Journal of the Nigerian Association of Mathematical Physics
Volume 60 **(April – June** ***2021 Issue*****), pp9 –14**

NON-RELATIVISTIC STUDY OF GENERALIZED YUKAWA POTENTIAL TO PREDICT THE MASS-SPECTRA OF HEAVY QUARKONIUM SYSTEM

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Abstract

We solved the radial Schrödinger equation analytically using the series expansion method to obtain the energy eigenvalues and unnormalized wavefunction with the Generalized Yukawa potential (GYP). The present results are applied for calculating the mass spectra of heavy mesons such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) for different quantum states. The present potential provides excellent results in comparison with experimental data with a maximum error of $0.0023\ GeV$ ***and work of other researchers.***

Keywords**:** Generalized Yukawa potential; Schrödinger equation; Heavy Quarkonium; Series Expansion method

1. Introduction

The solution of the Schrödinger equation (SE) for a physical system in quantum mechanics is of great importance, because the knowledge of Eigen energy and wavefunction contains all possible information about the physical properties of a system under study [1]. The study of behavior of several physical problems in physics requires us to solve the SE or Klein-Gordon equation (KGE). The solutions can be established only if we know the confining potential for a particular physical system [2]. The confining potentials may be in different forms depending upon the interaction of the particles within the system. Harmonic oscillator and hydrogen atom are the two potentials which solutions to the SE are found exactly. On the other hand, to obtain the approximate solutions, some techniques are employed. Example of such techniques include, asymptotic iteration method(AIM)[3,4], Laplace transformation method [5], super symmetric quantum mechanics method (SUSQM)[6-8], Nikiforov-Uvarov(NU) method [9-24],series expansion method (SEM) [25-29], analytical exact iterative method(AEIM)[30], and others [31]. The potential model used in predicting mass spectra of heavy mesons is the so called Cornell potential which contain two important terms. A confinement term and Coulomb term [32]. In past, this type of potential has been studied by many researchers using different techniques [33-40].The generalized Yukawa potential (GYP) takes the form [41],

$$V(r) = -a\frac{e^{-\alpha r}}{r} + \frac{b}{r^2} \tag{1}$$

where a and b are potential strength, r is interparticle distance and α is a potential range. It application cut across Nuclear and particle physics. Many researchers have obtained bond state solutions of relativistic and non- relativistic regime of GYP [42-45].

With this in mind, we aim at studying the SE with GYP using the series expansion method to obtain the mass spectra of heavy mesons such as charmonium and bottomonium. To the best of our knowledge this study is not in literature. The study will be carried out in three folds. We will first model the GYP to behave like the Cornell potential, thereafter we obtain the solutions via series expansion method and finally we obtain the mass spectra of heavy mesons.

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2. Approximate solutions of the Schrödinger equation with generalized Yukawa potential model

We consider the radial SE of the form [26],

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{2}$$

where l is angular quantum number taking the values 0,1,2,3,4…, μ is the reduced mass for the quarkonium particle, r is the inter-nuclear separation and E_{nl} denotes the energy eigenvalues of the system.

We carry out series expansion of the exponential term in Eq.(1) up to order three, in order to model the potential to interact in the quark-antiquark system and this yields,

$$\frac{e^{-\alpha r}}{r}=\frac{1}{r}-\alpha+\frac{\alpha^2 r}{2}-\frac{\alpha^3 r^2}{6}+\ldots \tag{3}$$

By substituting Eq. (3) into Eq. (1) we have

$$V(r)=-\frac{\alpha_0}{r}+\alpha_1 r+\alpha_2 r^2+\frac{\alpha_3}{r^2}+\alpha_4 \tag{4}$$

where

$$\left.\begin{aligned}&\alpha_0=a,\ \alpha_1=-\frac{a\alpha^2}{2}\\&\alpha_2=\frac{a\alpha^3}{6},\ \alpha_3=b,\ \alpha_4=a\alpha\end{aligned}\right\} \tag{5}$$

We substitute Eq.(4) into Eq.(2) and obtain

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\varepsilon+\frac{\xi_1}{r}-\xi_2 r-\xi_3 r^2-\frac{L(L+1)}{r^2}\right]R(r)=0 \tag{6}$$

where

$$\left.\begin{aligned}&\varepsilon=\frac{2\mu}{\hbar^2}\left(E_{nl}-\alpha_4\right),\ \xi_1=\frac{2\mu\alpha_0}{\hbar^2}\\&\xi_2=\frac{2\mu\alpha_1}{\hbar^2},\ \xi_3=\frac{2\mu\alpha_2}{\hbar^2}\end{aligned}\right\} \tag{7}$$

$$L(L+1)=\frac{2\mu\alpha_3}{\hbar^2}+l(l+1) \tag{8}$$

From Eq. (8), we have

$$L=-\frac{1}{2}+\frac{1}{2}\sqrt{(2l+1)^2+\frac{8\mu\alpha_3}{\hbar^2}} \tag{9}$$

Now make an anzats wave function [46],

$$R(r)=e^{-\alpha r^2-\beta r}F(r) \tag{10}$$

where α and β are positive constants whose values are determined in terms of potential parameters.

