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A NUMERICAL SCHEME FOR A FLUID DYNAMIC TRAFFIC FLOW MODEL WITH ONE-POINT BOUNDARY CONDITION

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Abstract

We modified the Greenberg model and used the modified version to predict density, velocity and flow profile at certain points of a highway using hypothetical initial and boundary condition data. We discovered that at any given time, there are hundreds of vehicles clustered within a given interval on our roadways.

***Keywords*:** Finite Difference Formula, Finite Difference Scheme, Numerical Simulation, Upwind Scheme, Early time behavior, Late time behavior, Traffic Flow Model.

Introduction

Traffic flow is a physical phenomenon that is complicated to model mathematically. Traffic flow models play an important role in both today's traffic research and many traffic applications such as traffic monitoring, flow prediction, incident detection, and traffic control. One of the important public transport problems especially in developing countries such as Nigeria is traffic jam, otherwise known as road congestion, and this is because in Nigeria, the basic traffic scenario is predominantly a one-dimensional road with one-way traffic. Nevertheless, engineers find this typical scenario easy to model due to its simplicity. Traffic congestion is one of the greatest problems in Nigeria like some other countries of the world. In this respect, countries managing traffic in congested networks require a good understanding of traffic operations [1]. Many models have been developed for the study of traffic flow problems such as traffic congestion, vehicle cluster and accidents. Traffic flow modeling and optimization have been traditionally studied under three main approaches; the Microscopic, Mesoscopic, and Macroscopic models. The first and most basic of the models, the microscopic or car following model describes each individual vehicle separately [2]. This approach considers driving behavior and vehicle dynamics. The problem with the microscopic model becomes mathematically intractable when a considerable volume of traffic flow is considered. One example of the microscopic approach is the GM family of car-following models developed in the 1960's [3]. In a mesoscopic model, individual vehicles with the same characteristics are grouped into a package. So, each vehicle within a package has the same origin and destination, the same route, the same driver characteristics, and so on. In that way the computation time needed for the simulation is reduced compared to microscopic models. The macroscopic models after its introduction in the 1950s by the work of Ligthill and Whitham [4], have seen an extensive attention over the years. In these models, individual vehicles are aggregated and described as fluid flows, which are then characterized by average spacemean speeds, densities, and flow rate or throughput (flux). In that way the computation time needed for the simulation is reduced even further, at the cost of accuracy. In general, there is therefore a trade-off between the accuracy of the model and the computation time required to simulate the model [5]. The macroscopic approach is adopted for this research.

We now discretize the one-dimensional traffic flow model [4] given as

$$\frac{\partial \rho}{\partial t}+\frac{\partial q(\rho)}{\partial x}=0 \qquad (1)$$

which describes a traffic wave propagating along the x-axis, with flux, $q(\rho)$ given as

$$(\rho)=\rho V(\rho) \qquad (2)$$

and a nonlinear velocity-density relationship of the form

$$V(\rho)=v_{max}\ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right] \qquad (3)$$

Which was used by [6] in a previous publication tagged "Analytical Solution of a Fluid Dynamic Traffic Flow Model Equation with an Initial Condition".

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Finite difference methods for first order nonlinear Partial differential equations (PDEs) are presented in [7], [1] and [8]. Based on their methods, we investigated a finite difference scheme for our modified Greenberg traffic flow model as an (IBVP) of the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left[\rho v_{max}\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)\right]=0 \qquad x\in(a,b)\,;\,t\in(0,T) \tag{4}$$

$$with\ I.C \qquad (x,0)=\rho_0(x) \tag{5}$$

$$and\ B.C \qquad \rho\,(a,t)=\rho_a(t) \tag{6}$$

We develop the scheme by discretization of the space and time. The discretization of $\frac{\partial\rho(x,t)}{\partial t}$ is obtained by first order forward difference in time and the discretization of $\frac{\partial\rho}{\partial t}$ is obtained by first order backward difference in space.

