6}$	0.46650635	0.47320508	0.466878559	3.7 X E-4	6.326522XE-3
$\frac{\pi}{3}$	0.808012701	0.819615241	0.808657385	6.4 X E-4	1.095785XE-2
$\frac{\pi}{2}$	0.933012701	0.946410161	0.933757118	7.4 X E-4	1.265300XE-2
$\frac{2\pi}{3}$	0.808012701	0.819615241	0.808657385	6.4 X E-4	1.095785XE-2
$\frac{5\pi}{6}$	0.46650635	0.47320508	0.466878559	3.7 X E-4	6.326522XE-3
π	0	0	0	0	0

Example 4.2

Use the scheme to approximate the solution to the heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 2$$

$$U(-1,t) = U(1,t) = 0, \qquad t > 0$$

$$U(x,0) = \cos\left(\frac{\pi x}{2}\right), \qquad -1 \le x \le 1, t = 0$$

Table 2: Result of action of Eqn. (2.21) on problem 4.2

x	New Method $U(x,t)$	Schmidt Method $U(x,t)$	Exact Solution $U(x,t)$	Errors	
				New Method	Schmidt Method
-1.00	0	0	0	0	0
-0.75	0.368125	0.371031	0.368211	8.5 X E-5	2.8204X E -3
-0.50	0.680200	0.685577	0.680364	1.6 X E-4	5.21266XE-3
-0.25	0.888725	0.895749	0.888939	2.1 x E-4	6.81008XE-3
0	0.9619500	0.969552	0.962181	2.3 X E-4	7.37081XE-3
0.25	0.888725	0.895749	0.888939	2.1 x E-4	6.81008XE-3
0.50	0.680200	0.685577	0.680364	1.6 X E-4	5.21266XE-3
0.75	0.3681250	0.371031	0.368211	8.5 X E-4	2.8204X E -3
1.00	0	0	0	0	0

5. Conclusion

A continuous interpolant is proposed for solving parabolic partial differential equation in one space variable without descretization. To check the numerical method, it is applied to solve two different test problems with known exact solutions. The numerical results confirm the validity of the new numerical scheme and suggested that it is an interesting and viable numerical method which does not involve the reduction of PDE to a system of ODEs.

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