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SOLUTION OF FIRST-ORDER LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATION BY METHOD OF LAPLACE TRANSFORM

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Abstract

The exponential matrix $e^{\propto}$ plays a central role in linear system and control theory. In this work, it is shown that by method of Laplace transform, a differential matrix can have meaning only when we considered the infinite series and taking the powers of the matrix, this can be achieved only when the matrix is a diagonal matrix. We give an example to illustrate our result.

Keywords: exponential matrix, Laplace transforms, First-order linear system of equation.

1. Introduction

It is obvious that some of the basic functions such as exponential functions, polynomial functions, logarithmic functions, as well as trigonometric functions have been useful in the development of Mathematical, Biological, Physical as well as Engineering sciences [1], the exponential matrix have been found to be very useful tool on solving linear systems of first order, it provides a method for closed solutions, with the help of this, we can analyse controllability and observability of a linear system [2], there are different methods for calculating the exponential matrix, neither computationally efficient [3], series methods [4], differential methods [5], matrix decomposition methods [6], polynomial methods [7], as well as splitting methods [8], none of which is entirely satisfactory from either a theorical or a computational point of view. Techniques involving the calculation of generalized Laplce transforms and eigenvectors have been used in some extend. The motivation of the study is based on the fact that in illustrating several methods to calculate the exponential matrix of a square matrix, most of the methods do not use the calculation of eigenvalues which has been a standard method of tackling the problem in several types at the level of initiation; this is because some of the methods are ineffective when dealing with large matrices or inaccurate entries.

2. Definition and Main Result

Definition 1

Let $\propto\in R^{n\times n}$, the exponential of $\propto$, denoted by $e^{\propto}$ is the $n\times n$ matrix given by the power series

$$e^{\propto}=I+\propto+\frac{1}{2!}\propto^2+\frac{1}{3!}\propto^3+\frac{1}{4!}\propto^4+\cdots+\frac{1}{(n-1)!}\propto^{n-1}=\sum_{n=0}^{\infty}\frac{1}{n!}\propto^n \quad (1)$$

$$\|e^{\propto}\|=\left\|I+\propto+\frac{1}{2!}\propto^2+\frac{1}{3!}\propto^3+\frac{1}{4!}\propto^4+\cdots+\frac{1}{(n-1)!}\propto^{n-1}\right\|=\left\|\sum_{n=0}^{\infty}\frac{1}{n!}\propto^n\right\|\leq\sum_{n=0}^{\infty}\left\|\frac{1}{n!}\propto^n\right\|$$

$$\leq 1+\|\propto\|+\frac{1}{2!}\|\propto\|^2+\frac{1}{3!}\|\propto\|^3+\frac{1}{4!}\|\propto\|^4+\cdots+\frac{1}{(n-1)!}\|\propto\|^{n-1}=e^{\|\propto\|} \quad (2)$$

The series in (1) converges absolutely for all $\propto\in R^{n\times n}$. Moreover, let $\|.\|$ be a normalized sub multiplicative norm, then $\|e^{\propto}\|\leq e^{\|\propto\|}$.

Proof:

The nth partial sum is

$$S_n=\sum_{k=0}^{\infty}\frac{1}{k!}\propto^k$$

So that

$$\|e^{\propto}-S_n\|\equiv\left\|\sum_{k=0}^{\infty}\frac{1}{k!}\propto^k-\sum_{k=0}^{m}\frac{1}{k!}\propto^k\right\|=\left\|\sum_{k=m+1}^{\infty}\frac{1}{k!}\propto^k\right\|\leq\sum_{k=m+1}^{\infty}\left\|\frac{1}{k!}\propto^k\right\|\leq\sum_{k=m+1}^{\infty}\frac{1}{k!}\|\propto\|^k \quad (3)$$

Since $\|\propto\|$ is a real number and the right-hand side is a part of the convergent series of real numbers, it follows that

$$e^{\|\propto\|}=\sum_{k=0}^{\infty}\frac{1}{k!}\|\propto\|^k \quad (4)$$

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Then the equation (4) is convergent, also,

$$\|e^{\propto}\| = \left\|\sum_{k=0}^{\infty}\frac{1}{k!}\propto^k\right\| \leq \sum_{k=0}^{\infty}\frac{1}{k!}\|\propto^k\| \leq \sum_{k=0}^{\infty}\frac{1}{k!}\|\propto\|^k = e^{\|\propto\|} \qquad (5)$$

Letting $\propto = At$, from equation (1), we have that

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots + \frac{1}{n!}A^nt^n + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^nt^n \qquad (6)$$

$$\Rightarrow \frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots + \frac{1}{(n-1)!}A^2t^{n-1} + \cdots$$

$$= A\left(1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots\right) = Ae^{At}$$

Note that, to make meaning of e^{At}, we need to make powers of the matrix A to any order. This can be cumbersome except we employ diagonal matrices. Note also that, for us to diagonalized any none diagonal matrix, we employ the following theorems.

