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NOTES ON SEMINORMS AND CONTINUITY

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Abstract

We establish three *Notes* the proofs (and precise statements of some) of which are not easily located in the literature.

Keywords: seminorm, (τ, τ')-continuous, the topology τ_p of the seminorm p.

1 LANGUAGE, NOTATION, SOME RECAP Our language and notation shall be pretty standard, as found, for example, in [1 – 5] $\mathbb{N} = \{1, 2, \ldots\ldots..\}$- the *natural numbers*, $\mathbb{R}$-the *real numbers*, $\mathfrak{C}$- the *complex numbers*, while we denote by K either of $\mathbb{R}$ and $\mathfrak{C}$. We indicate by /// the end or absence of proof.

Our vector space $(V, +, \theta)_K = V_K$, is over the field K ; it is an additive Abelina group with *ground set V* and additive identity θ, called its *zero*.

A topological space (V_K, τ), with ground set a vector space V_K, and a topology τ *compatible* with the *addition* and *external multiplication* (*scalar multiplication*) of V_K, is called a *topological vector space*. We assume familiarity with some elements of General Topology **(GT)** and Topological Vector Spaces **(TVS)**, and so freely employ results on (*continuity*, *net convergence, filter of neighbourhoods*, $N_\theta(\tau)$, seminorms, etc, etc) from **GT** and **TVS**. Of course, more than 75% of the subject of **TVS is** *undiluted*, *naked*, *unadulterated* **GT**.

If on the vector space V_K, $p : V_K \to \mathbb{R}$ is a seminorm, p induces on V_K the pseudometric

$dp : V_K \times V_K \to \mathbb{R}$

$(v, w) \mapsto p(v - w)$

The topology τ_{dp}, of this pseudometric is called the *topology of p*, and *here* denoted τ_p.

FACT 1 (TVS) [1] τ_p is a vector topology. ///

The *modulus* / *absolute value*, $|\ |$, on $K = \mathfrak{C} / \mathbb{R}$ is a seminorm (indeed, a norm); its topology $\tau_K = \tau_{\mathfrak{C}}/\tau_{\mathbb{R}}$ is called the *usual topology of* $K = \mathfrak{C} / \mathbb{R}$. By FACT 1, above $(K, +, 0))_K, \tau_{|\ |} = (K, \tau_{|\ |})$ is a topological vector space. Observe that the zero of this space is 0; don't mix it up with the notation θ. *FACT* : $\tau_K = \tau_{|\ |}$. In what follows, the topology on K shall always be τ_K.

Let *I, X* be non-empty sets. If *I* is directed by $\leq$, we *here write* $(x_i)_{i\in(I, \leq)}$ for a *net in X based on the directed set* $(I, \leq)$. If (X, τ) is a topological space, $x_0 \in X$, and a net $(x_i)_{i\in(I, \leq)}$ in X [| we also say *in* (X, τ)|] converges [|also say τ-converges|] to x_0, we may write

$x_i \xrightarrow{\tau} x_0$

Example If p is a seminorm on the vector space V_K, then τ_p is a vector topology on V_K and (V_K, τ_p) is a topological vector space. If $v_0 \in V_K$ and a net $(x_i)_{i\in(I, \leq)}$ in V_K τ_p-converges to v_0, we may write $x_i \xrightarrow{\tau_p} v_0$. A popular instance is

$x_i \xrightarrow{\tau_p} \theta$

where θ is the zero of V_K. If $((V, +, \theta)_K, \tau) = (V_K, \tau)$ is a topological vector space a net $(x_i)_{i\in(I, \leq)}$ in (V_K, τ) converging to θ is called a *null net* (or, a τ-*null net*) and, of course, we write

$x_i \xrightarrow{\tau} \theta$.

If (X, τ) is a topological space and $x_0 \in X$, we denote by $N_{x0}(\tau)$ the filter of neighbourhoods of x_0. *Definition* : **(GT)** For topological spaces (X, τ) and (X', τ'), $x_0 \in X$, and $f : (X, \tau) \to (X', \tau')$ a map [|function|] we say that f is *continuous at* x_0 [| f is (τ, τ')-*continuous at* x_0|] provided for every $W \in N_{f(x0)}(\tau')$ there exists $U \in N_{x0}(\tau)$ such that

$f(U) \subseteq W$.