Accordingly, we have $\frac{\partial\rho}{\partial t}$ as

$$\rho(x,t+h)=\rho(x,t)+h\frac{\partial\rho(x,t)}{\partial t}+\frac{h^2}{2!}\frac{\partial^2\rho(x,t)}{\partial t^2}+\frac{h^3}{3!}\frac{\partial^3\rho(x,t)}{\partial t^3}+\ldots$$

$$\therefore\ \frac{\partial\rho(x,t)}{\partial t}=\frac{\rho(x,t+h)-\rho(x,t)}{h} \tag{7}$$

Similarly, for the backward difference approximation in space for $\frac{\partial\rho}{\partial t}$ we have

$$\frac{\partial\rho(x,t)}{\partial x}\approx\frac{\rho(x,t)-\rho(x-k,t)}{k} \tag{8}$$

At any discrete point (t_n,x_i) for i=1,2,…,M; n= 0,1,…,N-1, we assume a fixed, regular mesh (grid) spacing with $h=t^{n+1}-t^n$ and $k=x_{i+1}-x_i$ representing respectively, the time step size and the spatial mesh size for the forward and backward difference formula (7) and (8).

If we write $\rho(t_n,x_i)\approx\rho_i^n$, then equation (7) and (8) become

$$\frac{\partial\rho(x_i,t_n)}{\partial x}\approx\frac{\rho_i^{n+1}-\rho_i^n}{\Delta x} \tag{9}$$

$$\frac{\partial\rho(x_i,t_n)}{\partial x}\approx\frac{\rho_i^n-\rho_{i-1}^n}{\Delta x} \tag{10}$$

Substituting (9) and (10) into (1) we have

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\frac{q_i^n-q_{i-1}^n}{\Delta x}=0 \tag{11}$$

$$\Rightarrow\rho_i^{n+1}=\rho_i^n-\frac{\Delta t}{\Delta x}(q_i^n-q_{i-1}^n); \qquad i=1,2,\ldots M;\ n=0,1,\ldots,N-1 \tag{12}$$

$$\text{Where } q_i^n=\rho_i^n\,.\,v_{max}\ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho_i^n}\right)^2\right] \tag{13}$$

This is the explicit upwind difference scheme for the equation (4) to (6). Therefore, equation (12), gives us the desired numerical scheme for our traffic model.

Stability Analysis and Physical Constraints Conditions

The model we are considering is a one – dimensional equation. Since the car is moving in one direction, the characteristic speed $\frac{\partial q}{\partial\rho}$ must be positive.

i.e $q'(\rho)=v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)-2\right]>0$

$$\Rightarrow v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)-2\right]>0$$

$$\rho_{max}\geq\rho\sqrt{2e^2} \tag{14}$$

$$\therefore q'(\rho)\leq v_{max} \tag{15}$$

Proposition 4.1: the stability and physical constraint condition of the explicit upwind difference scheme (12) is given by the simultaneous conditions respectively

$$\gamma=\frac{v_{max\Delta t}}{\Delta x}\leq 1\ and\ \rho_{max}=\max_i\rho_0(x_i);\ c\geq\sqrt{2e^2}$$

Proof: We recall the traffic flow model (1) as $\frac{\partial\rho}{\partial t}+\frac{\partial q(\rho)}{\partial x}=0$ (16)

with $q(\rho)=\rho V(\rho)=\rho V_{max}\ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]$

The PDE (16) can be written as

$\frac{\partial \rho}{\partial t}+q'(\rho)\frac{\partial \rho}{\partial x}=0$

Where $q'(\rho)=v_{max}\left[\ln\left(\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right)-2\right]$

Then the explicit upwind difference scheme (12) takes the form

$\rho_i^{n+1}-\rho_i^n = -q'(\rho_i^n)\,\frac{\Delta t}{\Delta x}\,[\rho_i^n-\rho_{i-1}^n]$

$\Rightarrow \rho_i^{n+1}=(1-\lambda)\,\rho_i^n+\lambda\rho_{i-1}^n$ (17)

Where $\lambda=q'\,(\rho_i^n)\,\frac{\Delta t}{\Delta x}$ (18)

Equation (17) implies that if $0\leq\lambda\leq 1$, the new solution is a convex combination of two previous solutions. That is, the solution ρ_i^{n+1} at new time step (n+1) at a spatial-vertex (node) i, is an average of the solutions at the previous time- step at the spatial-vertices i and i - 1. This means that the extreme values of the new solution ρ_i^{n+1} is the average of the extreme values of the previous two solutions at the two consecutive vertices. This means that the new solution continuously depends on the initial value ρ_i^0 for all i=1,2 ,…, M and the explicit upwind difference scheme is stable if

$\lambda=q'\,(\rho_i^n)\,\frac{\Delta t}{\Delta x}\leq 1$ (19)

Then by the condition in equation (15) the stability condition (19) can be guaranteed by

$\gamma=\frac{V_{max}\Delta t}{\Delta x}\leq 1$ (20)

In conclusion, whenever we employ the stability condition (20), the physical constraints condition (14) can be guaranteed instantaneously by choosing

$\rho_{max}=\max_i \rho_0(x_i);\ c\geq\sqrt{2e^2}$ (21)

Discussion of the Numerical Results

A numerical experiment was performed to apply the upwind difference scheme (12) for the modified model in order to obtain the density and the velocity profile. We developed a computer programs in MATLAB for the implementation of the numerical scheme and conducted some numerical experiments to verify some flow behavior for various traffic parameters. In order to use the scheme, we have made the following assumptions:

(i) The highway is of 20km range.

