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A VECTOR TOPOLOGY IS UNIFORMIZABLE

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Abstract

A vector topology is uniformizable. We here offer a simple proof of this fundamental fact.

Keywords: Vector topology, uniformity, neighbourhood system of zero (= the filter of neighbourhoods of zero).

1 LANGUAGE AND NOTATION Our language and notation shall be pretty standard, as found in [1 – 6]. $\mathbb{R}$ denotes the *real* (*numbers* $\mathfrak{C}$ the *complex numbers* and $\mathbb{N} = \{1, 2, \ldots..\}$ the *natural numbers*. We signify by /// the end or absence of a proof.

We assume the reader is familiar with the elements of *General Topology* (**GT**) and *Topological Vector Spaces* (**TVS**).

Let $X \neq \emptyset$. Form the Cartesian product $X\text{x}X$; its subset $\Delta_X \equiv \{(x, x) \in X\text{x}X : x \in X\}$is called its *diagonal*. Let $\emptyset \neq A, B \subseteq X\text{x}X$. Define

$A^{-1} \equiv \{(a, b) \in X\text{x}X : (b, a) \in A\}$

called the *inverse* of *A*. Also define $A\text{o}B \equiv \{(p, q) \in X\text{x}X :$ There exists $z \in X$ such that $(p, z) \in B$ and $(z, q) \in A\}$

which we here label the *nought product* of *A* and *B*, following one author that calls o the *nought operator*.

Let $X \neq \emptyset$. A *filter*,$\mathcal{U}$, in $X\text{x}X$ [| a non-empty collection of non-empty subsets of $X\text{x}X$ closed under finite intersections and the taking of supersets|] satisfying

UFT 1 If $U \in \mathcal{U}$, then $U \supseteq \Delta_X$,

UFT 2 $U \in \mathcal{U}$, implies $U^{-1} \in \mathcal{U}$,

UFT 3 For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, such that $V\text{o}V \subseteq U$,

is called a *uniformity*, on *X*.

Let $X \neq \emptyset$ and $\mathcal{U}$ a uniformity on *X*. A *filterbase* $\mathcal{B}$ in $X\text{x}X$ [| a non-empty collection of non-empty subsets of $X\text{x}X$ such that, for $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ satisfying $C \subseteq A \cap B$|] generating $\mathcal{U}$ [|Supersets of members of $\mathcal{B}$ constitute $\mathcal{U}$.|] is called a *base* for $\mathcal{U}$.

FACT 1 GT Let $X \neq \emptyset$. $\mathcal{U}$ a uniformity on *X*, and $\mathcal{B}$ a base for $\mathcal{U}$. Then, $\mathcal{B}$ satisfies

BUFT 1 $U \in \mathcal{B}$ implies $U \supseteq \Delta_X$

BUFT 2 For $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$.

and

BUFT 3 For $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V\text{o}V \subseteq U$. ///

FACT 2 (GT) Let $X \neq \emptyset$ and $\mathcal{B}$ a filterbase in $X\text{x}X$. If $\mathcal{B}$ satisfies BUFT 1, BUFT 2 and BUFT 3 of FACT 1, then $\mathcal{B}$ is a base for a uniformity on *X*. ///

Let (X, τ) be a topological space, $x_0 \in X$ and $\mathcal{N}_{x0}(\tau)$ the filter of τ-neighbourhoods of x_0.

FACT 3 (GT) If (X, τ) is a topological space, and $\emptyset \neq G \in \tau$, then

$$G = \bigcup_{\substack{U \subseteq G \\ x \in U \in N_x(\tau)}} U$$

That is, *G* is a union of neighbourhoods of its points. ///

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Let $\mathcal{U}$ be a uniformity on $X \neq \varnothing$. The pair $(X, \mathcal{U})$ is then called a *uniform space*. $U \in \mathcal{U}$ is called an *entourage* of $\mathcal{U}$, and, for $x \in X$, we define
$U(x) \equiv \{z \in X : (x, z) \in U\}$. Clearly, since $U \supseteq \Delta_X$, $x \in U(X)$.

FACT 4 (GT) Let $(X, \mathcal{U})$ be a uniform space. Then,
$\tau_{\mathcal{U}} = \{\varnothing\} \cup \{\varnothing \neq G \subseteq X$: For each $x \in G$, there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G\} = \{\varnothing\} \cup \left\{\varnothing \neq G \subseteq X : G = \bigcup_{\substack{U \subseteq G \\ x \in U(X) \subseteq G, u \in U}} U(X)\right\}$ is a topology on X. ///

The topology $\tau_{\mathcal{U}}$ in the preceding is called a *uniform* topology, more precisely, the *uniform topology of the uniformity* $\mathcal{U}$ or of the space $(X, \mathcal{U})$.