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[| Equivalently: For every $W \in N_{f(x_0)}(\tau)$ $f^{-1}(W) \in N_{x_0}(\tau)$|]. If f is continuous at every $x \in X$, then f is simply called a *continuous map* [| a *continuous function* |].

FACT 2 (GT) Let (X, τ) and (X', τ') be topological spaces and $x_0 \in X$. A function $f : (X, \tau) \to (X', \tau')$ is continuous at x_0 if and only if for every net $(x_i)_{i\in(I,\leq)}$ in X, τ-converging to x_0, the net $(f(x_i))_{i\in(I,\leq)}$ τ'-converges to $f(x_0)$ [| if and only if
$x_i \xrightarrow{\tau} x_0 \Rightarrow f(x_i) \xrightarrow{\tau'} f(x_0)$|]. ///

FACT 3 (GT) Let (X, τ) ,(X', τ') and (X'', τ'') be topological spaces, and
$(X, \tau) \xrightarrow{f} (X', \tau')$,
$(X', \tau') \xrightarrow{g} (X'', \tau'')$
continuous maps. Then, their composition
$(X, \tau) \xrightarrow{g\circ f} (X'', \tau'')$
is also continuous. ///

FACT 4 If $(V, +, \theta)_{(K,+,\cdot,0,1)} = V_K$ is a vector space, and p a seminorm on V_K, then $p(\theta) = 0$. ///

FACT 5 Let $(V, +, \theta)_K$ and $(V', +, \theta')_K$ be vector spaces, and $f : (V, +, \theta)_K \to (V', +, \theta')_K$ a linear map. Then, $f(\theta) = \theta'$. ///

Let X be a non-empty set and Φ a collection of topologies on X. The coarset of all topologies on X finer than each member of Φ. Is called the *supremum* of Φ and denoted $\vee\Phi$. We have

FACT 6 (GT) Let (X, τ) be a topological space, X' a non-empty set, Φ a collection of topologies on X' and $f : (X, \tau) \to X'$ a map. Then, f is $(\tau, \vee\Phi)$-continuous if and only if f is (τ,τ')-continuous for each $\tau' \in \Phi$. ///

FACT 7 (TVS) If V_K is a vector space, and Φ is a collection of vector topologies on V_K, then the supremum $\vee\Phi$ is a vector topology. ///

Let V_K be a vector space, and P a collection of seminorms on V_K. By FACT 1, τ_p is a vector topology for each $p \in P$. Hence, by FACT 7, $\bigvee_{p\in P} \tau_p = \vee\{\tau_p : p \in P\}$, *here* denoted τ_p, is a vector topology.

The results, FACT1 – FACT 7 recalled for ease of reference may be used in what follows with or without citation.

2 CONTINUOUS SEMINORMS

We curl up four results from **GT** and **TVS** for ease of reference.

FACT 1 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space and $p : (V_K, \tau) \to (\mathbb{R}, \tau_{\mathbb{R}})$ a seminorm. p is $(\tau, \tau_{\mathbb{R}})$-contin- uous if and only if it is $(\tau, \tau_{\mathbb{R}})$-continuous at θ. ///

FACT 2 (GT) For nets $(x_i)_{i\in(I,\leq)}$ and $(y_i)_{i\in(I,\leq)}$ in $\mathbb{R}$ based on same directed set $(I, \leq)$, if
(i) $x_i \xrightarrow{\tau_{\mathbb{R}}} 0$
(ii) $0 \leq y_i \leq x_i$ for all $i \in I$,
then
$y_i \xrightarrow{\tau_{\mathbb{R}}} 0$
also. ///

FACT 3 (GT) (*Net Convergence in* $(\mathbb{R}, \tau_{\mathbb{R}})$) For nets $(x_i)_{i\in(I,\leq)}$ and $(y_i)_{i\in(I,\leq)}$ in $(\mathbb{R}, \tau_{\mathbb{R}})$ based on same directed set $(I, \leq)$, and $x, y \in \mathbb{R}$,
$x_i \xrightarrow{\tau_{\mathbb{R}}} x$
and
$y_i \xrightarrow{\tau_{\mathbb{R}}} y$
jointly imply
$x_i + y_i \xrightarrow{\tau_{\mathbb{R}}} x + y$. ///

Let V_K be a vector space, $\alpha > 0$, p a seminorm on V_K, and $P = \{p_1, p_2, …., p_n\}(n \in \mathbb{N})$ a finite collection of seminorms on V_K. Define, as follows, αp, P_{sum} and P_{max}.
$\alpha_P(\nu) = \alpha(p(\nu))$ for $\nu \in V_K$, and pa seminorm
$P_{sum}(\nu) = p_1(\nu) + p_2(\nu) + …… + p_n(\nu)$ for $\nu \in V_K$
and
$P_{max}(\nu) = \max_{1\leq i\leq n} p_i(\nu)$ for $\nu \in V_K$.