Theorem 1 (Schur Triangularization Theorem).

Let $A^{n\times n}$ be a square matrix, there is an unitary matrix say P a non-diagonal matrix, such that $PBP^{-1} = A$ is upper triangular, where the entries B is diagonal matrix consisting of the eigenvalues of the matrix A.

Theorem 2 (Cayley Hamilton Theorem)

For any square matrix $A^{n\times n}$, and $\Delta(\lambda) = |A - \lambda I|$ its characteristic polynomial, then $\Delta(A) = 0$.

Definition 2 (Diagonalizable Matrix)

Let $D = diag(\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n)$ be a $n \times n$ diagonal matrix, it is to say that $D^k = diag\left({\lambda_1}^k, {\lambda_2}^k, {\lambda_3}^k, \dots {\lambda_n}^k\right)$ is true for all $k \in \mathbb{Z}^+$, then we have

$$e^{At} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = \sum_{k=0}^{\infty}\frac{t^k}{k!}(PD^kP^{-1}) = P\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}D^k\right)P^{-1} = Pe^{Dt}P^{-1}$$

Definition 3 (None Diagonalizable Matrix)

Suppose matrix A is not diagonalizable matrix which it is not possible to find n linearly independent eigenvectors of the matrix A. A matrix is said to be none diagonalizable if it is not diagonalizable. A square matrix that is not diagonalizable is called defective.

Suppose A is a 3 by 3 diagonal matrix, i.e

$$\text{suppose } A = \begin{pmatrix} \propto & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}_{3\times 3} \qquad (7)$$

it is shown by Mathematical induction that

$$A^n = \begin{pmatrix} \propto & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}^n_{3\times 3} = \begin{pmatrix} \propto^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & \gamma^n \end{pmatrix}_{3\times 3} \qquad (8)$$

$$\Rightarrow e^{At} = \sum_{n=0}^{\infty}\frac{A^n}{n!}t^n = \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\begin{pmatrix} \propto & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}^n t^n = \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\begin{pmatrix} \propto^n & 0 & 0 \\ 0 & \beta^n & 0 \\ 0 & 0 & \gamma^n \end{pmatrix} t^n = \begin{pmatrix} \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\propto^n t^n & 0 & 0 \\ 0 & \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\beta^n t^n & 0 \\ 0 & 0 & \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{n!}\gamma^n t^n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty}\frac{1}{n!}\propto^n t^n & 0 & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{1}{n!}\beta^n t^n & 0 \\ 0 & 0 & \sum_{n=0}^{\infty}\frac{1}{n!}\gamma^n t^n \end{pmatrix} = \begin{pmatrix} e^{\propto t} & 0 & 0 \\ 0 & e^{\beta t} & 0 \\ 0 & 0 & e^{\gamma t} \end{pmatrix}$$

$$\therefore e^{At} = \begin{pmatrix} e^{\propto t} & 0 & 0 \\ 0 & e^{\beta t} & 0 \\ 0 & 0 & e^{\gamma t} \end{pmatrix} \qquad (9)$$

Thus, for the linear system,
$\dot{x} = Ax, x(0) = x_0$, where $A = (a_{ij})_{3\times3}$ and x a vector

$$e^{At}x_0 = \begin{pmatrix} e^{\propto t} & 0 & 0 \\ 0 & e^{\beta t} & 0 \\ 0 & 0 & e^{\gamma t} \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \\ x_{03} \end{pmatrix} = \begin{pmatrix} x_{01}e^{\propto t} \\ x_{02}e^{\beta t} \\ x_{03}e^{\gamma t} \end{pmatrix} \quad (10)$$