(ii) The number of vehicles at various points at a particular time are the initial data and constant boundary $\rho\ (0,\,t)=44/0.1$km.

(iii)This research considered vehicles within the length of 4.0m

(iv)The initial condition $\rho\ (x,0)=13$veh/km is set arbitrarily.

Case 1: Given the boundary and initial condition with maximum speed $V_{max}=50$km/hr, ρ_{max} =250veh/km in a spatial domain [0km, 20km], we perform the numerical experiment for 2, 4, 16 and 20 minutes respectively with $\Delta t=1$ time steps for a highway of 20km with step size $\Delta x=50$m. We wanted to simulate the early time traffic flow for two and four minutes and the late time traffic flow for sixteen and twenty minutes respectively. Fig 1 and Fig 2, show the early time propagating traffic wave density at 2 and 4 minutes respectively, and Fig 3 and Fig4 represents the late time traffic wave density at 16 and 20 minutes respectively. We saw in Fig 1 and Fig 2 that the traffic curve somewhat maintained a uniform density while after 20 minutes have passed, a spatial oscillation (cluster) in the density flow Fig 3 and Fig 4) near source ($3\leq x\leq 8$) was observed. The density cluster can be as a result of road bumps or accident.

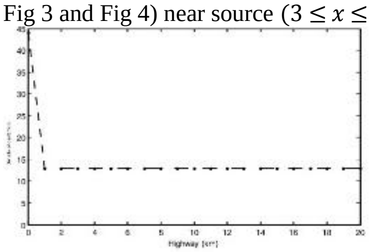

Fig. 1. Early Time Density of car for 2 minutes

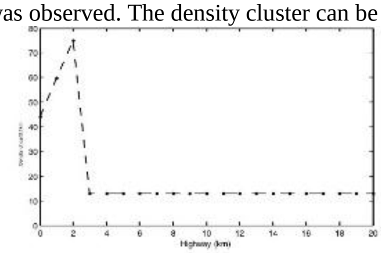

Fig. 2. Early Time Density of car for 4 minutes

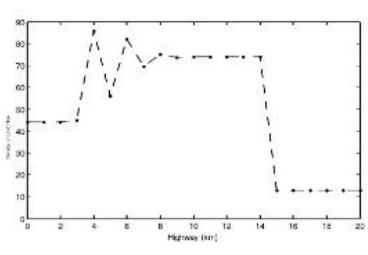

Fig. 3. Late Time Density of car for 16 minutes

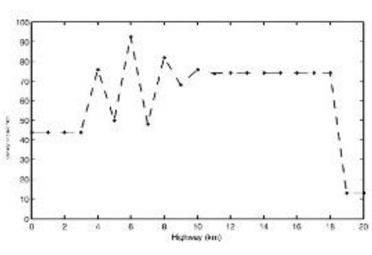

Fig. 4. Late Time Density of car for 20 minutes

Fig 5, Fig 6 and Fig 7, Fig 8 represent respectively the computed early time and late time velocity profile according to certain points of the highway. The velocity is computed by the following relation $= \rho V_{max} \ln\left[\frac{1}{2}\left(\frac{\rho_{max}}{\rho}\right)^2\right]$. We observed from Fig 3, Fig 4, Fig 7 and Fig8 that the density and velocity are maintaining the inverse relation as given by equation(3) throughout the computational process as required. Since we know the density and speed (velocity) for certain points, we calculated the early time and late time flux with the aid of the relation, $q\ (\rho) = \rho * V(\rho)$. Fig (9), Fig (10), Fig (11) and Fig (12) shows the computed flux with respect to distance.