FACT 4 (TVS) [2] Let V_K be a vector space, $\alpha > 0$, p a seminorm on V_K, and $P = \{p_1, p_2, \ldots, p_n\}(n \in \mathbb{N})$ a finite collection of semi- norms on V_K. Then, αp, P_{sum} and P_{max} are also seminorms on V_K. ///

We now state and establish our only theorem of this section.

Note 1 5 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space.

(i) If p is a continous seminorm on (V_K, τ) and $\alpha > 0$, then αp is also a continous seminorm.

(ii) If p, q are seminorms on (V_K, τ), $p \le q$, and q continuous, so is p.

(iii) if $P = \{p_1, p_2, \ldots, p_n\}(n \in \mathbb{N}, n \ge 2)$ is a finite collection of continuous seminorms on (V_K, τ), then P_{sum} is also continuous.

(iv) If $P = \{p_1, p_2, \ldots, p_n\}(n \in \mathbb{N}, n \ge 2)$ is a finite collection of continuous seminorms on (V_K, τ), P_{max} is also continuous.

Proof (i): That αp is a seminorm is one of the claims of FACT 4. It therefore suffices by FACT 1 to show that αp is continuous at θ. So, suppose $(x_i)_{i\in(I,\le)}$ is a net in (V_K, τ) and that

$$x_i \xrightarrow{\tau} \theta \qquad \ldots..(\Delta^1)$$

By the assumed continuity of p, from (Δ^1) and 1.2 follows that

$$p(x_i) \xrightarrow{\tau_R} p(\theta) = 0 \qquad \ldots..(\Delta^2)$$

And from arguments in Elementary Real Analysis, (Δ^2) gives

$$(\alpha p)(x_i) = \alpha p(x_i) \xrightarrow{\tau_R} 0 = \alpha\cdot 0 = \alpha\cdot p(\theta) = (\alpha p)(\theta)$$

(ii): By FACT 1, it suffices to show that p is continuous at θ. So, let $(x_i)_{i\in(I,\le)}$ be a null net in (V_K, τ). By the continuity of q, it follows from 1.2 that

$$q(x_i) \xrightarrow{\tau_R} q(\theta) = 0 \qquad \ldots..(\Delta^3)$$

The hypothesis $p \le q$, (Δ^3) and FACT 2 give

$$p(x_i) \xrightarrow{\tau_R} 0 = p(\theta).$$

iii: It suffices to establish the claim here for $n = 2$. So, suppose $P = \{p_1, p_2\}$. By FACT 4, $p_1 + p_2$ is a seminorm on V_K. Let $(x_i)_{i\in(I,\le)}$ be a null net in (V_K, τ). By hypothesis, p_1 and p_2 are continuous, and so continuous at θ. Hence,

$$p_1(x_i) \xrightarrow{\tau_R} p_1(\theta) = 0$$

and

$$p_2(x_i) \xrightarrow{\tau_R} p_2(\theta) = 0$$

By \FACT 3, therefore,

$$(p_1 + p_2)(x_i) = p_1(x_i) + p_2(x_i) \xrightarrow{\tau_R} 0 + 0 = 0$$

$$= p_1(\theta) + p_2(\theta) = (p_1 + p_2)(\theta).$$

(iv): Clearly, $P_{max} \le P_{sum}$, and so the claim here follows from (ii) and (iii). ///

3 LINEAR MAP/FUNCTIONAL AND CONTINUITY Let V_K be a vector space and p a seminorm on V_K. Recall that in Section 1 we denote by τ_p the pseudometric topology of the pseudometric

$$dp : V_K \mathrm{x} V_K \to \mathbb{R}$$

$$(v, w) \mapsto P(v - w) \qquad \ldots\ldots (\Delta)$$

And called it the topology of p. We have from (Δ) and net convergence in **GT**,

FACT 1 If $p : V_K \to ((\mathbb{R}, +, \cdot, 0, 1),), \tau_{\mathbb{R}})$ is a seminorm on the vector space V_K, $x_0 \in V_K$, and $(x_i)_{i\in(I,\le)}$ is a net in V_K, then,

$$x_i \xrightarrow{\tau_p} x_0$$

if and only if

$$p(x_i - x_0) \xrightarrow{\tau_R} 0 \text{ ///}$$

Note 2 2 Let V_K and V_K' be vector spaces, p a seminorm on V_K' and $f : V_K \to V_K'$ a linear map. Then, the composition $p \circ f : V_K \to \mathbb{R}$ is a seminorm on V_K.