FACT 5 (GT) Let $(X, \mathcal{U})$ be a uniform space, $x \in X$ and $U \in \mathcal{U}$. then, $U(x) \in \mathcal{N}_x(\tau_{\mathcal{U}})$. ///
Noted in the literature is the next theorem 6, but a proof is difficult to locate. We here furnish a proof.
THEOREM 6 GT Let $(X, \mathcal{U})$ be a uniform space, $x \in X$ and suppose $W \in \mathcal{N}_x(\tau_{\mathcal{U}})$. Then, $W = U(x)$ for some $U \in \mathcal{U}$.
Proof By FACT 4, $W \supseteq U'(x)$ for some $U' \in \mathcal{U}$.
If $W = U'(x)$, then we have nothing more to show. So, suppose $w \in W$ but $w \notin U'(X)$. Hence, $(x, w) \notin U'$. From
$U' \cup \{(x, w)\}$
Add all such (x, w) to U' to obtain U. Clearly, $U \supseteq U'$, and since $\mathcal{U}$ is a filter, $U \in \mathcal{U}$. Clearly also,
$W = U(x)$. ///
Immediate from FACT 3 is
FACT 7 (GT) Let $X \neq \varnothing$ and τ_1, τ_2 topologies on X. Then, $\tau_1 = \tau_2$ if and only if $\mathcal{N}_x(\tau_1) = \mathcal{N}_x(\tau_2)$ for all $x \in X$. ///

By a *vector space* we shall mean an additive Abelian group $(V, +, \theta)_{\mathsf{K}} = V_{\mathsf{K}}$ with scalar multiplication by the elements of the field $\mathsf{K} = \mathbb{R}$ or $\mathfrak{C}$; the additive identity θ is called the *zero* of $(V, +, \theta)_{\mathsf{K}}$.
FACT 8 (TVS) Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space, and $x \in V_{\mathsf{K}}$. Then, $\mathcal{N}_x(\tau) = \{x + W : W \in \mathcal{N}_\theta(\tau)\}$. ///

Let $(V, +, \theta)_{\mathsf{K}} = V_{\mathsf{K}}$ be a vector space and $\varnothing \neq A \subseteq V_{\mathsf{K}}$. The ser A is said to be *balanced* if $\lambda A \subseteq A$ for all $\lambda \in \mathsf{K}$, $|\lambda| \leq 1$. Hence,
(i) $\theta \in A$,
and
(ii) $\frac{1}{2} A \subseteq A$.

FACT 9 (TVS) Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space. Then,
(i) $\lambda U \in \mathcal{N}_\theta(\tau)$ if $U \in \mathcal{N}_\theta(\tau)$ and $\lambda \in \mathsf{K}$, $\lambda \neq 0$ [*Note* : 0 is the *zero* of the field K]
(ii) For $U \in \mathcal{N}_\theta(\tau)$, there exists $V \in \mathcal{N}_\theta(\tau)$ such that $V + V \subseteq U$.
(iii) For $U \in \mathcal{N}_\theta(\tau)$, there exists a balanced $V \in \mathcal{N}_\theta(\tau)$ such that $V \subseteq U$. ///

FACT 10 (TVS) Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space. Then,
$\mathcal{N}_\theta(\tau) = \{-W : W \in \mathcal{N}_\theta(\tau)\}$. ///

2 A VECTOR TOPOLOGY IS UNIFORMIZABLE Let $X \neq \varnothing$ and τ a topology on X. The topology τ is said to be *uniformizable* if there exists a uniformity $\mathcal{U}$, say, on X such that $\tau = \tau_{\mathcal{U}}$.

THEOREM 1[5,(11.10), p.50][6, first paragraph, p.134] Let $((V, +, \theta)_{\mathsf{K}}, \tau) = (V_{\mathsf{K}}, \tau)$ be a topological vector space, and
$\mathcal{B} \equiv \{B_W : W \in \mathcal{N}_\theta(\tau)\}$
where

$B_W \equiv \{(x, y) \in V_K x V_K : x - y \in W\}$.
Then,

(i) $\mathcal{B}$ is a base for a uniformity $\mathcal{U}$, say on V_K,
and

(ii) $\tau_{\mathcal{U}} = \tau$.

Proof (i): We show that $\mathcal{B}$ is a filterbase in $V_K x V_K$ satisfying BUFT 1, BUFT 2 and BUFT 3 of 1.1, and then evoke 1.2. Clearly, $V_K \in \mathcal{N}_\theta(\tau)$, and so, $B_{V_K} \in \mathcal{B}$, from which follows that $\mathcal{B} \neq \varnothing$. Also, if $W \in \mathcal{N}_\theta(\tau)$ then, $\theta \in W$ and $x - x = \theta$ for all $x \in X$, from which follows that

$\Delta_X \subseteq B_W$, and so, $B_W \neq \varnothing$. Hence, $\mathcal{B}$ is a non-empty family of non-empty subsets of $V_K x V_K$. Clearly, if $W_1, W_2 \in \mathcal{N}_\theta(\tau)$, then $W_1 \cap W_2 \in \mathcal{N}_\theta(\tau)$ and $B_{W1 \cap W2} \subseteq Bw_1 \cap Bw_2$. Thus, we have shown that $\mathcal{B}$ is a filterbase in $V_K x V_K$.