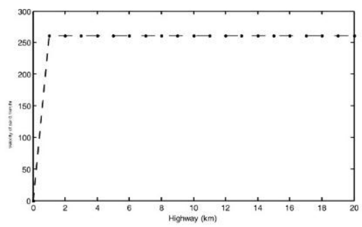

Fig. 5. Early Time Velocity of car for 2 minutes

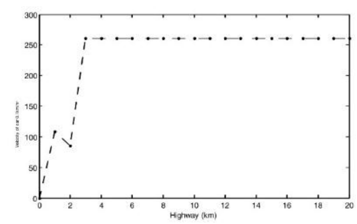

Fig. 6. Early Time Velocity of car for 4 minutes

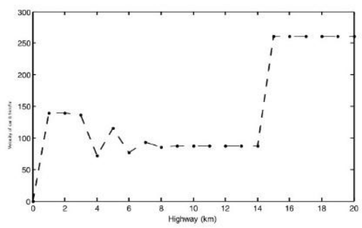

Fig. 7. Late Time Velocity of car for 16 minutes

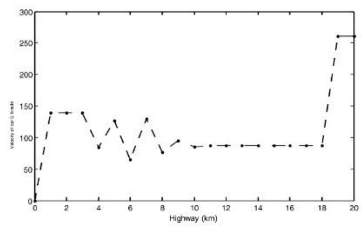

Fig. 8. Late Time Velocity of car for 20 minutes

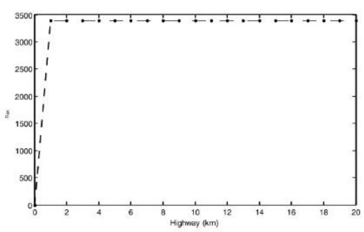

Fig.9. Early TimeTraffic Flux for 2 minutes

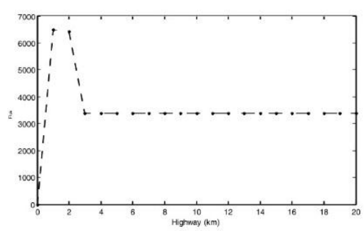

Fig.10. Early TimeTraffic Flux for 4 minutes

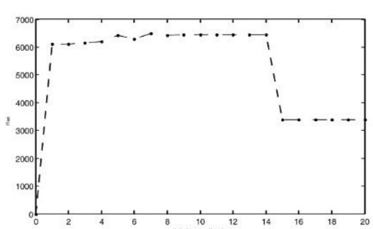

Fig.11. Late TimeTraffic Flux for 16 minutes

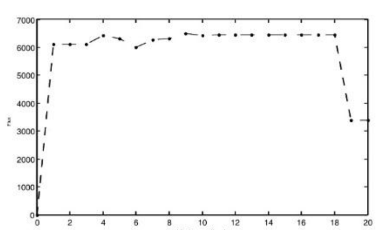

Fig.12. Late Time Traffic Flux for 20 minutes

Case2: The density and velocity profiles for the modified traffic flow model when the parameter $V_{max} = 75$km/hr is increase is presented in Fig 13, Fig 14, Fig 15, Fig 16, Fig 17, Fig 18, Fig 19 and Fig 20. We discovered that when the maximum speed V_{max} which was previously 50km/hr in **case1** is increase to 75km/hr on a highway of 20km, vehicles can travel with such speed for the time range of (0-6) minutes as seen in Fig 13 and Fig 14. After 6minutes have passed, that is as from the 7th minutes, the density and the velocity profiles (Fig 15, Fig 16 and Fig 19, Fig 20), where $V_{max} = 75$km/hr goes negative. This indicates that a car cannot travel constantly on that same speed on a highway of 20km, else it will hit other vehicles. This model has therefore allows for sensitivity analysis.

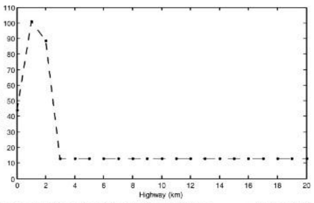

Fig.13. Density of car for 4 minutes (where = 75km/hr)

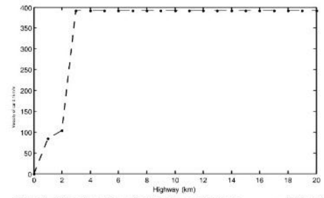

Fig.17. Velocity of car for 4 minutes (where = 75km/hr)

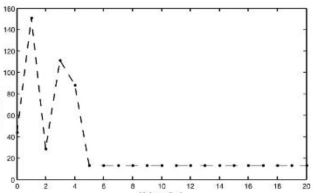

Fig.14. Density of car for 6 minutes (where = 75km/hr)