If we consider the initial value problem $\dot{x} = Ax, x(0) = x_0$ (11)
Solving the above problem, where A is an $n \times n$ and $x \in \mathbb{R}^n$,

$$\dot{x} = Ax \Rightarrow \dot{x} - Ax = 0 \Rightarrow \frac{dx}{dt} - Ax = 0 \quad (12)$$

by integrating factor method, we have $IF = e^{-\int Adt} = e^{-At}$
multiplying (12) by the integrating factor, we have

$$e^{-At}\frac{dx}{dt} - Axe^{-At} = 0 \Rightarrow \frac{d}{dt}(e^{-At}x) = 0 \Rightarrow \int \frac{d}{dt}(e^{-At}x)dt = \int 0dt + c \Rightarrow (e^{-At}x) = c$$

$$\Rightarrow x(t) = ce^{At} \quad (13)$$

But, $x(0) = x_0 \Rightarrow x(0) = ce^0 = c \Rightarrow c = x_0$

$$\therefore x(t) = x_0 e^{At} \quad (14)$$

Taking the Laplace transform of (12), gives

$$L\{\dot{x}(t)\} - L\{Ax(t)\} = L\{0\} \quad (15)$$

Where, $L\{\dot{x}(t)\} = SL\{x(t)\} - x(0)$, and $x(0) = x_0$
let $L\{x(t)\} = \emptyset(s)$, then equation (14) becomes

$$S\emptyset(s) - x_0 - A\emptyset(s) = 0 \Rightarrow S\emptyset(s) - A\emptyset(s) = x_0 \quad (16)$$

$\Rightarrow (SI - A)\emptyset(s) = x_0$, where I is an identity matrix

$$\Rightarrow \emptyset(s) = (SI - A)^{-1}x_0 \Rightarrow L\{x(t)\} = (SI - A)^{-1}x_0$$

$$\therefore x(t) = L^{-1}\{(SI - A)^{-1}\}x_0 \quad (17)$$

Comparing (14) and (17), we have

$$x(t) = x_0 e^{At} = L^{-1}\{(SI - A)^{-1}\}x_0 \quad (18)$$

Where $x(t) = L^{-1}\{(SI - A)^{-1}\} = e^{At}$ is a state-transition matrix

$$\therefore e^{At} = L^{-1}\{(SI - A)^{-1}\} \quad (19)$$

Example. Find the general solution to the system

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}x \quad (20)$$

Solution

$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}x \Rightarrow \dot{x} = Ax$, where $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, using the fact that $e^{At} = L^{-1}\{(SI - A)^{-1}\}$, we have that

$$(SI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 & 1 \\ 0 & S-1 & 0 \\ 1 & 0 & S+1 \end{bmatrix} \quad (21)$$

$$|(SI - A)| = \begin{vmatrix} S-1 & 0 & 1 \\ 0 & S-1 & 0 \\ 1 & 0 & S+1 \end{vmatrix} = (S-1)[(S-1)(S+1)] + 1[0 - (S-1)]$$

$$= (S-1)(S^2 + S - S - 1) + (-S + 1) = (S-1)(S^2 - 1) - (S-1)$$

$$= (S-1)(S^2 - 1 - 1) = (S-1)(S^2 - 2)$$

$$(SI - A)^{-1} = \frac{1}{(S-1)(S^2-2)}\begin{bmatrix} S^2 - 1 & 0 & -(S-1) \\ 0 & S^2 - 2 & 0 \\ -(S-1) & 0 & (S-1)^2 \end{bmatrix} = \begin{bmatrix} \frac{S+1}{S^2-2} & 0 & \frac{-1}{S^2-2} \\ 0 & \frac{1}{S-1} & 0 \\ \frac{-1}{S^2-2} & 0 & \frac{S-1}{S^2-2} \end{bmatrix}$$

$$\therefore L^{-1}\{(SI - A)^{-1}\} = L^{-1}\begin{bmatrix} \frac{S+1}{S^2-2} & 0 & \frac{-1}{S^2-2} \\ 0 & \frac{1}{S-1} & 0 \\ \frac{-1}{S^2-2} & 0 & \frac{S-1}{S^2-2} \end{bmatrix} = \begin{bmatrix} L^{-1}\left\{\frac{S+1}{S^2-2}\right\} & 0 & L^{-1}\left\{\frac{-1}{S^2-2}\right\} \\ 0 & L^{-1}\left\{\frac{1}{S-1}\right\} & 0 \\ L^{-1}\left\{\frac{-1}{S^2-2}\right\} & 0 & L^{-1}\left\{\frac{S-1}{S^2-2}\right\} \end{bmatrix} \quad (22)$$