Now let $W \in \mathcal{N}_\theta(\tau)$, and so, for $x \in V_K$, $x - x = \theta \in W$, and from this follows that $(x, x) \in Bw$. That is, $\Delta_X \subseteq Bw$. And BUFT 1 is satisfied by $\mathcal{B}$.

Again, if $W \in \mathcal{N}_\theta(\tau)$, then - $W \in \mathcal{N}_\theta(\tau)$, and one checks easily that

$(B_{-W})^{-1} = B_W \subseteq B_W$.

And thus, $\mathcal{B}$ satisfies BUFT 2.

Again, let $W \in \mathcal{N}_\theta(\tau)$. By 1.9, there exists balanced $U \in \mathcal{N}_\theta(\tau)$ such that $U + U \subseteq W$. Because U is balanced, $\frac{1}{2} U \subseteq U$, and so,

$$\frac{1}{2} U + \frac{1}{2} U \subseteq U + U \subseteq W \qquad \text{….(1)}$$

From (1), one sees easily that

$$B_{(\frac{1}{2})U} \circ B_{(\frac{1}{2})U} \subseteq BW.$$

By 1.9(i), $(1/2)U \in \mathcal{N}_\theta(\tau)$. Thus, we have shown that $\mathcal{B}$ satisfies BUFT 3.

(ii): By 1.7 it suffices to show that $\mathcal{N}_X(\tau_{\mathcal{U}}) = \mathcal{N}_X(\tau)$ for all $x \in V_K$; and for this we shall show that

(1) $\mathcal{N}_X(\tau_{\mathcal{U}}) \subseteq \mathcal{N}_X(\tau)$

and

(2) $\mathcal{N}_X(\tau) \subseteq \mathcal{N}_X(\tau_{\mathcal{U}})$;

for all $x \in V_K$.
So, let

$x \in V_K$ and $N \in \mathcal{N}_X(\tau_{\mathcal{U}})$ …..(*)

Then, by 1.6, $N = U(x)$ for some $U \in \mathcal{U}$. Since $\mathcal{B}$ is a base for $\mathcal{U}$, $U \supseteq B_W$ for some $W \in \mathcal{N}_\theta(\tau)$, and so, $U(x) \supseteq B_W(x)$. Hence

$N = U(x) \supseteq B_W(x)$ ….(Δ^1)

Now,

$B_W(x) = \{y \in V_K : x - y \in W\}$

$= \{y \in V_K : y \in x - W\}$
$= x - W$
$= x + (- W)$
That is,
$B_W(x) = x + (- W)$
which by 1.10 and 1.8,
$\in \mathcal{N}_X(\tau)$.
That is,
$B_W(x) \in \mathcal{N}_X(\tau)$ ….(Δ^2)

Clearly, (Δ^1) and (Δ^2) therefore give

$N \in \mathcal{N}_X(\tau)$ ….(Δ^3)

Clearly, (*) and (Δ^3) give (1).

Now suppose

$M \in \mathcal{N}_X(\tau)$ ….(**)

By 1.8, $M = x + W$ for some $W \in \mathcal{N}_\theta(\tau)$. That is,

$M = x + W = B_{-W}(x)$

which by 1.10 and 1.5,

$\in \mathcal{N}_X(\tau_{\mathcal{U}})$.

And so,

$M \in \mathcal{N}_X(\tau_{\mathcal{U}})$ …..(∇)

Clearly, (**) and (∇) give (2). ///

3 REMARK We have established a highly fundamental theorem which is: A vector topology is uniformizable.

Question Why *highly fundamental*?

Answer (a) Let (X, τ) be a topological space. A uniformity $\mathcal{U}$ on X has a topology $\tau_{\mathcal{U}}$ associated with it.

If $\tau_{\mathcal{U}} = \tau$, we say that the topology τ is *uniformizable*.

(b) A *uniformity* $\mathcal{U}$ and its *uniform topology* $\tau_{\mathcal{U}}$ are *Uniform Space* concepts. The theory of Uniform Spaces is part of *General Education* in *General Topology* – Courtesy, Sterling K. Berberian in a private communication to me.

(c) *Equicontinuity, Total Boundedness, Completeness*, etc, are Uniform Space concepts, that become applicable to topological vector spaces due

to the admission of topological vector spaces into the prestigious class of uniform spaces by our THEOREM above. Incidentally, some of these concepts, e.g., *equicontinuity*, assume a serious notoriety in topological

Vector Space Theory [e.g., the equicontinuous sets of linear functionals on a locally convex space, (V_K, τ), determine the topology τ of the space.]

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