By differentiating Eq.(12) , we have

$$R'(\mathrm{r})=F'(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r} \tag{11}$$

$$\begin{aligned}R''(\mathrm{r})=&\ \mathrm{F}''(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F'(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r}\\&+\left[\left(-2\alpha\right)+\left(-2\alpha r-\beta\right)\left(-2\alpha r-\beta\right)\right]F(r)e^{-\alpha r^2-\beta r}\end{aligned} \tag{12}$$

Upon substituting Eqs. (10), (11) and (12) into Eq.(6) and dividing through by $e^{-\alpha r^2-\beta r}$ we obtain

$$F''(\mathrm{r})+\left[-4\alpha r-2\beta+\frac{2}{r}\right]F'(\mathrm{r})+\left[\begin{aligned}&\left(4\alpha^2-\xi_3\right)r^2+\left(4\alpha\beta-\xi_2\right)r\\&+\left(\xi_1-2\beta\right)\frac{1}{r}-\frac{L(L+1)}{r^2}+\left(\varepsilon+\beta^2-6\alpha\right)\end{aligned}\right]F(r)=0 \tag{13}$$

The function $F(r)$ is considered as a series of the form

$$F(r)=\sum_{n=0}^{\infty}a_n r^{2n+L} \tag{14}$$

Taking the first and second derivatives of Eq.(14) we obtain

$$F'(r) = \sum_{n=0}^{\infty} (2n+L) a_n r^{2n+L-1} \tag{15}$$

$$F''(r) = \sum_{n=0}^{\infty} (2n+L)(2n+L-1) a_n r^{2n+L-2} \tag{16}$$

We substitute Eqs. (14),(15) and (16) into Eq.(13) and obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} (2n+L)(2n+L-1) a_n r^{2n+L-2} + \left[-4\alpha r - 2\beta + \frac{2}{r}\right] \sum_{n=0}^{\infty} (2n+L) a_n r^{2n+L-1} \\ &+ \left[\left(4\alpha^2 - \xi_3\right) r^2 + \left(4\alpha\beta - \xi_2\right) r + \frac{\left(\xi_1 - 2\beta\right)}{r} - \frac{L(L+1)}{r^2} + \left(\varepsilon + \beta^2 - 6\alpha\right)\right] \sum_{n=0}^{\infty} a_n r^{2n+L} = 0 \end{aligned} \tag{17}$$

By collecting powers of r in Eq.(17) we have

$$\sum_{n=0}^{\infty} a_n \left\{ \begin{array}{l} \left[(2n+L)(2n+L-1) + 2(2n+L) - L(L+1)\right] r^{2n+L-2} \\ + \left[-2\beta(2n+L) + (\xi_1 - 2\beta)\right] r^{2n+L-1} \\ + \left[-4\alpha(2n+L) + \varepsilon + \beta^2 - 6\alpha\right] r^{2n+L} \\ + \left[4\alpha\beta - \xi_2\right] r^{2n+L+1} + \left[4\alpha^2 - \xi_3\right] r^{2n+L+2} \end{array} \right\} = 0 \tag{18}$$

Equation (18) is linearly independent, noting that r is a non-zero function; therefore, it is the coefficient of r that is zero. With this in mind, we obtain the relation for each of the terms.

$$(2n+L)(2n+L-1) + 2(2n+L) - L(L+1) = 0 \tag{19}$$

$$-2\beta(2n+L) + \xi_1 - 2\beta = 0 \tag{20}$$

$$-4\alpha(2n+L) + \varepsilon + \beta^2 - 6\alpha = 0 \tag{21}$$

$$4\alpha\beta - \xi_2 = 0 \tag{22}$$

$$4\alpha^2 - \xi_3 = 0 \tag{23}$$

From Eqs.(20) and (23) we have

$$\beta = \frac{\xi_1}{4n + 2L + 2} \tag{24}$$

$$\alpha = \frac{\sqrt{\xi_3}}{2} \tag{25}$$

We proceed to obtaining the energy eigenvalues equation using Eq.(21) and have

$$\varepsilon = 2\alpha(4n + 2L + 3) - \beta^2 \tag{26}$$

By substituting Eqs. (7), (9), (24) and (275) into Eq.(26) and simplifying we obtain

$$E_{nl} = \sqrt{\frac{\hbar^2 \alpha_2}{2\mu}} \left(4n + 2 + \sqrt{(2l+1)^2 + \frac{8\mu\alpha_3}{\hbar^2}}\right) - \frac{2\mu\alpha_0^2}{\hbar^2}\left(4n + 1 + \sqrt{(2l+1)^2 + \frac{8\mu\alpha_3}{\hbar^2}}\right)^{-2} + \alpha_4 \tag{27}$$