Where,

$L^{-1}\left\{\frac{S+1}{S^2-2}\right\} = L^{-1}\left\{\frac{S}{S^2-2}\right\} + L^{-1}\left\{\frac{1}{S^2-2}\right\} = L^{-1}\left\{\frac{S}{S^2-(\sqrt{2})^2}\right\} + L^{-1}\left\{\frac{1}{S^2-(\sqrt{2})^2}\right\} = L^{-1}\left\{\frac{S}{S^2-(\sqrt{2})^2}\right\} + \frac{1}{\sqrt{2}}L^{-1}\left\{\frac{\sqrt{2}}{S^2-(\sqrt{2})^2}\right\} = cosh\sqrt{2}t + \frac{1}{\sqrt{2}}sinh\sqrt{2}t$, $L^{-1}\left\{\frac{-1}{S^2-2}\right\} = -\frac{1}{\sqrt{2}}sinh\sqrt{2}t$, $L^{-1}\left\{\frac{1}{S-1}\right\} = e^t$, $L^{-1}\left\{\frac{-1}{S^2-2}\right\} = cosh\sqrt{2}t - \frac{1}{\sqrt{2}}sinh\sqrt{2}t$

$$\therefore L^{-1}\{(SI-A)^{-1}\} = \begin{bmatrix} cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t & 0 & -\frac{1}{\sqrt{2}}sinh\sqrt{2}t \\ 0 & e^t & 0 \\ -\frac{1}{\sqrt{2}}sinh\sqrt{2}t & 0 & cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t \end{bmatrix} \quad (23)$$

But, $x\,(t) = e^{At}x_0$, where $e^{At} = L^{-1}\{(SI-A)^{-1}\}$, let $x_0 = (x_{01}, x_{02}, x_{03})^T$, then the general solution of the system is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{bmatrix} cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t & 0 & -\frac{1}{\sqrt{2}}sinh\sqrt{2}t \\ 0 & e^t & 0 \\ -\frac{1}{\sqrt{2}}sinh\sqrt{2}t & 0 & cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} \quad (24)$$

where

$x_1(t) = \left(cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t\right)x_{01} - \frac{1}{\sqrt{2}}x_{03}sinh\sqrt{2}t,$

$x_2(t) = x_{02}e^t$ and

$x_3(t) = -\frac{1}{\sqrt{2}}x_{01}sinh\sqrt{2}t + \left(cosh\sqrt{2}t + \frac{1}{2}sinh\sqrt{2}t\right)x_{03}$

3. Conclusion

The solution of the differential equation $\dot{x}(t) = Ax(t)$ plays an important role in linear system and control theory. It is known that e^{At} can be defined by a convergent power series $e^{At} = \sum_{k=0}^{\infty}\frac{(At)^k}{j!}$. The infinite series $\sum_{k=0}^{\infty}\frac{(At)^k}{j!}$, makes researchers design accurate controllers difficultly in theory and application, so is it important to develop a frame work to get the accurate solution of the exponential matrix e^{At}.

References

[1]. Mothmmed, A.S.S., Broker, V.C. (2016). Exponential matrix and their properties. International Journal of Scientific and Innovative Mathematical Research. Vol. 4, pp. 53-63.

[2]. Dhaigude, D.B., Dhaigude, S.B. (2013). Prefunctions and System of Differential Equation via Laplace Transforms. Journal of Mathematics and Computer Science, Vol. 7, pp. 293-304.

[3]. Horn, R.A., Johnson, C.R. (1985). Matrix Analysis. Cambridge University Press, ISBN 0-521-30586-2, pp. 25-60.

[4]. Plou, S.H. and Pang, W.K. (2006). On the matrix exponential function. Internal Journal of Mathematical Education in Science and Technology, Vol. 37, pp. 70.

[5]. Al Zhour, Z.A., Kilicman, A. (2007). Some new connections between matrix products for portioned and non-partitioned matrices. Computers and Mathematics with Applications. 54(6), 763-784.

[6]. Ding, F. and Chen, T. (2006). On initiative solutions of general coupled matrix equations. SIAM Journal on Control and Optimization. 44(6), 2269-2284.

[7]. Ding, F. (2010). Transformation between some special matrices. Computers and Mathematics with Applications. 59(8), 2676-2695.

[8]. Leonard, I.E. (1996). The matrix exponential. Society for Industrial and Applied Mathematics. Vol. 38, No. 3, pp. 507-512.