Proof *Positivity* For $v \in V_K$,

$(p \circ f)(v) = p(f(v)) \ge 0$

by the positivity of p.

Absolute Homogeneity For $v \in V_K$,
$(p \circ f)(\lambda v) = p(f(\lambda v)) = p(\lambda f(v))$
$= |\lambda| p(f(v)) = |\lambda| (p \circ f)(v)$.
Triangle Inequality Let $v, w \in V_K$. Then,
$(p \circ f)(v + w) = p(f(v + w)) = p(f(v) + (f(w))$
which by the Triangle Inequality applied to p.
$\leq p(f(v)) + p(f(w)) = (p \circ f)(v) + (p \circ f)(w)$. ///

FACT 3 (TVS) Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ and $((V', +, \theta')_K, \tau') = (V_K', \tau')$ be topological vector spaces and $f : (V_K, \tau) \to (V_K', \tau')$ a linear map. Then, f is continuous if and only if it is continuous at θ. ///
FACT 4 (TVS) Let W_K be a vector space, and p a seminorm on W_K. Then, $p : (W_K, \tau_p) \to (\mathbb{R}, \tau_{\mathbb{R}})$ is continuous. ///
Next, we have
Note 3 5 Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space, p a smeinorm on a vector space $(V', +, \theta')_K = V_K'$, and $f : (V_K, \tau) \to (V_K', \tau_p)$ a linear map. Then, f is continuous if and only if the seminorm $p \circ f$ is continuous.
Proof ⇒: *Hypothesis* f is continuous.
By FACT 4, p is $(\tau_p, \tau_{\mathbb{R}})$ continuous. By 1.3, therefore, $p \circ f$ is contin- uous.
⇐: *Hypothesis* The seminorm (***Note* 2**) $p \circ f$ is continuous.
We want to show that the linear map f is (τ, τ_p)-continuous. By FACT 3, it suffices to show that f is (τ, τ_p)-continuous at θ. So (1.2), let $(x_i)_{i\in(I, \leq)}$ be a net in (V_K, τ) converging to θ. That is, let

$$x_i \xrightarrow{\tau} \theta \qquad \ldots(\Delta^1)$$

By the *Hypothesis* and 2.1, we have from (Δ^1),

$$(p \circ f)(x_i) \xrightarrow{\tau_R} (p \circ f)(\theta) = p(f(\theta)) = p(\theta') = 0.$$

That is,

$$p(f(x_i)) \xrightarrow{\tau_R} 0.$$

That is,

$$p(f(x_i) - \theta') \xrightarrow{\tau_R} 0.$$

That is,

$$p(f(x_i) - f(\theta)) \xrightarrow{\tau_R} 0 \qquad \ldots(\Delta^2)$$

But by FACT 1, (Δ^2) means

$$f(x_i) \xrightarrow{\tau_p} f(\theta) \qquad \ldots(\Delta^3)$$

Clearly, (Δ^1), (Δ^3) and 1.2, jointly say that f is (τ, τ_p)-continuous at θ. ///
COROLLARY 6 [5] Let $((V, +, \theta)_K, \tau) = (V_K, \tau)$ be a topological vector space, P a collection of seminorms on a vector space $(V', +, \theta')_K = V_K'$, and $f : (V_K, \tau) \to (V_K', \tau_P)$ a linear map. Then, f is continuous if and only if the seminorms
$p \circ f : (V_K, \tau) \to (\mathbb{R}, \tau_{\mathbb{R}})$,
$p \in P$ are each continuous.

Proof : $\tau_P = \vee\{\tau_p : p \in P\}$. The claim is immediate from ***Note* 3** above and 1.6. ///
COROLLARY 7 Let $f : (V_K, \tau) \to (K_K, \tau_K)$ be a linear functional on the topological vector space (V_K, τ). Then, f is continuous if and only if $|f|$ is continuous. ///

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