Upon substituting Eq.(5) into Eq.(27) we obtain the energy eigenvalues of the generalized Yukawa potential as,

$$E_{nl} = \sqrt{\frac{\hbar^2 a\alpha^3}{12\mu}} \left(4n + 2 + \sqrt{(2l+1)^2 + \frac{8\mu b}{\hbar^2}}\right) - \frac{2\mu a^2}{\hbar^2}\left(4n + 1 + \sqrt{(2l+1)^2 + \frac{8\mu b}{\hbar^2}}\right)^{-2} + a\alpha \tag{28}$$

The unnormalized wavefunction is given as

$$R(r) = \sum_{n=0}^{\infty} a_n r^{2n - \frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 + \frac{8\mu b}{\hbar^2}}} e^{-\frac{\sqrt{\frac{\mu a \alpha^3}{3\hbar^2}}}{2} r^2 - \frac{\frac{2\mu a}{\hbar^2}}{4n + 1 + \sqrt{(2l+1)^2 + \frac{8\mu b}{\hbar^2}}} r} \tag{29}$$

3. Results and discussion

3.1 Results

The mass spectra of the heavy quarkonium system such as charmonium and bottomonium that have the quark and antiquark flavor is calculated by applying the following relation [47, 48]

$$M = 2m + E_{nl}, \tag{30}$$

where m is quarkonium bare mass, and E_{nl} is energy eigenvalues. By substituting Eq. (28) into Eq. (30) we obtain the mass spectra for GYP as:

$$M = 2m + \sqrt{\frac{\hbar^2 a\alpha^3}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2+\frac{8\mu b}{\hbar^2}}\right) - \frac{2\mu a^2}{\hbar^2}\left(4n+1+\sqrt{(2l+1)^2+\frac{8\mu b}{\hbar^2}}\right)^{-2} + a\alpha \tag{31}$$

Table1. Mass spectra of charmonium in GeV
$(m_c = 1.209\ GeV, \hbar = 1, a = 3.65583 GeV, b = -0.159206 GeV, \alpha = 0.23, \mu = 0.6045\ GeV)$

State	Present work	NU[40]	AIM[3]	Experiment[49]
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.524	3.255	3.521	3.525
2P	3.772	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.273	4.269	4.433	4.263
1D	3.727	3.504	3.800	3.770
2D	4.157	-	-	4.159
1F	3.895	-	-	-

Table 2. Mass spectra of bottomonium in (GeV),
$(m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = 4.555837\ GeV, b = 0.469016\ GeV, \alpha = 0.23, \hbar = 1)$

State	Present work	NU[40]	AIM[3]	Experiment[49]
1S	9.460	9.460	9.462	9.460
2S	10.023	10.023	10.027	10.023
1P	9.871	9.619	9.963	9.899
2P	10.260	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.579	10.567	10.624	10.580
1D	10.157	9.864	10.209	10.164
2D	10.698	-	-	-
1F	10.460	-	-	-

3.2 Discussion of results

We calculate mass spectra of heavy quarkonium system such as charmonium and bottomonium for states from 1S, 2S, 1P, 2P, 3S, 4S, 1D, 2D, and 1F, by using Eq. (34). The free parameters of Eq. (34) were then obtained by solving two algebraic equations in the case of charmonium and bottomonium, respectively.

The experimental data were taken from [49]. For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems we adopt the numerical values of these masses as $m_b = 4.823 GeV$ and $m_c = 1.209 GeV$ [50]. Then, the corresponding reduced mass are $\mu_b =$ 2.4115 GeV and $\mu_c = 0.6045 GeV$, respectively. We note that calculation of mass spectra of charmonium and bottomonium are in good agreement with experimental data and work of other researchers, in Refs.[3,40] as presented in Tables 1 and 2. In order to test for the accuracy of the predicted results determined numerically, we used a Chi square function to determine the error between the experimental data and theoretical predicted values. The maximum error in comparison with the experimental data is found to be 0.0023 GeV .

4. Conclusion

In this study, we model the adopted Generalized Yukawa potential to interact in quark-antiquark system. We obtained the solutions of the Schrödinger equation for energy eigenvalues and unnormalized wave function using the series expansion method. We applied the present results to compute heavy-meson masses of charmonium and bottomonium for different quantum states. The result agreed with experimental data with a maximum error of 0.0023 GeV and work of other researchers.

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