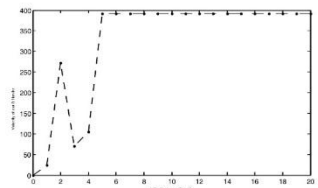

Fig.18. Velocity of car for 6 minutes (where = 75km/hr)

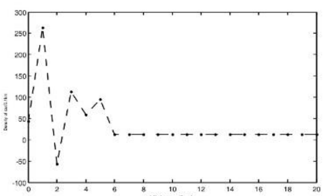

Fig.15. Density of car for 7 minutes (where = 75km/hr)

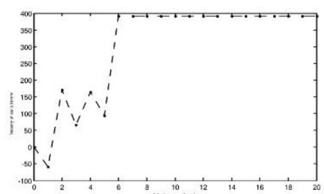

Fig.19. Velocity of car for 7 minutes (where = 75km/hr)

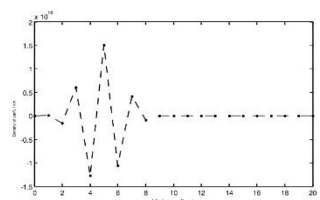

Fig.16. Density of car for 16 minutes (where = 75km/hr)

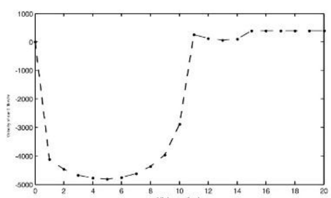

Fig.20. Velocity of car for 16 minutes (where = 75km/hr)

Case3: We re-set the initial and boundary condition. The new value for the initial and boundary condition is set as $\rho\ (x,0) = 47$veh/km and $\rho\ (0,\ t) = 65$veh/km as against $\rho\ (x,\ 0) = 13$veh/km and $\rho\ (0,\ t) = 44$veh/km. The density curve (Fig 21) sampled at the 16th minutes appears quite good and practicable and also appears as an exponential distribution. The density and speed at which the flow (Fig 23) is maximum (6500veh/hr) is 65veh/km (Fig 21) and 102km/hr (Fig 22) respectively. Though our model could not overcome the initial criticism of Greenberg model [9].

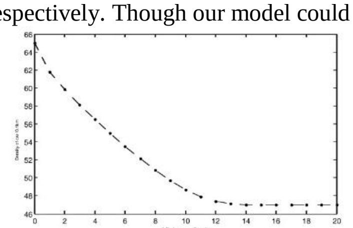

Fig. 21. Density of car sampled at the 16th minutes with new initial and boundary 47veh/km and $(0,t) = 65$veh/km

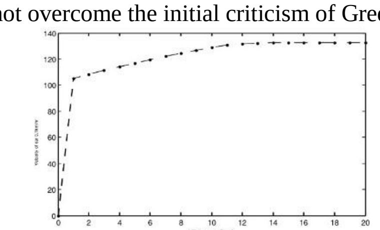

Fig. 22. Velocity of car sampled at the 16th minutes with new initial and condition as $(x,0) =$ boundary condition as $\rho(x,0) = 47$veh/km and $\rho(0,t) = 65$veh/km

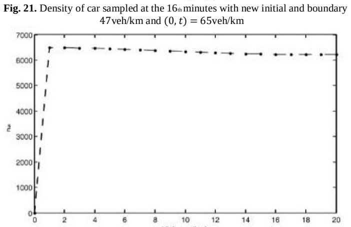

Figure 23 Traffic Flux sampled at the 16th minutes with new initial and boundary condition as $\rho(x,0) = 47$veh/km and $\rho(0,t) = 65$veh/km

Conclusion and Recommendation

The model has been presented as an IBVP and we derived the explicit upwind difference scheme for the modified traffic flow model and have used it to predict density, velocity and flow profile (flux) at certain points of a highway using hypothetical initial and boundary data. We have obtained the stability condition of the numerical scheme. Computer programs in MATLAB has been developed and used to implement the upwind difference scheme and we have verified the behavior of the difference flow variables of the traffic flow model. Based on the findings in this research work and the evaluation of the traffic flow, we discovered that at any given time, there are hundreds of vehicles clustered within a given interval on our roadways. This vehicle interact with each other and impact the overall movement of traffic flow. The flow rate (flux) with respect to some distance in an interrupted flow, where flow is regulated by an external means such as traffic signal is very high. Such high flow rate causes a high traffic jam and thereby delay movement. Our model is a one-dimensional traffic flow model which gives these results and therefore suggests that a multi-lane traffic flow will reduce the problems associated with one dimensional traffic flow model.

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